(k,1)-CONTRA HARMONIC MEAN LABELING OF GRAPHS

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ABSTRACT

Let \( G = (V,E) \) be a graph with \( p \) vertices and \( q \) edges. Let \( f : V(G) \rightarrow \{0,1,2,\ldots,k+(q-1)\} \) be an injective function such that the induced edge labeling \( f(e = uv) \) is defined by \( f(e) = \left[ \frac{f(u)^2 + f(v)^2}{f(u) + f(v)} \right] \) or \( \left[ \frac{f(u)^2 + f(v)^2}{f(u) + f(v)} \right] \) is a bijection from \( E \) to \( \{k, k+1, k+2,\ldots,k+(q-1)\}. \) Then \( f \) has a \((k,1)\)-contra harmonic mean labeling. Any graph which admits a \((k,1)\)-contra harmonic mean labeling is called a \((k,1)\)-contra harmonic mean graph. In this paper we investigate the \((k,1)\)-Contra Harmonic mean labeling for some path related graphs.

Keywords: Contra Harmonic mean labeling, \((k, 1)\) - Contra Harmonic mean labeling.

1. INTRODUCTION

Let \( G = (V,E) \) be a finite, simple, undirected graph with \( p \) vertices and \( q \) edges. For all detailed survey of graph labeling we refer to Gallian [1]. For all other standard terminology and notation we follow Harary [4]. We will give a brief summary of definition and other information which are useful for the present investigation.

A graph \( G = (V,E) \) with \( p \) vertices and \( q \) edges is called a Contra Harmonic mean graph if it is possible to label the vertices \( x \in V \) with distinct elements \( f(x) \) from \( 0,1,2,\ldots,q \) in such a way that each edge \( e=uv \) is labeled with \( f(e=uv) = \left[ \frac{f(u)^2 + f(v)^2}{f(u) + f(v)} \right] \) or \( \left[ \frac{f(u)^2 + f(v)^2}{f(u) + f(v)} \right] \) with distinct edge labels . Here \( f \) is called a Contra Harmonic mean labeling of \( G \).

S. Somasundaram and R. Ponraj introduced mean labeling of graphs and investigated mean labeling for some standard graph in [6]. These labeling patterns motivated us to introduced Contra Harmonic mean labeling [5]. In this paper we prove that some path related graphs admits \((k,1)\)-Contra Harmonic mean labeling where \( k \) is any positive integer greater than or equal to 1.

Definition 1.1: Let \( G = (V,E) \) be a graph with \( p \) vertices and \( q \) edges . Let \( f : V(G) \rightarrow \{0,1,2,\ldots,k+(q-1)\} \) be an injective function such that the induced edge labeling \( f(e = uv) \) is defined by \( f(e) = \left[ \frac{f(u)^2 + f(v)^2}{f(u) + f(v)} \right] \) or \( \left[ \frac{f(u)^2 + f(v)^2}{f(u) + f(v)} \right] \) is a bijection from \( E \) to \( \{k, k+1, k+2,\ldots,k+(q-1)\}. \) Then \( f \) has a \((k,1)\)-contra harmonic mean labeling. Any graph which admits a \((k,1)\)-contra harmonic mean labeling is called a \((k,1)\)-contra harmonic mean graph. Here \( k \) is a positive integer greater than or equal to 1.

Definition 1.2: A Triangular snake \( T_n \) is obtained from a path \( u_1\ldots u_n \) by joining \( u_i \) and to a vertex \( v_i \) for \( 1 \leq i \leq n-1 \).
Definition 1.3: A Quadrilateral snake $Q_n$ is obtained from a path $u_1,\ldots,u_n$ by joining $u_i$ and $u_{i+1}$ to new vertices $v_i$, $w_i$, $1 \leq i \leq n-1$.

Definition 1.4: A middle graph $M(G)$ of a graph $G$ is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of $G$ or one is a vertex of $G$ and the other is an edge incident on it.

2. MAIN RESULTS

Theorem 2.1: A path has a $(k, 1)$-Contra Harmonic mean labeling for all $k$.

Proof: Let $u_1,\ldots,u_n$ be the vertices of the path $P_n$.

We define an injective function $f: V(P_n) \rightarrow \{0, 1, 2,\ldots,k+(q-1)\}$ as follows
$$f(u_1)=k-1, \quad f(u_i)=k+i-2, \quad 2 \leq i \leq n$$

The distinct edge labeling are as follows
$$f(u_iu_{i+1}) = i-1+k, \quad 1 \leq i \leq n-1$$

Hence, $f$ is a $(k, 1)$-Contra Harmonic mean labeling of $G$.

Thus, the path admits a $(k, 1)$-Contra Harmonic mean graph for all $k$.

Figure-1: $(17, 1)$ Contra Harmonic mean labeling of $P_5$

Theorem 2.2: A comb is a $(k, 1)$-Contra Harmonic mean graph for all $k$.

Proof: Let $u_1,u_2,u_3,\ldots,u_n$ be the vertices of the path and let $v_i$ be the pendant vertices attached to each $u_i$, $1 \leq i \leq n$.

Let $G$ be a comb graph $P_n \circ K_1$.

We define $f: V(G) \rightarrow \{0, 1, 2,\ldots,k+(q-1)\}$ as follows
$$f(u_i)=2i-3+k, \quad 1 \leq i \leq n$$
$$f(v_i)=2i-2+k, \quad 1 \leq i \leq n$$

The distinct edge labeling are as follows
$$f(u_iu_{i+1}) = 2i-1+k, \quad 1 \leq i \leq n-1$$
$$f(u_iv_i) = 2i-2+k, \quad 1 \leq i \leq n$$

Hence, $f$ is a $(k, 1)$-Contra Harmonic mean labeling of $G$.

Then, the comb is a $(k, 1)$-Contra Harmonic mean graph for all $k$.

Figure-2: $(51, 1)$-Contra Harmonic mean labeling of $P_4 \circ K_1$
Theorem 2.3: $P_n \circ K_{1,2}$ admits $(k,1)$-Contra Harmonic mean labeling for all $k$.

Proof: Let $P_n$ be the path $u_1, u_2, u_3, \ldots, u_n$ and $v_i, w_i$ be the vertices of $K_{1,2}$ which are attached to the vertex $u_i$ of $P_n$, $1 \leq i \leq n$

Let $G = P_n \circ K_{1,2}$

We define $f: V(G) \rightarrow \{0, 1, 2, \ldots, k+(q-1)\}$ as

$f(u_i) = 3i-3+k$, $1 \leq i \leq n$

$f(v_i) = 3i-4+k$, $1 \leq i \leq n$

$f(x_i) = 3i-2+k$, $1 \leq i \leq n$

The distinct edge labeling are as follows

$f(u_iu_{i+1}) = 3i-1+k$, $1 \leq i \leq n-1$

$f(u_iv_i) = 3i-3+k$, $1 \leq i \leq n$

$f(u_ix_i) = 3i-2+k$, $1 \leq i \leq n$

Hence, the function $f$ is a $(k,1)$-Contra Harmonic mean labeling of $G$.

Thus, $P_n \circ K_{1,2}$ is a $(k-1)$-Contra Harmonic mean graph for all $k$.

![Figure-3: (34, 1)-Contra Harmonic mean labeling of $P_n \circ K_{1,2}$](image)

Theorem 2.4: $P_n \circ K_{1,3}$ admits $(k,1)$-Contra Harmonic mean labeling for all $k$.

Proof: Let $P_n$ be the path with vertices $u_1, u_2, u_3, \ldots, u_n$ and $v_i, w_i, z_i$ be the vertices of $K_{1,3}$ which are joined to the vertices $u_i$ of the path $P_n$, $1 \leq i \leq n$.

Let $G = P_n \circ K_{1,3}$

Let $f: V(G) \rightarrow \{0, 1, 2, \ldots, k+(q-1)\}$ be defined by

$f(u_i) = 4i-4+k$, $1 \leq i \leq n$

$f(v_i) = 4i-4+k-1$, $1 \leq i \leq n$

$f(w_i) = 4i-3+k$, $1 \leq i \leq n$

$f(x_i) = 4i-2+k$, $1 \leq i \leq n$

The distinct edge labeling are as follows

$f(u_iu_{i+1}) = 4i-1+k$, $1 \leq i \leq n-1$

$f(u_iv_i) = 4i-4+k$, $1 \leq i \leq n$

$f(u_iw_i) = 4i-3+k$, $1 \leq i \leq n$

$f(u_ix_i) = 4i-2+k$, $1 \leq i \leq n$

Hence, the function $f$ is a $(k,1)$-Contra Harmonic mean labeling of $G$.

Then, $P_n \circ K_{1,3}$ is a $(k-1)$-Contra Harmonic mean graph.
Figure 4: (72, 1)- Contra Harmonic mean labeling of $P_3 \odot K_{1,3}$

**Theorem 2.5:** A Ladder is a $(k, 1)$- Contra Harmonic mean graph for all $k$.

**Proof:** Let $G = P_2 \times P_n$ be a ladder graph. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices of $G$.

Let us define $f: V(G) \rightarrow \{0, 1, 2, \ldots, k+(q-1)\}$ as follows:

- $f(u_i) = 3i-3+k$, $1 \leq i \leq n$
- $f(v_i) = k+3i-4$, if $i$ is odd
- $f(v_i) = k+3i-5$, if $i$ is even

The distinct edge labeling are as follows:

- $f(u_iu_{i+1}) = 3i-1+k$, $1 \leq i \leq n-1$
- $f(u_iv_i) = 3i-3+k$, $1 \leq i \leq n$
- $f(v_iw_i) = 3i-2+k$, $1 \leq i \leq n$

Hence, the function $f$ is a $(k, 1)$- Contra Harmonic mean labeling of $G$.

Then, $P_2 \times P_n$ is a $(k-1)$-Contra Harmonic mean graph for all $k$.

Figure 5: (64, 1) – Contra Harmonic mean labeling of $P_2 \times P_3$

**Theorem 2.6:** $(P_n \odot K_1) \odot K_{1,2}$ admits $(k, 1)$-Contra Harmonic mean graph for all $k$.

**Proof:** Let $G = (P_n \odot K_1) \odot K_{1,2}$, where $P_n$ is a path with vertices $u_1, u_2, u_3, \ldots, u_n$. Let $v_i$ be a vertex adjacent to $u_i$, $1 \leq i \leq n$.

The resultant graph is $(P_n \odot K_1)$. Let $x_i, w_i, z_i$ be the vertices of $i^{th}$ copy of $K_{1,2}$ with $z_i$ the central vertex. Identify the vertex $z_i$ with $v_i$ we get the resultant graph $G$.

That is, $G$ is a graph obtained by attaching the central vertex of $K_{1,2}$ at each pendant vertex of a comb.

Let us define $f: V(G) \rightarrow \{0, 1, 2, \ldots, k+(q-1)\}$ as:

- $f(u_i) = 4i-3+k$, $1 \leq i \leq n$
- $f(v_i) = 4i-4+k$, $1 \leq i \leq n$
- $f(w_i) = 4i-5+k$, $1 \leq i \leq n$
- $f(x_i) = 4i-2+k$, $1 \leq i \leq n$

Then the distinct edge labeling are:

- $f(u_iu_{i+1}) = 4i-1+k$, $1 \leq i \leq n-1$
- $f(u_iv_i) = 4i-3+k$, $1 \leq i \leq n$
- $f(v_iw_i) = 4i-4+k$, $1 \leq i \leq n$
- $f(v_ix_i) = 4i-2+k$, $1 \leq i \leq n$
Then \( f \) provides a \((k, 1)\) – Contra Harmonic mean labeling of \( G \).

Hence \((P_n \odot K_1) \odot K_{1,2}\) is a \((k-1)\) Contra Harmonic mean graph.

**Theorem 2.7:** Any Triangular snake is a \((k, 1)\)-Contra Harmonic mean graph for all \( k \).

**Proof:** Let \( G = T_n \), where \( T_n \) is a Triangular snake obtained from a path \( u_1, u_2, u_3, \ldots, u_n \) by joining \( u_i \) to \( v_{i+1} \) to a new vertex \( v_i \) for \( 1 \leq i \leq n-1 \).

Let us define \( f: V(G) \rightarrow \{0, 1, 2, \ldots, k+(q-1)\} \) as follows

\[
\begin{align*}
    f(u_i) &= 3i-4+k, \quad 1 \leq i \leq n \\
    f(v_i) &= 3i-2+k, \quad 1 \leq i \leq n-1
\end{align*}
\]

The distinct edge labeling are as follows

\[
\begin{align*}
    f(u_iu_{i+1}) &= 3i-2+k, \quad 1 \leq i \leq n-1 \\
    f(u_iv_i) &= 3i-3+k, \quad 1 \leq i \leq n-1 \\
    f(v_iv_{i+1}) &= 3i-1+k, \quad 1 \leq i \leq n-1
\end{align*}
\]

Then \( f \) is a \((k, 1)\)- Contra Harmonic mean labeling of \( G \).

Hence, any Triangular snake is a \((k, 1)\)-Contra Harmonic mean Graph.

**Theorem 2.1.8:** Any Quadrilateral Snake is a \((k, 1)\) – Contra Harmonic mean graph for all \( k \).

**Proof:** Let \( G \) be a Quadrilateral snake obtained from a path \( u_1, u_2, u_3, \ldots, u_n \) by joining \( u_i \) and \( u_{i+1} \) to new vertices \( v_i \) and \( w_i \) respectively and joining the vertices \( v_i \) and \( w_i \) for \( 1 \leq i \leq n-1 \).

Let us define \( f: V(G) \rightarrow \{0, 1, 2, \ldots, k+(q-1)\} \) as follows

\[
\begin{align*}
    f(u_i) &= 4i-5+k, \quad 1 \leq i \leq n \\
    f(v_i) &= 4i-4+k, \quad 1 \leq i \leq n-1 \\
    f(w_i) &= 4i-3+k, \quad 1 \leq i \leq n-1
\end{align*}
\]

Then the distinct edge labeling are as follows

\[
\begin{align*}
    f(u_iu_{i+1}) &= 4i-2+k, \quad 1 \leq i \leq n-1 \\
    f(u_iv_i) &= 4i-4+k, \quad 1 \leq i \leq n-1 \\
    f(v_iw_i) &= 4i-3+k, \quad 1 \leq i \leq n-1 \\
    f(w_iu_{i+1}) &= 4i-1+k, \quad 1 \leq i \leq n-1
\end{align*}
\]

Hence, \( f \) is a \((k, 1)\)-Contra Harmonic mean labeling for \( G \).

Thus any Quadrilateral snake is a \((k, 1)\) – Contra Harmonic mean graph.
Theorem: 2.9: The middle graph of path $P_n$ ($n \geq 3$) is a $(k, 1)$ – Contra Harmonic mean graph for all $k$.

Proof: Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{e_i = v_i v_{i+1}, 1 \leq i \leq n-1\}$ be the vertex set and edge set of the path $P_n$.

Then $V(G) = \{v_1, v_2, \ldots, v_n, e_1, e_2, e_3, \ldots, e_n\}$ and $E(G) = \{v_i e_i, e_i v_{i+1}, 1 \leq i \leq n-1\} U \{e_i e_{i+1}, 1 \leq i \leq n-2\}$.

Let us define $f: V(G) \rightarrow \{0, 1, 2, \ldots, k+(q-1)\}$ by

$f(e_i) = 3i-3+k, 1 \leq i \leq n$

$f(v_i) = 3i-5+k, 1 \leq i \leq n+1$

Then the distinct edge labels are

$f(e_\iota e_{\iota+1}) = 3\iota-1+k, 1 \leq \iota \leq n-1$

$f(e_\iota v_{\iota+1}) = 3\iota-3+k, 1 \leq \iota \leq n, f(e_{\iota} v_{\iota+1}) = 3\iota-2+k, 1 \leq \iota \leq n$

Clearly, $f$ provides a $(k,1)$- contra Harmonic mean labeling for $G$. 

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