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ON STATIONARY CONVECTION IN ROTATORY HYDRODYNAMIC TRIPLY DIFFUSIVE CONVECTION IN A DENSELY PACKED POROUS MEDIUM

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ABSTRACT

The present paper mathematically establishes that 'the principle of the exchange of stabilities' for rotatory hydrodynamic triply diffusive convection analogous to Stern (Tellus, 12, (1960), pp. 172-175) type in a densely packed porous medium is valid in the regime $\frac{|R|A\sigma}{2\pi^4}+T_aP_1^2\leq 1$, *where R is the thermal Rayleigh number, A is a porous parameter,* T_a *is the Taylor number,* P_1 *is a constant and* σ *is the thermal Prandtl number. It is further proved that the above result is uniformly valid for any combination of rigid and free boundaries.*

Keywords: *Triply diffusive convection; Principle of the exchange of stabilities; Rayleigh number; Taylor number; Porous medium; Darcy Model.*

INTRODUCTION

Research on convective fluid motion in porous media has been an area of great activity due to its importance in the predication of ground water movement in aquifers, in engineering geology, in assessing the effectiveness of fibrous materials, in nuclear engineering, in chemical process industry, food processing industry, solidification and centrifugal costing of metals and rotating machinery, geophysics, petroleum industry and biomechanics. Double diffusive convection in porous medium has been extensively studied. For a broad view of the subject one may be referred to Nield and Bezan [11], Murray and Chen [9], Nield [10], Taunton *et al.* [25], Kuznetsov and Nield [6], Vafai [30] and Kellner and Tilgner [5], Vadasz [29], Nield and Bezan [11], Tagare *et al.* [24], Malashetty and Begum [8].

Only double diffusive convection has been investigated by these researchers. However, it has been recognized later that there are many fluid systems, in which more than two components are present. The oceans contain many salts having concentrations less than a few percent of the sodium chloride concentration. Multi-component concentrations can also be found in magmas and substratum of water reservoirs. The subject with more than two components (in porous and non porous medium) has attracted the attention of many researchers Griffiths [2, 3], Poulikakos [14], Pearlstein and Harris [12], Terrones and Pearlstein [26], Rudraiah and Vortmeyer [20], Lopez *et al.* [7], Tracey [27, 28], Straughan and Tracey [23] and Prakash *et al.* [16, 18, 19]. The essence of the works of these researchers is that small salinity of a third component with a smaller mass diffusivity can have a significant effect upon the nature of convection; and 'oscillatory' and direct 'salt finger' modes are simultaneous possible under a wide range of conditions.

The establishment of the nonoccurrence of any slow oscillatory motions which may be neutral or unstable implies the validity of the principle of the exchange of stabilities (PES). The validity of this principle in stability problems eliminates the unsteady terms from the linearized perturbation equations which results in notable mathematical simplicity since the transition from stability to instability occurs via a marginal state which is characterized by the vanishing of both real and imaginary parts of the complex time eigenvalue associated with the perturbation. Pellew and Southwell [13] proved the validity of PES (i.e. occurrence of stationary convection) for the classical Rayleigh-Benard instability problem. However no such results existed for other more general hydrodynamic configurations. Banerjee *et al.* [1] established such a criterion for magnetohydrodynamic Rayleigh-Benard convection problem which has further been extended by Gupta *et al.* [4] for thermohaline convection problems. Recently Prakash *et al.* [15] extended these results to some triply diffusive configurations.

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The aim of the present paper is to establish criteria for characterizing non oscillatory motions which may be neutral or unstable for rotatory hydrodynamic triply diffusive configuration in a densely packed porous medium analogous to Stern [22] type. It is proved that for rotatory hydrodynamic triply diffusive convection analogous to Stern [22] in densely packed porous medium, if $\frac{|R|A\sigma}{2\pi^4} + T_a P_1^2 \leq 1$, then an arbitrary neutral or unstable mode of the system is definitely non oscillatory in character and in particular PES is valid where R is thermal Rayleigh number, A is a porous parameter, T_a is the Taylor number and σ is the Prandtl number. It is further proved that the above result is uniformly valid for all the combinations of rigid and free boundaries.

MATHEMATICAL FORMULATION OF THE PROBLEM

An infinite horizontal porous layer filled with a viscous and Boussinesq fluid, statically confined between two horizontal boundaries $z = 0$ and $z = d$, respectively maintained at uniform constant temperatures T_0 and T_1 $(> T_0)$ and uniform concentrations S_{10} , S_{20} and $S_{11}(> S_{10})$, $S_{21}(> S_{20})$ is kept rotating at a constant rate $\vec{\Omega}$ about the vertical (see fig. 1). It is further assumed that the cross-diffusion effects of the stratifying agencies can be neglected. The Darcy model has been used to investigate the triply diffusive convection in a densely packed porous medium.

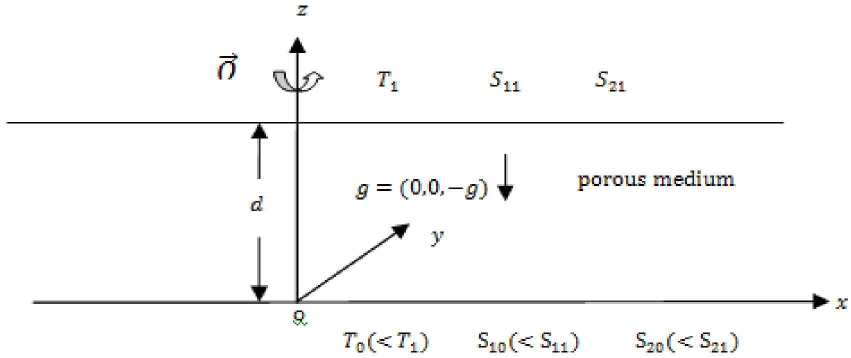

Figure-1: Geometrical Configuration

The equations that govern the motion of triply diffusive fluid layer in a densely packed porous medium (Darcy Model) under the action of a uniform vertical rotation, in the non-dimensional form, are as follows (Prakash *et al.* [17], with $R < 0, R_1 < 0$ and $R_2 < 0$):

$$\left(\frac{p}{\sigma} + \frac{1}{P_1}\right)(D^2 - a^2)w = |R|a^2\theta - |R_1|a^2\phi_1 - |R_2|a^2\phi_2 - T_a D\zeta, \tag{1}$$

$$(D^2 - a^2 - Ap)\theta = -w, \tag{2}$$

$$\left(D^2 - a^2 - \frac{p}{\tau_1}\right)\phi_1 = -\frac{w}{\tau_1}, \tag{3}$$

$$\left(D^2 - a^2 - \frac{p}{\tau_2}\right)\phi_2 = -\frac{w}{\tau_2}, \tag{4}$$

and
$$\left(\frac{p}{\sigma} + \frac{1}{P_1}\right)\zeta = Dw. \tag{5}$$

Eqs. (1) - (5) are to be solved using the following boundary conditions:

$$w = 0 = \theta = \phi_1 = \phi_2 = D^2 w = D\zeta \text{ at } z = 0 \text{ and } z = 1, \tag{6}$$

(Both the boundaries are dynamically free)

Or
$$w = 0 = \theta = \phi_1 = \phi_2 = Dw = \zeta \text{ at } z = 0 \text{ and } z = 1, \tag{7}$$

(Both the boundaries are rigid)

$$w = 0 = \theta = \phi_1 = \phi_2 = D^2 w = D\zeta \text{ at } z = 0, \tag{8}$$

(lower boundary is dynamically free)

and
$$w = 0 = \theta = \phi_1 = \phi_2 = Dw = \zeta \text{ at } z = 1, \tag{9}$$

(upper boundary is rigid)

Eqs. (1) – (5) together with the boundary conditions (6) – (9) present an eigenvalue problem for p for the given values of the other parameters and govern rotatory triply diffusive convection in a porous medium.

The meaning of the symbols involved in equations (1)-(9) from the physical point of view are as follows : z is the vertical coordinate, D is the differentiation w.r.t. z , a^2 is square of the wave number, $\sigma > 0$ is the Prandtl number, $\tau_1 > 0$ and $\tau_2 > 0$ are the Lewis numbers for the two concentration components with mass diffusivity κ_1, κ_2 respectively, $R < 0$ is the thermal Rayleigh number, $R_1 < 0$ and $R_2 < 0$ are the two concentration Rayleigh numbers, $T_a > 0$ is Taylor number, $p = p_r + i\,p_i$ is the complex growth rate where p_r and p_i are real constants, w is the vertical velocity, θ is the temperature and ϕ_1 and ϕ_2 are the two concentrations. The governing equations also involve two more positive constants namely $P_1 = \frac{k_1}{\epsilon d^2}$ and $A = 1 + \left(\frac{\rho_{s_0} c_{s_0}}{\rho_0 c_0}\right)\frac{1-\epsilon}{\epsilon}$, where k_1 is the permeability, ϵ is the porosity of the medium, d is the depth of the fluid layer, ρ_{s_0} is the solid density, c_{s_0} is the heat capacity of the solid. The suffix '0' denotes the values of various parameters involved in the governing equations at some properly chosen temperature T_0 .

We prove the following theorem:

Theorem: If $(w, \theta, \phi_1, \phi_2, p, \zeta)$, $p_r \geq 0$ is a solution of Eqs. (1)– (9) with < 0, $R_1 < 0$, $R_2 < 0, T_a > 0$ and $\frac{|R|A\sigma}{2\pi^4} + T_a P_1^2 \leq 1$, then $p_i = 0$. In particular $p_r = 0$ implies $p_i = 0$, if

$$\frac{|R|A\sigma}{2\pi^4} + T_a P_1^2 \leq 1.$$

Proof: Multiplying Eq. (1) by w^* (the complex conjugate of w) and integrating the resulting equation over the vertical range of z, we obtain

$$\left(\frac{p}{\sigma} + \frac{1}{P_1}\right) \int_0^1 w^*(D^2 - a^2) w dz = |R| a^2 \int_0^1 w^* \theta \, dz - |R_1| a^2 \int_0^1 w^* \phi_1 \, dz - |R_2| a^2 \int_0^1 w^* \phi_2 \, dz - T_a \int_0^1 w^* D\zeta \, dz. \quad (10)$$

Making use of Eqs. (2) – (5) and the fact that $w(0) = 0 = w(1)$, we can write

$$|R| a^2 \int_0^1 w^* \theta \, dz = -|R| a^2 \int_0^1 \theta \, (D^2 - a^2 - Ap^*) \theta^* \, dz, \quad (11)$$

$$-|R_1| a^2 \int_0^1 w^* \phi_1 \, dz = |R_1| a^2 \tau_1 \int_0^1 \phi_1 \left(D^2 - a^2 - \frac{p^*}{\tau_1}\right) \phi_1^* \, dz, \quad (12)$$

$$-|R_2| a^2 \int_0^1 w^* \phi_2 \, dz = |R_2| a^2 \tau_2 \int_0^1 \phi_2 \, (D^2 - a^2 - \frac{p^*}{\tau_2}) \phi_2^* \, dz, \quad (13)$$

$$-T_a \int_0^1 w^* D\zeta \, dz = T_a \left(\frac{p^*}{\sigma} + \frac{1}{P_1}\right) \int_0^1 |\zeta|^2 \, dz. \quad (14)$$

Combining Eqs. (10) – (14), we get

$$\left(\frac{p}{\sigma} + \frac{1}{P_1}\right) \int_0^1 w^*(D^2 - a^2) w dz = -|R| \, a^2 \int_0^1 \theta \, (D^2 - a^2 - Ap^*) \theta^* \, dz + |R_1| a^2 \tau_1$$
$$\int_0^1 \phi_1 \left(D^2 - a^2 - \frac{p^*}{\tau_1}\right) \phi_1^* \, dz + |R_2| a^2 \tau_2 \int_0^1 \phi_2 \, (D^2 - a^2 - \frac{p^*}{\tau_2}) \phi_2^* dz + T_a \left(\frac{p^*}{\sigma} + \frac{1}{P_1}\right) \int_0^1 |\zeta|^2 \, dz. \quad (15)$$

Integrating the various terms of Eq. (15), by parts, for a suitable number of times and utilizing the boundary conditions (6) - (9), we obtain

$$\left(\frac{p}{\sigma} + \frac{1}{P_1}\right) \int_0^1 (|Dw|^2 + a^2 |w|^2) dz = -|R| \, a^2$$
$$\int_0^1 (|D\theta|^2 + a^2 |\theta|^2 + Ap^* |\theta|^2) dz + |R_1| a^2 \tau_1 \int_0^1 \left(|D\phi_1|^2 + a^2 |\phi_1|^2 + \frac{p^*}{\tau_1} |\phi_1|^2\right) dz + |R_2| a^2 \tau_2 \int_0^1 \left(|D\phi_2|^2 + \right.$$

a2ϕ22+p∗τ2ϕ22dz−Tap∗σ+1P101ζ2 dz. (16)

Equating the imaginary parts of both sides of Eq. (16) and cancelling $p_i \ (\neq 0)$ throughout from the resulting equation, we have

$$\frac{1}{\sigma} \int_0^1 (|Dw|^2 + a^2 |w|^2) dz = |R| A a^2 \int_0^1 |\theta|^2 dz - |R_1| a^2 \int_0^1 |\phi_1|^2 dz - |R_2| a^2 \int_0^1 |\phi_2|^2 dz + \frac{T_a}{\sigma} \int_0^1 |\zeta|^2 \, dz. \quad (17)$$

Now, multiplying Eq. (2) by its complex conjugate and integrating the various terms of resulting equation, by parts, for an appropriate number of times and making use of the boundary conditions on θ, it follows that

$$\int_0^1 (|D^2\theta|^2 + 2a^2 |D\theta|^2 + a^4 |\theta|^2) dz + 2Ap_r \int_0^1 (|D\theta|^2 + a^2 |\theta|^2) dz + A^2 |p|^2 \int_0^1 |\theta|^2 \, dz = \int_0^1 |w|^2 \, dz. \quad (18)$$

Since $p_r \geq 0$, it follows from Eq. (18), that

$$2a^2 \int_0^1 |D\theta|^2 dz < \int_0^1 |w|^2 \, dz. \quad (19)$$

Now, since θ and w satisfy the boundary conditions, namely, $\theta(0) = 0 = \theta(1)$, $w(0) = 0 = w(1)$ respectively, we have by Rayleigh-Ritz inequality (Schultz [21])

$$\int_0^1 |D\theta|^2 dz \geq \pi^2 \int_0^1 |\theta|^2 dz, \quad (20)$$

and $$\int_0^1 |Dw|^2 dz \geq \pi^2 \int_0^1 |w|^2 dz. \quad (21)$$

Utilizing inequalities (20) and (21) in inequality (19), we get

$$a^2 \int_0^1 |\theta|^2 dz \leq \frac{1}{2\pi^4} \int_0^1 |Dw|^2 dz. \quad (22)$$

Multiplying Eq. (5) by ζ^* on both sides and equating real parts on both sides, we obtain

$$\frac{p_r}{\sigma} \int_0^1 |\zeta|^2 \, dz + \frac{1}{P_1} \int_0^1 |\zeta|^2 \, dz = \text{real part of} \left(\int_0^1 \zeta^* Dw \, dz\right)$$
$$\leq \left|\int_0^1 \zeta^* Dw dz\right| \leq \int_0^1 |\zeta^* Dw| dz$$
$$\leq \left(\int_0^1 |Dw|^2 dz\right)^{1/2} \left(\int_0^1 |\zeta|^2 dz\right)^{1/2}. \quad \text{(using Schwartz inequality)}$$

Since $p_r \geq 0$, above inequality implies that

$$\frac{1}{P_1}\left(\int_0^1 |\zeta|^2 dz\right)^{1/2} \leq \left(\int_0^1 |Dw|^2 dz\right)^{1/2}, \tag{23}$$

which gives

$$\int_0^1 |\zeta|^2 dz \leq P_1^2 \int_0^1 |Dw|^2 dz. \tag{24}$$

Using inequalities (22) and (24) in Eq. (17), we get

$$\left[\frac{1}{\sigma} - \left(\frac{|R|A}{2\pi^4} + \frac{T_a P_1^2}{\sigma}\right)\right] \int_0^1 |Dw|^2 dz + \frac{a^2}{\sigma}\int_0^1 |w|^2 dz + |R_1| a^2 \int_0^1 |\phi_1|^2 dz + |R_2| a^2 \int_0^1 |\phi_2|^2 dz < 0, \tag{25}$$

which clearly implies (for $p_i \neq 0$) that

$$\frac{|R|A\sigma}{2\pi^4} + T_a P_1^2 > 1. \tag{26}$$

Hence if $\frac{|R|A\sigma}{2\pi^4} + T_a P_1^2 \leq 1$, then we must have $p_i = 0$.

This establishes the desired result.

The essential content of the theorem from the physical point of view are that for the problem of rotatory hydrodynamic triply diffusive convection in a porous medium analogous to Stern [22] type of an arbitrary neutral or unstable mode of the system is definitely nonoscillatory in character if $\frac{|R|A\sigma}{2\pi^4} + T_a P_1^2 \leq 1$ and in particular PES is valid if

$$\frac{|R|A\sigma}{2\pi^4} + T_a P_1^2 \leq 1.$$

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