International Journal of Mathematical Archive-9(4), 2018, 183-185
Available online through www.ijma.info ISSN 2229 – 5046

A UNIQUE FIXED POINT THEOREM IN CONE METRIC SPACE

S. VIJAYA LAKSHMI*

**Department of Mathematics,
University College of Science, Osmania University,
Hyderabad-500004, Telangana State, India.**

(Received On: 01-02-18; Revised & Accepted On: 16-04-18)

ABSTRACT

In this paper, we prove a unique common fixed point theorem for generated contractive in complete cone metric spaces without normal cone. Our results generalize and extension of some of the recent results existing in the literature.

***Key words**: Cone metric space, normal and non normal cone, fixed point.*

***AMS Subject Classification:** 47H10, 54H25.*

1. INTRODUCTION AND PRELIMINARIES

In 2007, Huang and Zhang [3] have generalized the concept of a metric space, they replacing the set of real numbers by ordered Banach space and obftained some fixed point theorems for mapping satisfying different contractive conditions. Subsequently many authors like Abbas and Jungck [1] and Abbas and Rhoades [2] have studied common fixed point theorems in cone metric spaces. (see also [4.5]). In 2008, Rezapour, and Hamlbarni [4] generalized fixed point results of Huang and Zhang [3] proved fixed point theorems without normal cone. In this paper, we proved a unique common fixed point theorem for generalized contractive condition without normal cone. Our result extended and generalized the results of Rezapour and Halbarani [4].

The following definitions are due to Huang and Zhang [3].

Definition 1.1: Let B be a real Banach space and P a subset of B .The set P is called a cone if and only if:
(a) P is closed, non –empty and $P\neq\{0\}$
(b) a, b $\in R$, a, b ≥ 0, x, y $\in P$ implies ax + by $\in P$;
(c) x $\in$ P and -x $\in P$ implies x = 0.

Definition 1.2: Let P be a cone in a Banach space B, define partial ordering '$\leq$' with respect to P by x $\leq$ y if and only if y-x $\in P$. We shall write x<y to indicate $x\leq y$ but $x \neq y$ while x<<y will stand for y-x $\in$ Int P, where Int P denotes the interior of the set P. This cone P is called an order cone.

Definition 1.3: Let B be a Banach space and $P\subset B$ be an order cone .The order cone P is called normal if there exists L>0 such that for all x, y $\in$ B, $0 \leq x \leq y$ implies $\|x\|\leq L\|y\|$.

The least positive number L satisfying the above inequality is called the normal constant of P.

Definition 1.4: Let X be a nonempty set of B .Suppose that the map d: $X \times X \to B$ satisfies:
(d1). $0 \leq d(x, y)$ for all x, y $\in X$ and
d(x, y) = 0 if and only if x = y ;
(d2). d(x, y) = d(y, x) for all x, y $\in X$;
(d3). $d(x, y)\leq d(x, z) + d(y, z)$ for all x, y, z $\in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Corresponding Author: S. Vijaya Lakshmi, Department of Mathematics,
University College of Science, Osmania University, Hyderabad-500004, Telangana State, India.*

The concept of a cone metric space is more general than that of a metric space.

Example 1.5: ([3]) Let $B=R^2$, $P = \{(x, y)\in B$ such that : $x, y \geq 0\} \subset R^2$, $X = R$ and $d: X \times X \rightarrow B$ such that $d(x, y) = (|x - y|, \alpha\, |x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 1.6: Let (X, d) be a cone metric space. We say that $\{x_n\}$ is

(i) a Cauchy sequence if for every c in B with $c>>0$,there is N such that for all $n, m>N$, $d(x_n, x_m)<<c$;

(ii) convergent sequence if for any $c >>0$, there is an N such that for all $n>N$, $d(x_n, x) <<c$, for some fixed x in X . We denote this $x_n \rightarrow x$ (as $n \rightarrow \infty$) .

Lemma 1.7: Let (X, d) be a cone metric space, and let P be a normal cone with normal constant L. Let $\{x_n\}$ be a sequence in X .Then

(i) $\{x_n\}$ converges to x if and only if $d(x_n, x)\rightarrow 0$ $(n \rightarrow \infty)$.

(ii) $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m)\rightarrow 0$ (as $n, m\rightarrow\infty$).

2. MAIN RESULTS

Theorem 2.1: Let (X,d) be a complete cone metric space and the mapping $T:X\rightarrow X$ satisfy the contractive condition $d(Tx, Ty) \leq \alpha d(x, y) + \beta\, [d(x, Tx) + d(y, Ty)] + \gamma\, [d(x, Tx) + d(y, Ty)]$ for all $x, y\in X$. where $\alpha, \beta, \gamma \in [0,1)$ is constant $\alpha + 2\beta + 2\gamma < 1$. Then T has a fixed point in X. Also the fixed point of T is unique whenever $\alpha + \beta + \gamma < 1$.

Proof: Let $x_0\in X$ and $n\geq 1$, constants $x_1=Tx_0$ and $x_{n+1}=Tx_n=T^{n+1}x_0$ then

$$
\begin{aligned}
d(x_{n+1},x_n) &= d(Tx_n,x_{n-1})\\
&\leq \alpha d(x_n,x_{n-1}) + \beta[d(x_n,Tx_n) + d(x_{n-1},Tx_{n-1})] + \gamma[d(x_n,Tx_{n-1}) + d(x_{n-1},Tx_n)]\\
&\leq \alpha d(x_n,x_{n-1}) + \beta[d(x_n,x_{n+1}) + d(x_{n-1},x_n)] + \gamma[d(x_n,x_n) + d(x_{n-1},x_{n+1})]\\
&\leq \alpha d(x_n,x_{n-1}) + \beta[d(x_n,Tx_{n+1}) + d(x_{n-1},x_n)] + \gamma[d(x_{n-1},x_n) + d(x_n,x_{n+1})]\\
&\leq (\alpha+\beta+\gamma)\, d(x_n,Tx_{n+1}) + (\beta+\gamma)\, d(x_{n+1},x_{n+1})
\end{aligned}
$$

$\Rightarrow$ $1-(\beta+\gamma)\, d(x_{n+1},x_n) \leq (\alpha+\beta+\gamma)d(x_n,x_{n-1})$

$$d(x_{n+1},x_n) \leq \frac{(\alpha+\beta+\gamma)}{(1-(\beta+\gamma)} < 1$$

$$
\begin{aligned}
\Rightarrow d(x_{n+1},x_n) &\leq bd(x_n,x_{n-1})\\
&\leq b^n\, d(x_1,x_0)
\end{aligned}
$$

Then for $n > m$, we have

$$
\begin{aligned}
d(x_n,x_m) &\leq d(x_n,x_{n-1}) + d(x_{n-1},x_{n-2})+\ldots\ldots\ldots + d(x_{m+1},x_m)\\
&\leq b(s^{n-1} + s^{n-2} + s^{n-3} +\ldots+s^m)\, d(x_1,x_0)\\
&\leq \frac{b^m}{1-b}\, d(x_1,x_0)\\
&\leq \alpha d(x_{n-1},x) + \beta[d(x_{n-1},x) + d(x,x_n)+d(x,Tx)] + \gamma[d(x_{n-1},x) + d(x,x_n)] + d(x_n,x)]\\
&\leq (\alpha+\beta+\gamma)\, d(x_{n-1},x) + (1+\beta+\gamma)\, d(x_n,x) + (\beta+\gamma)d(x,Tx)
\end{aligned}
$$

$\Rightarrow$ $1-(\beta+\gamma)\, d(Tx, x) \leq (\alpha+\beta+\gamma)\, d(x_{n-1}, x) + (1+\beta+\gamma)d(x_n, x)$

$$d(Tx, x) \leq \frac{(\alpha+\beta+\gamma)}{(1-(\beta+\gamma)}\, d(x_{n-1},x) + \frac{(1+\beta+\gamma)}{(1-(\beta+\gamma)}\, d(x_n, x)$$

Let $0<<c$, choose a natural number n, such that

$d(x_{n-1}, x_n) << \frac{c}{2}\, \frac{1-(\beta+\gamma)}{(\alpha+\beta+\gamma)}$ there exists $n_2 \in N$ such that

$d(x_n, x) << \frac{c}{2}\, \frac{1-(\beta+\gamma)}{(1+\beta+\gamma)}$

$\Rightarrow$ $d(Tx, x) << \frac{c}{2} + \frac{c}{2} = c$

Let $0<<c$ be given, choose a natural number k_1, Such that $\frac{b^m}{1-b}\, d(x_1, x_0) <<$ for all $m \geq k_1$.

Thus $d(x_n, x_m) << c$, $n > m$

Therefore $\{x_n\}$ is a Cauchy sequence in (x, d)

Since (x, d) is a complete,

There exists $x\in X$, $x_n \rightarrow x$,

$$
\begin{aligned}
d(Tx, x) &\leq d(x_n,Tx) + d(x_n,x)\\
&\leq d(Tx_{n-1},Tx) + d(x_n, x)\\
&\leq \alpha d(x_{n-1},x) + \beta[d(x_{n-1},Tx_{n-1}) + d(x, T x)] + \gamma[d(x_{n-1},Tx) + d(x,Tx_{n-1})] + d(x_n, x)\\
&\leq \alpha d(x_{n-1},x) + \beta[d(x_{n-1},x_n) + d(x, Tx)] + \gamma[d(x_{n-1},x) + d(x, Tx) + d(x, x_n)] + d(x_n, x)
\end{aligned}
$$

Thus $d(Tx, x) << \frac{c}{r}$ for all $r \geq 1$

Hence $\frac{c}{r}$ - d(Tx, x) $\in$ p for all $r \geq 1$.

Since $\frac{c}{r} \to 0$ (as $r \to \infty$) and p is closed, - d(Tx, x) $\in$ p.

But d(Tx, x) $\in$ p

Therefore d(Tx, x) = p.
$\Rightarrow$ Tx = x.

Therefore T has a fixed point.

Uniqueness: Let x, be another fixed point of X. Then

$$
\begin{aligned}
d(Tx, x) &= d(Tx, x) \\
&\leq \alpha d(x, x_1) + \beta[d(x, Tx) + d(x, Tx)] + \gamma\,[d(x, Tx_1) + d(x_1, Tx)] \\
&\leq \alpha d(x, x_1) + \beta[d(x, x) + d(x, x_1)] + \gamma[d(x, x_1) + d(x_1, x)] \\
&\leq (\alpha+2\gamma)\, d(x, x_1) \\
&\leq (\alpha+2\gamma)\, d(x, x_1)
\end{aligned}
$$

$\Rightarrow$ $d(x, x_1) = 0$
$\Rightarrow$ $x = x_1$ since $\alpha+2\gamma < 1$.

Therefore T has a unique fixed point.

Remarks 2.2: If we choose $\alpha=\beta=0$ at $\gamma = k$ and $k\in[0,1/2)$ in the above theorem 2.1, then we get Theorem 2.6 of [4]

Remarks 2.3: If we choose $\alpha=\gamma=0$ at $\beta = k$ and $k\in[0,1/2)$ in the above theorem 2.1, then we get Theorem 2.7 of [4]

CONCLUSION

We have generalized and extended the results of [4].

REFERENCES

1. M.Abbas and G.Jungck, Common fixed point results for non commuting mappings without continuity in cone metric spaces, J.Math.Anal.Appl. 341(2008), 416-420.
2. M.Abbas and B.E. Rhoades, Fixed and periodic point results in cone metric spaces, Appl. Math. Lett. 22 (2009), 511-515.
3. L.G.Huang, X.Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332(2)(2007), 1468-1476.
4. S.Rezapour and Hamlbarani, Some notes on the paper cone metric spaces and fixed point theorem of contractive mappings, J. Math. Anal. Appl. 345(2008), 719-724.
5. Stojan Radenovi, Common fixed points under contractive conditions in cone metric spaces, Computer And Mathematics with Applications 582009), 1273-1278.

Source of support: Nil, Conflict of interest: None Declared.