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GEODETIC GLOBAL DOMINATION IN GRAPHS

S. ROBINSON CHELLATHURAI[1] AND X. LENIN XAVIOUR[2]

[1]Department of Mathematics,
Scott Christian College, Nagercoil - 629 003, Tamil Nadu, India.

[2]Research Scholar, Department of Mathematics,
Scott Christian College, Nagercoil -629 003, Tamil Nadu, India.

E-mail: robinchel@rediffmail.com[1], leninxaviour93@gmail.com[2]

ABSTRACT

In this paper, we introduce the concept geodetic global domination number of a graph. Also, geodetic global domination number of certain classes of graphs are determined and some of its general properties are studied. It is shown that for any two integers a and b, where $2 \le a \le b$, there exists a connected graph G with $\gamma_g(G) = a$ and $\overline{\gamma}_g(G) = b$.

Keywords: *Geodetic set/Dominating set/ Global domination/Geodetic domination/ Geodetic global domination.*

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1. INTRODUCTION

We consider only finite simple connected graphs with at least two vertices. For any graph G, the vertex set is denoted by V(G) and the edge set by E(G). The order and size of G are denoted by p and q, respectively. For a vertex $v \in V(G)$, the open neighbourhood N(v) is the set of all vertices adjacent to v, and $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of v. The degree deg(v) of a vertex v is defined by $\deg(v) = |N(v)|$. The minimum and maximum degrees of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. For $X \in V(G)$, let G[X] be the subgraph of G induced by X, $N(X) = \bigcup_{x \in X} N(x)$ and $N[X] = \bigcup_{x \in X} N[x]$. If G is a connected graph, then the distance d(x, y) is the length of a shortest $x - y$ path in G. The diameter of a connected graph G is defined by diam (G) = $\max_{x,y \in V(G)} d(x, y)$.

The complement $\overline{G}$ of G is the graph with vertex set V and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. A full vertex of G is a vertex that is adjacent to all other vertices of G. The set of all full vertices is denoted by $F_x(G)$.A vertex v in a connected graph G is a cut vertex of G, if G- v is disconnected. The girth of a graph G is the length of a shortest cycle contained in the graph and is denoted by c(G). An acyclic connected graph is called a tree. For the basic graph theoretic notations and terminology we refer to Buckley and Harary [2]. A vertex of G is said to be an extreme vertex if the subgraph induced by its neighborhood is complete. An x- y path of length d(x, y) is called and x– y geodesic. A vertex v is said to lie on an x- y geodesic P if v is an internal vertex of P. The closed interval I[x, y] consists of x, y and all vertices lying on some x–y geodesic of G, and for a nonempty set $S \subseteq V(G)$, $I[S] = \bigcup_{x,y \in S} I[x, y]$.

The concept of geodetic number of a graph was introduced in [2, 3]. A set $S \subseteq V(G)$ is a geodetic set of G if I[S] = V(G). The minimum cardinality of a geodetic set of G is the geodetic number g(G) of G. For any integer $k \ge 1$, a geodesic in a connected graph G of length k is called a k– geodesic. A set $S \subseteq V(G)$ is called a k- geodetic set of G if each vertex V\S lies on a k- geodesic of vertices in S. The minimum cardinality of a k– geodetic set of G is the k– geodetic number $g_k(G)$ of G. The k– geodetic number of a graph was referred to as k- geo domination number and studies in [1].

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The concept of domination number and global domination number of a graph was introduced in [6, 8]. A set of vertices S in a graph G is a dominating set if N[S] = V(G). A dominating set S of G is a global dominating set of G if N[S] = V($\overline{G}$). The domination number γ(G) of G and global domination number $\overline{\gamma}$(G) of G is the minimum cardinality of a dominating set and global dominating set of G. The concept of geodetic domination number of a graph was introduced in [4]. A set S$\subseteq$V(G) is a geodetic dominating set of G if S is both geodetic and dominating set of G. The minimum cardinality of a geodetic dominating set of a graph G is its geodetic domination number γ_g(G).

It is easily seen that a global dominating set is not in general is a geodetic set in a graph G. Also the converse is not a valid in general. This has motivated us to study the new domination conception of geodetic global domination. We investigate those subsets of vertices of a graph that are both a geodetic set and a global dominating set. We call these sets geodetic global dominating sets. We call the minimum cardinality of a geodetic global dominating set of G, the geodetic global domination number of G.

2. PRELIMINARY NOTES

In this section we cite some results to be used in the sequel.

Theorem 2.1[3]: Each extreme vertex of a connected graph G belongs to every geodetic set of G.

Theorem 2.2[4]: If G is a connected graph of order $p \geq 2$, then $2 \leq \max\{\gamma(G), g(G)\} \leq \gamma_g(G) \leq p$.

Theorem 2.3[2]: A vertex v of a connected graph G is a cut vertex of G if and only if there exists vertices u and w distinct from v lies on every u- w path of G.

3. GEODETIC GLOBAL DOMINATION NUMBER OF A GRAPH.

Definition 3.1: Let G = (V, E) be a connected graph. A Set S $\subseteq$ V is said to be a geodetic global dominating set if S is both geodetic set and global dominating set of G. The minimum cardinality of geodetic global dominating set of G is the geodetic global domination number of G and is denoted by $\overline{\gamma}_g$(G). A geodetic global dominating set of cardinality $\overline{\gamma}_g$(G) is called a $\overline{\gamma}_g$ - set of G.

Example 3.2: For the graph G given in figure 3.1, S = {v_1, v_4} is a minimum geodetic and minimum geodetic dominating set of G. So g(G) = 2 and γ_g(G) = 2. Here S is not a dominating set of $\overline{G}$, and So S is not a geodetic global dominating set of G.

Figure–3.1

Now, it is clear that S$_1$={v_1, v_2,v_4}, S$_2$ = {v_1, v_3, v_4}, S$_3$ = {v_1, v_2, v_3} and S$_4$ = {v_2, v_3, v_4}are four different $\overline{\gamma}_g$ - sets of G. There is no geodetic global dominating set with two vertices and so $\overline{\gamma}_g$(G) = 3. Thus geodetic global domination number is different from geodetic number as well as geodetic domination number of G.

Observation 3.3: Let G be a connected graph of order $p \geq 2$. Then, $\max\{\overline{\gamma}(G), g(G)\} \leq \overline{\gamma}_g(G) \leq g(G) + \overline{\gamma}(G)$.

Observation 3.4:

i) Path P$_p$ of p vertices, $\overline{\gamma}_g(P_p) = \left\lceil \frac{p+2}{3} \right\rceil$, $p \geq 4$

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ii) Cycle C_p of p vertices, $\overline{\gamma}_g(C_p) = \left\lceil \frac{p}{3} \right\rceil$, $p \geq 6$

iii) Complete graph K_p of p vertices, $\overline{\gamma}_g(K_p) = p$

iv) Star graph $K_{1,p-1}$ of p vertices, $\overline{\gamma}_g(K_{1,p-1}) = p$

v) Peterson graph G, $\overline{\gamma}_g(G) = 4$.

vi) Fan graph F_p of p vertices, $\overline{\gamma}_g(F_p) = \left\lceil \frac{p+2}{2} \right\rceil$, $p \geq 5$.

vii) Wheel graph W_p of p vertices, $\overline{\gamma}_g(W_p) = \left\lceil \frac{p+1}{2} \right\rceil$, $p \geq 6$.

viii) Middle graph M(G) of a connected graph G of p vertices, $\overline{\gamma}_g(M(G)) = p$.

Observation 3.5: Let G be a connected graph of order $p \geq 2$, then $2 \leq g(G) \leq \overline{\gamma}_g(G) \leq p$.

Proof: Any geodetic set has at least two vertices. Therefore $2 \leq g(G)$. Since every geodetic global dominating set is a geodetic set, so $g(G) \leq \overline{\gamma}_g(G)$. Clearly, $\overline{\gamma}_g(G) \leq p$.

Observation 3.6: For any connected graph G of order p, $2 \leq \gamma_g(G) \leq \overline{\gamma}_g(G) \leq p$

Proof: Since every geodetic dominating set contain atleast two vertices and so $\gamma_g(G) \geq 2$. Since every geodetic global dominating set is also a geodetic dominating set, it follows that $\gamma_g(G) \leq \overline{\gamma}_g(G)$. Also, the set of all vertices of G is a geodetic global dominating set of G and so $\overline{\gamma}_g(G) \leq p$. Thus $2 \leq \gamma_g(G) \leq \overline{\gamma}_g(G) \leq p$.

Theorem 3.7: Let G be a connected graph of order p. Then, a) every geodetic global dominating set of G contains its extreme vertices. b) Every geodetic global dominating set of G contains its full vertices. c) If the set S contains only full and extreme vertices is a geodetic global dominating set of G, then S is the unique minimum geodetic global dominating set of G and $\overline{\gamma}_g(G) = |S|$.

Proof: a) Let u be an extreme vertex and S be a geodetic global dominating set of a connected graph G. Suppose that $u \notin S$, then by theorem 2.1, S is not a geodetic set of G. Thus S is not a geodetic global dominating set of G, which is a contradiction. Hence each extreme vertex of G belongs to every geodetic global dominating set of G. b) Let v be a full vertex and S be a geodetic global dominating set of a connected graph G. Suppose that $v \notin S$. Since $\deg(v) = p-1$ in G, v is isolate vertex in $\overline{G}$ .Hence S is not a dominating set of $\overline{G}$. It follows that, S is not a geodetic global dominating set of G, which is a contradiction. Hence every full vertex of G belong to every geodetic global dominating set of G.
c) Follows directly from (a) and (b).

Theorem 3.8: Let G be a connected graph of order p. $\overline{\gamma}_g(G) = 2$ if and only of $G = K_2$ or there exists a geodetic set $S=\{u, v\}$ such that $d(u,v) = 3$.

Proof: Let G be a connected graph of order $p \geq 2$. Suppose $G = K_2$, then $\overline{\gamma}_g(G) = 2$. Assume $G \neq K_2$ and there exists a geodetic set $S = \{u, v\}$ such that $d(u, v) = 3$. To prove S is a global dominating set of G. Since $d(u, v) = 3$, every vertex in $V \setminus S$ is adjacent to some vertex in S. Therefore S is dominating set of G. Now to prove S is a dominating set of $\overline{G}$. Suppose S is not a dominating set of $\overline{G}$. Then there is a vertex w in $V \setminus S$ is adjacent to every vertex in S and so $d(u, v) \leq 2$, which is a contradiction to be fact that $d(u,v) = 3$. Thus S is a geodetic global dominating set of G and so $\overline{\gamma}_g(G) \leq |S| = 2$. Always $\overline{\gamma}_g(G) \geq 2$ implies that $\overline{\gamma}_g(G) = 2$. Conversely, suppose $\overline{\gamma}_g(G) = 2$. Let $S = \{u, v\}$ be a minimum geodetic global dominating set of G. Then there are two cases.

Case-i): u and v are adjacent in G. Then the only possibility is $G = K_2$.

Case-ii): u and v are non-adjacent in G. Since S is a global dominating set of G, every vertex in V\S is adjacent to either u or v not both, It follows that $d(u, v) = 3$. Therefore there exists a geodetics set S = $\{u, v\}$ of G such that $d(u, v) = 3$ or $G = K_2$.

Theorem 3.9: Let G be a connected graph of order $p \geq 2$. Then $\overline{\gamma}_g(G) = p$ if and only if G contains only the extreme and full vertices.

Proof: The result holds for $p = 2$. Now we consider the case where $p \geq 3$. Assume that $\overline{\gamma}_g(G) = p$. To prove G contains only extreme and full vertices. Suppose G contain a vertex v which is neither an extreme vertex nor a full vertex. Since v is not an extreme vertex, there exists two non-adjacent vertices x, y in $N(v)$ such that v lies on x-y geodesic and so $V(G)\backslash\{v\}$ is a geodetic set of G. Since G is connected, $V(G)\backslash\{v\}$ is a dominating set of G. Since v is a not a full vertex, v is non-adjacent to at least one vertex in G which implies that v is adjacent to at least one vertex in $\overline{G}$. It follows that $V(G)\backslash\{v\}$ is a global dominating set of G. Therefore $V(G)\backslash\{v\}$ is a geodetic global dominating set of G, contradicting the fact that $\overline{\gamma}_g(G) = p$. Hence G contains only the extreme and full vertices. Converse follows by theorem. 3.7, $\overline{\gamma}_g(G) = p$.

Theorem3.10: Let G be a connected graph with a cut vertex v and let S be a geodetic global dominating set of G. Then i) every component of G-v contains atleast one element of S. ii) Every branch of G at v contains element of S.

Proof: (i) Let v be a cut vertex of a connected graph G and S be a geodetic global dominating set of G. Suppose, to the contrary, that there exists a components H of G – v such that H contains no vertex of S. By theorem3.7, S contain its extreme vertices and hence it follows that H does not contain any extreme vertex of G. Let $u \in V(H)$. Since S is a geodetic global dominating set of G, there exists a pair of vertices $x, y \in$ S such that $u \in I[x, y] \subseteq I[S]$. Also $u \in N[S]$ in G and $\overline{G}$. Let the x- y geodesic in G be P: $x = u_0, u_1, \ldots, u, u_p = y$. Since v is a cut vertex of G, by theorem 2.3, the x- u sub path of P and the $u - y$ sub path of P both contains v, it follows that P is not a path, which is a contradiction. Thus every component of G- v contain an element of S. (ii) Since every branch of G at v is a component H of G – v with the vertex together with all edges joining v to V(H). By (i) we conclude that every branch of G at v contains an element of S.

Theorem 3.11: If G is a connected graph with $\delta(G) \geq 2$ and $c(G) \geq 6$, then $\overline{\gamma}_g(G) = \overline{\gamma}(G)$

Proof: Let S be a global dominating set of G such that $\overline{\gamma}(G) = |S|$. To prove that S is a geodetic set of G. Suppose S is not a geodetic set of G. Let $x \in V(G)\backslash I[S]$. Since S is a global dominating set of G, x is adjacent to a vertex u in S. Since $\delta \geq 2$, x is adjacent to a vertex v in G other than u. Since $c(G) \geq 6$, u and v are non- adjacent in G. If $v \in$ S, then x lies on $u - v$ geodesic, which is a contradiction. Hence $v \notin$ S. Since $\delta \geq 2$, v is adjacent to a vertex w in G other than x. If $w \in$ S, then x, u lies on u-w geodesic, which is a contradiction. Hence $w \notin$S. Continuing this process we obtained that $N(v) \nsubseteq$ S. Thus S is not a global dominating set of G, which is a contradiction. Therefore S is a geodetic global dominating set of G and so $\overline{\gamma}_g(G) \leq |S| = \overline{\gamma}(G)$. By observation 3.3, we conclude that $\overline{\gamma}_g(G) = \overline{\gamma}(G)$.

Remark 3.12: The converse of the theorem 3.11 not true. For the cycle C_5, $\overline{\gamma}_g(C_5) = \overline{\gamma}(C_5)$ but c(G) = 5. For the path P_4, $\overline{\gamma}_g(P_4) = \overline{\gamma}(P_4)$ but $\delta = 1$.

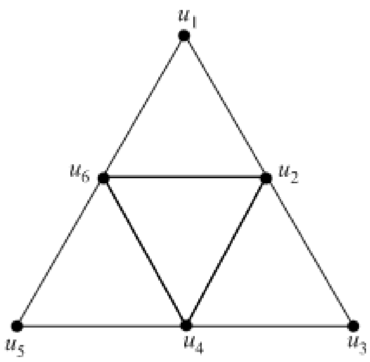
Remark 3.13: Theorem 3.11 is not true if $c(G) < 6$ and $\delta = 1$. For the cycle C_4, $\overline{\gamma}_g(C_4) \neq \overline{\gamma}(C_4)$.

For the path P_6, $\gamma_g(P_6) \neq \overline{\gamma}(P_6)$.

Theorem 3.14: Let G be a connected graph of order p ≥ 2. If $\gamma_g(G) \neq g_2(G)$, then $\overline{\gamma}_g(G) = \gamma_g(G)$.

Proof: Let G be a connected graph of order p. Let S be a minimum geodetic dominating set in G which is not a 2-geodetic set of G. To prove that S is a geodetic global dominating set of G, it is enough to prove that S is a dominating set in $\overline{G}$. Suppose S is not a dominating set in $\overline{G}$. Then there exist a vertex v in V\S such that v is adjacent to every vertices of S in G, so the distance between any two vertices in S is atmost two. It follows that S is a 2- geodetic set in G, which is a contradiction. Thus S is a dominating set in $\overline{G}$. Therefore S is a geodetic global dominating set of G and so $\overline{\gamma}_g(G) \leq |S| = \gamma_g(G)$. By observation 3.6 we conclude that $\overline{\gamma}_g(G) = \gamma_g(G)$.

Remark 3.15: The converse of Theorem 3.14 not true. For the graph G given in Figure3.2, $\overline{\gamma}_g(G) = \gamma_g(G) = 3$, but $\gamma_g(G) = g_2(G)$.



G
Figure-3.2

Theorem 3.16: Let G be a connected graph with diam(G) > 4. Then every geodetic dominating set in G is a geodetic global dominating set in G.

Proof: Let G be a connected graph with diam(G) > 4. Let S be a geodetic dominating set in G. We show that S is a geodetic global dominating set of G. It is enough to prove S is a dominating set of $\overline{G}$. Suppose not, then there exists a vertex v in V\S such that v is adjacent to every vertex of S in G, which implies that for every x, y in S, $d(x, y) \leq 2$. Since S is a geodetic dominating set of G, every vertex in V\S is adjacent to some vertex of S in G. Thus, for every u,v in V\S, $d(u, v) \leq 4$. It follows that diam(G)≤ 4 which is a contradiction. Therefore S is a geodetic global dominating set of G.

Corollary 3.17: Let G be a connected graph of diam(G) >4. Then $\overline{\gamma}_g(G) = \gamma_g(G)$.

Proof: Let S be a geodetic dominating set of G. Since diam(G) > 4, by theorem 3.16,S is a geodetic global dominating set of G. Therefore $\overline{\gamma}_g(G) \leq |S| = \gamma_g(G)$. By observation 3.6, we conclude that $\overline{\gamma}_g(G) = \gamma_g(G)$.

Theorem 3.18: For any two integers $p, q \geq 2$, the geodetic global domination number of a complete bipartite graph $K_{p,q}$ is

$$\overline{\gamma}_g(K_{p,q}) = \begin{cases} \min\{p, q\} + 1 & \text{if } 2 \leq p, q \leq 3 \\ 4 & \text{if } p, q \geq 4. \end{cases}$$

Proof: Let $G = K_{p,q}$. Let $X = \{x_1, x_2, \ldots, x_p\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$ be a bipartition of G. Let $2 \leq p,q \leq 3$. First we assume that $p < q$. Then $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$.Since $y_j \in I[x_1,x_2] = I[X]$, we have $I[X] = V(G)$ and hence X is a geodetic set of G. Since $y_j \in N[x_1] \subseteq N[X]$, we have $N[X] = V(G)$ and hence X is a dominating set of G. Since $\overline{G}$ is a disconnected component of two complete graph induced by $\langle X \rangle$ and $\langle Y \rangle$, we have $N[X] \neq V(\overline{G})$. Therefore, X is not a global dominating set of G. Now let $S = X \cup \{y_1\}$. Since $N[S] = V(\overline{G})$, S is a minimum geodetic global dominating set of G. So that $\overline{\gamma}_g(G) = |S| = p+1$. Now, if $p = q$, then we can prove similarly that $S = X \cup \{y_1\}$ is a minimum geodetic global dominating set of G. Thus $\overline{\gamma}_g(G) = |S| = p+1$. Hence $\overline{\gamma}_g(G) = \min\{p, q\} + 1$. Let $p, q \geq 4$ and $S = \{x_1, x_2, y_1, y_2\}$. Since $x_i \in I[y_1,y_2] \subseteq I[S]$ and $y_i \in I[x_1,x_2] \subseteq I[S]$, we have $I[S] = V(G)$ and hence S is a geodetic set of G. Since $x_i \in N[y_1] \subseteq N[S]$ for all $x_i \in X$ $y_i \in N[x_1] \subseteq N[S]$ for all $y_i \in Y$, we have $N[S]=V(G)$. Also $N[S] = V(\overline{G})$. Therefore S is a geodetic global dominating set of G. It remains to show that no 3- element subset of V is a geodetic global dominating set of G. Suppose to the contrary, there exists a 3- element subset Z of V such that Z is a geodetic global dominating set of G. Now we consider three cases.

Case-(i): Let $Z \subset X$. Since G is complete bipartite it is clear that $I[Z] = Z \cup Y \neq V$. Also since $Z \subset X$, there is a vertex x in X such that x is not adjacent to any vertex in Z. Therefore, Z is not a geodetic global dominating set of G.

Case-(ii): Let $Z \subset Y$. Similarly as case (i) we obtained that Z is not a geodetic global dominating set of G.

Case-(iii): Let $Z \subset X \cup Y$. Without loss of generality, we assume that $Z \cap X = \{x_i, x_j\}$ and $Z \cap Y = \{y_k\}$. Then it is clear that Z is a global dominating set of G. But $I[Z] = \{x_i, x_j\} \cup Y \neq V$. If follows that Z is not a geodetic global dominating set of G. In all the three cases, we attain a contradiction. Hence S is a minimum geodetic global dominating set of G and so $\overline{\gamma}_g(G) = |S| = 4$. Hence the proof is complete.

Theorem 3.19: If G is a connected graph with $\Delta(G) = p$-1.Then $\overline{\gamma}_g(G) = g(G)$ if and only if G is complete.

Proof: Let G be a connected graph with $\Delta(G) = p$-1.Assume G is complete. Then $\overline{\gamma}_g(G) = p = g(G)$.Conversely, Assume $\overline{\gamma}_g(G) = g(G)$. To prove G is complete. Suppose G is non- complete. G $\neq K_p$ and $\Delta(G) = p$-1 shows that G has at least two non- adjacent vertices and so diam(G)=2. Let S be a geodetic set such that $g(G) = |S|$. Let x be a vertex of degree p-1 (such a vertex exists as $\Delta(G) = p$-1). Since S is a geodetic set, there exists vertices $x_1, x_2 \in S$ such that x belong to an x_1- x_2 geodesic. But diam(G) = 2 implies that x_1-x_2 geodesic containing x must be the path $x_1 x x_2$. Thus $x \notin S$. Since diam(G) = 2, S is a dominating set of G. Since in G, deg(v) = p-1, v is an isolate vertex in $\overline{G}$. Therefore S is not a global dominating set of G, it follows that $\overline{\gamma}_g(G) > |S| = g(G)$, which is a contradiction. Therefore we conclude that G is complete.

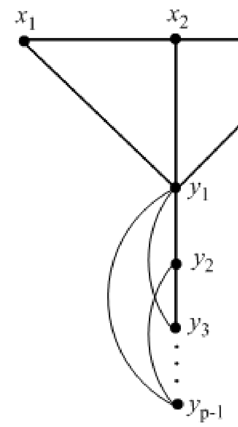
4. REALIZATION RESULTS

Theorem 4.1: For any two positive integers $3 \leq a \leq$ p there exists a connected graph G with $\overline{\gamma}_g(G) = a$ and $|V(G)| = p$.

Proof: It can be verified that the result is true for $3 \leq a \leq 4$. Since if p=3, then $G \in \{P_3, K_3\}$ while if $p = 4$, then $G \in \{C_4, K_4\}$. Let us now consider the case that $p \geq 5$. If $a = p$, let G = K_p or G = $K_{1, p-1}$.For $a \leq p$ -1 prove by considering two cases.

Case-1: $a = p$ -1. Let $p_3 : x_1, x_2, x_3$ be a path on three vertices. Add new vertices $y_1, y_2, y_3, \ldots, y_{p-3}$and join vertex y_1 with x_1, x_2, x_3. Also, join each y_i($1 \leq I \leq p$-3) with y_j ($i + 1 \leq j \leq p$-3), thereby obtaining the connected graph G given in figure 4.1. Then the vertex set of G is $v(G) = \{x_1, x_2, x_3, y_1, y_2, \ldots, y_{p-3}\}$ and the set $S = \{x_1, x_3, y_1, y_2, \ldots, y_{p-3}\}$ is a minimum geodetic global dominating set of G. Therefore, $|V(G)| = p$ and $\overline{\gamma}_g(G) = |S| = 2 + p-3 = p-1 = a$.

Case-2: $a \leq p - 2$. Consider the star $K_{1, p-2}$ with end vertices $x_1, x_2, \ldots, x_{p-2}$. Add a new vertex y and join each x_i ($a \leq i \leq p$ -2) with y, there by obtaining the connected graph G of order p. Then the set $S = \{x_1, x_2, \ldots, x_{a-1}, y\}$ is a minimum geodetic global dominating set of G. Therefore, $|V(G)| = p$ and $\overline{\gamma}_g(G) = |S| = a-1+1 = a$.



G

G

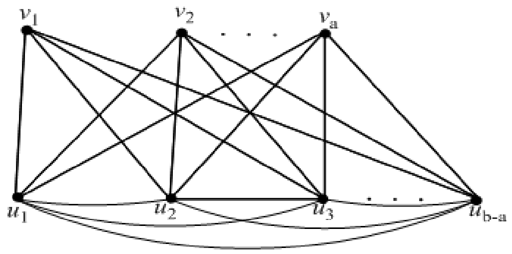
Figure-4.1

Theorem 4.2: For any two integers a and b such that $2 \leq a \leq b$ there exists a connected graph G with $\gamma_g(G) = a$ and $\overline{\gamma}_g(G) = b$

Proof: We prove this theorem by two cases.

Case-1: Let $2 \leq a = b$. Take G as the complete graph K_b. Then $\gamma_g(G) = \overline{\gamma}_g(G) = b$.

Case-2: Let $2 \leq a < b$. Let P_{b-a} : $u_1, u_2, \ldots, u_{b-a}$ be a path on $b- a$ vertices and join each $u_i (1 \leq i \leq b-a)$ with u_j ($i + 1 \leq j \leq b- a$). Also, add new vertices $v_1, v_2, \ldots, v_a$ and join each $v_i (1 \leq i \leq a)$ with $u_i (1 \leq i \leq b-a)$, thereby obtaining the connected graph of order b given in figure 4.2. Then $S_1 = \{v_1, v_2, \ldots, v_a\}$ be the set of extreme vertices of G, so $\gamma_g(G) \geq |S_1| = a$ and $S_2 = \{u_1, u_2, \ldots, u_{b-a}\}$ be the set of full vertices of G, So $\overline{\gamma}_g(G) \geq |S_1 \cup S_2| = a + b - a = b$. Since, $|V(G)| = b$, $\overline{\gamma}_g(G) = b$. Also, $u_i \in N[S_1]$, S_1 is the minimum geodetic dominating set of G and so $\gamma_g(G) = a$. Hence $\gamma_g(G) = a$ and $\overline{\gamma}_g(G) = b$.



G

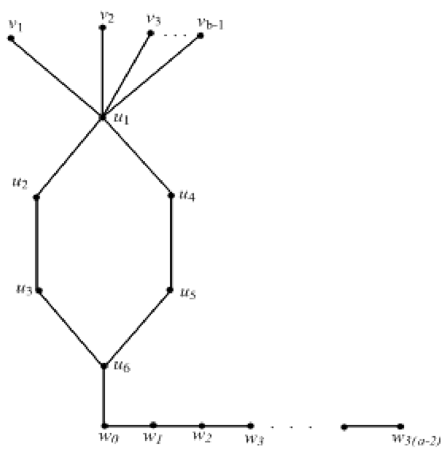
G

Figure-4.2

Theorem 4.3: For any two integers a, $b \geq 2$, there is a connected graph G such that $\overline{\gamma}(G) = a$, $g(G) = b$ and $\overline{\gamma}_g(G) = a + b$.

Proof: Let C : $u_1, u_2, u_3, u_4, u_5, u_6$ be a copy of C_6. Let H be a graph obtained from C by adding the new vertices $v_1, v_2, \ldots, v_{b-1}$ and join each to the vertex u_1. Let G be the graph obtained from H by taking a copy of the path on $3(a-2)+1$ vertices $w_0, w_1, w_2, \ldots, w_{3(a-2)}$ and joining w_0 to the vertex u_6 as shown in figure 4.3.

Let $S_1 = \{u_1, u_6, w_2, w_5, \ldots, w_{3(a-2)-1}\}$. Then it is clear that S_1 is the minimum global dominating set of G. Clearly S_1 contains a vertices and so $\overline{\gamma}(G) = a$. Take $S_2 = \{v_1, v_2, \ldots, v_{b-1}, w_{3(a-2)}\}$. Then S_2 is a minimum geodetic set of G, so $g(G) = b$. Now, let $S = S_1 \cup S_2$ and clearly S is a minimum geodetic global dominating set of G, it follows that $\overline{\gamma}_g(G) = a + b$



G

G

Figure-4.3

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