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# SOME PROPERTIES OF CAYLEY GRAPHS OF FULL TRANSFORMATION SEMIGROUPS

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***ABSTRACT***

*Let $G$ be a finite group and let $S$ be a non-empty subset of $G$. The Cayley graph $Cay(G,S)$ of $G$ relative to $S$ is defined as the graph with vertex set $G$ and edge set $\{(x,y): sx = y \text{ for some } s \in S, x \neq y\}$. A Full Transformation semigroup $T_X$ on a set $X$ is the set of all mappings of a set $X$ onto itself with composition of transformations as semigroup operation. It is also named as Symmetric semigroupof $X$. $T_X$ is unit regular monoid. In this paper, we study the Cayley graphs of Full transformation semigroup $T_X$ relative to the set of idempotents $E(T_X)$ and the existence of Hamiltonian cycles in it.*

***Key Words:*** *Full Transformation semigroup, Cayley graph, Hamiltonian cycle, Green's Equivalent classes (L-class, R- class).*

***AMS Subject Classification:*** *05C25, 20M18.*

## 1. INTRODUCTION

The Cayley graph of groups was introduced by Arthur Cayley in 1878 and the Cayley graphs of groups have received serious attention since then. The Cayley graphs of semigroups are generalizations of Cayley graphs of groups. The whole section 2.4 of the book [7] is devoted to Cayley graphs of semigroups. In 1964, Bosak [2] and in 1981, Zelinka [10] studied certain graphs over semigroups. The description of generators and factorizations of transformation semigroup has been given Howie and Nikola Ruskuc in [5].The Green's relations play an important role in the theory of semigroups. Inthis paper, we study the Cayley graphs of Full transformation semigroup $T_X$ relative to the set of idempotents $E(T_X)$.

## 2. PRELIMINARIES

In this section we describe some basic definitions and results in Semigroup theory and Graph theory which are needed in the sequel.

**Definition 2.1:** A pair $(S,.)$ consisting of a non-empty set $S$ and an associative binary operation. on $S$ is called a semigroup. A semigroup with identity is called a monoid.

**Definition 2.2:** If $S$ is a monoid with $1$ as the identity, then $G(S) = \{u \in S: \text{ there is } v \in S \text{ with } uv = 1 = vu\}$ is called the group of units of $S$. Here and elsewhere we denote $u.v$ as $uv$.

**Definition 2.3:** An element $x$ in $S$ is said to be an idempotent if $x^2 = x$ and the set of all idempotents in $S$ is denoted as $E(S)$.



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**Definition 2.4:** (c.f. [6]) Let $S$ be a semigroup. We define $a\,L\,b\ (a,b \in S)$ if and only if $a$ and $b$ generates the same principal left ideal, that is, if and only if $S^1a = S^1b$. Similarly we define $a\,R\,b$ if and only if $a$ and $b$ generates the same principal right ideal, that is, if and only if $a\,S^1 = b\,S^1$. We define $a\,H\,b$ if and only if $a\,L\,b$ and $a\,R\,b$.

**Lemma 2.5:** (c. f. [6]) Let $a, b$ be elements of a semigroup $S$. Then $a\,L\,b$ if and only if there exist $x, y \in S^1$ such that $x\,a = b$, $y\,b = a$ and $a\,R\,b$ if and only if $u, v \in S^1$ such that $a\,u = b,\ b\,v = a$.

**Notation 2.6:** The $L$-class ($R$-class, $H$-class) containing an element $a$ in a semigroup $S$ will be written as $L_a\ (R_a, H_a)$.

**Definition 2.7:** An element $a$ of a semigroup $S$ is said to be regular if there exists $x \in S$ such that $a\,x\,a = a$. The semigroup $S$ is called regular if all its elements are regular.

**Theorem 2.8:** (c.f. [3]) Let $S$ be a semigroup, $G$ be a subgroup of $S$ and $E = E(S)$. Then the following conditions are equivalent

(i) $S = GE$

(ii) $L_e = Ge$ for every $e \in E$.

(iii) $R_e = eG$ for every $e \in E$.

**Definition 2.9:** A regular monoid $S$ is said to be unit regular if for each element $s \in S$ there exists an element $u$ in the group of units $G(S) = G$ of $S$ such that $s = s\,u\,s$.

**Lemma 2.10:** (c.f.[8]) Equivalence relations $R_G$, $L_G$ and $H_G$ are defined on $S$ as

(i) $xR_Gy$ if and only if $x = y\,u$ for some $u \in G$

(ii) $xR_Gy$ if and only if $x = u\,y$ for some $u \in G$

(iii) $xH_Gy$ if and only if $xR_Gy$ and $xR_Gy$.

Evidently $R_G \subseteq R$, $L_G \subseteq L$ and $H_G \subseteq H$, where $R$, $L$ and $H$ are Green's equivalences on $S$.

**Definition 2.11:** A unit regular monoid $S$ is said to be $L-$strongly unit regular if $L = L_G$ and $R-$ strongly unit regular if $R = R_G$ on S.

**Theorem 2.12:** (c.f. [9]) Let $S$ be a unit regular monoid. Then $S$ is $L-$strongly unit regular if $L_e = Ge = \{ue; u \in G\}$ and $R-$strongly unit regular if $R_e = eG = \{eu; u \in G\}$ for every $e \in E(S)$.

**Definition 2.13:** A graph $G^*$ is a pair $(V, E)$ where $V$ is a non-empty set whose elements are called vertices of $G*$ and $E$ is a subset of $V \times V$ whose elementsare called edges of $G^*$. The vertex set of a graph $G^*$ is denoted by $V(G^*)$ and edge set is denoted by $E(G^*)$.

**Definition 2.14:** A subgraph $H^* = (U, F)$ of a graph $G^* = (V, E)$ is said to be vertex induced subgraph if $F$ consists of all the edges of $G^*$ joining pairs of vertices of $U$.

**Definition 2.15:** A Hamiltonian path is a path in $G^*$ which goes through all the vertices in $G^*$ exactly once. A Hamiltonian cycle is a closed Hamiltonian path. A graph $G^*$ is said to be Hamiltonian if it possesses a Hamiltonian cycle.

 

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**Definition 2.16:** A bipartite graph $G^*$ is a graph whose vertex set $V$ can be partitioned into two disjoint subsets $V_1$ and $V_2$ such that every line of $G^*$ joins $V_1$ and $V_2$.

**Definition 2.17:** Let $S$ be a finite semigroup and let $T$ be a non-empty subset of $S$. The Cayley graph $Cay(S,T)$ of $S$ relative to $T$ is defined as the graph with vertex set $S$ and edge set $\{(x,y) : tx = y \text{ for some } t \in T,\ x \neq y\}$.

## 3. MAIN RESULTS

To determine the global structure of a regular semigroup, we must first determine the structure of its idempotents and for this way introduce the concept of a biordered set. In, general, a biordered set is a structure $(X, w^r, w^l, T)$ consisting of a set $X$ together with two quasiorders $w^r$ and $w^l$ and a family $T$ of partial transformations of $X$ satisfying certain axioms.

By a partial algebra $X$, we mean a set $X$ together with a partial binary operation on $X$. The domain of the partial binary operation will be denoted by $D_X$. Then $D_X$ is a relation on $X$ and $(x,y) \in D_X$ if and only if the product $xy$ exists in the partial algebra $X$ .On $X$ we define:

$$w^r = \{(x,y) : yx = y\}, w^l = \{(x,y) : xy = x\} \text{ and } R = w^r \cap (w^r)^{-1},\ L = w^l \cap (w^l)^{-1} \text{ and } w = w^r \cap w^l.$$

Throughout this paper $X$ is considered as a finite set.

**Proposition 3.1:** Let $T_X$ be the full transformation semigroup on a set $X$ with group of units $G$ and set of idempotents $E(T_X)$. Then for $x, y \in E(T_X)$ with $x \neq y$, there is an edge from $x$ to $y$ in the Cayley graph $Cay(E(T_X), E(T_X))$ if and only if $y\, w^l x$.

**Proof:** Let $x, y \in E(T_X)$ with $x \neq y$. Suppose that there is an edge from $x$ to $y$ in the Cayley graph $Cay(E(T_X), E(T_X))$.Then by Definition 2.17, there exist an $e \in E(T_X)$ such that $ex = y$. Therefore $yx = ex^2 = y$. Hence we have $y\, w^l x$.

Conversely assume that $y\, w^l x$. Then by Definition of $w^l$, we have $yx = y$. Since $y \in E(T_X)$, we get an edge from $x$ to $y$ in $Cay(E(T_X), E(T_X))$.

**Proposition 3.2:** Let $T_X$ be the full transformation semigroup on a set $X$ with group of units $G$ and set of idempotents $E(T_X)$. Then for $x \in G, y \in E(T_X)$ with $x \neq y$, there is an edge from $x$ to $y$ in the Cayley graph $Cay(E(T_X), E(T_X))$ if and only if $y\, w^r x$.

**Proof:** Let $x \in G, y \in E(T_X)$ with $x \neq y$. Suppose that there is an edge from $x$ to $y$ in the Cayley graph $Cay(E(T_X), E(T_X))$. Then by Definition 2.17, there exist an $e' \in E(T_X)$ such that $e'x = y$. Thus $xy = (xe')x$. Since $x \in G$ and $x \in E(T_X)$, we have $x = e$, the identity of $G$. Therefore $xy = y$. Hence we have $y\, w^r x$.

Conversely assume that $y\, w^r x$. Then by Definition of $w^r$, we have $xy = y$. Thus $yx = x(yx)$. Since $x = e \in G$, we have $yx = y$. Since $y \in E(T_X)$, we get an edge from $x$ to $y$ in $Cay(E(T_X), E(T_X))$.

**Proposition 3.3:** Let $T_X$ be the full transformation semigroup on a set $X$ with group of units $G$ and set of idempotents $E(T_X)$. Then for $x, y \in E(T_X)$ with $x \neq y$, there is an edge between $x$ and $y$ in the Cayley graph $Cay(E(T_X), E(T_X))$ if and only if $x\, L\, y$.




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**Proof:** Let $x, y \in E(T_X)$ with $x \neq y$. Suppose that there is an edge between $x$ and $y$ in the Cayley graph $Cay(E(T_X), E(T_X))$. Then by Definition 2.17, there exist an $e, e' \in E(T_X)$ such that $ex = y$ and $e'y = x$. By Proposition 3.1, we have $y\, w^l x$. Also we have $xy = (e'y)y = e'y = x$. Thus $xw^l y$. Hence $x\, L\, y$, where $L = w^l \cap (w^l)^{-1}$.

Conversely assume that $x\, L\, y$. Then we have $xw^l y$ and $yw^l x$. Therefore we get $xy = x$ and $yx = y$. Since $x, y \in E(T_X)$, there exist an edge between $x$ and $y$ in $Cay(E(T_X), E(T_X))$.

**Remark 3.4:** Let $T_X$ be the full transformation semigroup on a set $X$ with group of units $G$ and set of idempotents $E(T_X)$. Then for $x \in G, y \in E(T_X)$ with $x \neq y$, there is an edge from $x$ to $y$ in the induced subgraph with vertex set $E(T_X)$ of the Cayley graph $Cay(T_X, E(T_X))$ if and only if $y\, wx$.

**Remark 3.5:** Let $T_X$ be the full transformation semigroup on a set $X$ with group of units $G$ and set of idempotents $E(T_X)$. Then for $x, y \in E(T_X)$ with $x \neq y$, there is an edge between $x$ and $y$ in the induced subgraph with vertex set $E(T_X)$ of the Cayley graph $Cay(T_X, E(T_X))$ if and only if $x\, L\, y$.

**Proposition 3.6:** Let $T_X$ be the full transformation semigroup on a set $X$ with group of units $G$ and set of idempotents $E(T_X)$. Then for $x \in G, y \in E(T_X)$ with $x \neq y$, there is an edge from $x$ to $y$ in the induced subgraph with vertex set $xE(T_X)$ of the Cayley graph $Cay(T_X, E(T_X))$ if and only if $yx\, R\, y$.

**Proof:** Let $x \in G$ and $y \in T_X$ with $x \neq y$. Suppose that there is an edge from $x$ to $y$ in $Cay(T_X, E(T_X))$. Then by Definition 2.17, there exist an $e \in E(T_X)$ such that $ex = y$. Now $yx = (ex)x = ex'$ where $x' = x^2 \in G$. By Proposition 2.12, we have $ex' \in R_e$. Thus $yx\, R\, e$. Also by Definition 2.10, we have $yR_G e$. Thus $yxR_G y$. Since $T_X$ is $R-$strongly unit regular monoid, we have $R = R_G$ on $T_X$. Hence $yx\, R\, y$.

Conversely assume that $yx\, R\, y$. Since $y \in T_X$ and $x \in G$, we have $yx \in R_e$ and so $y\, \mathrm{Re}$. Since $R = R_G$ on $T_X$, we have $yR_G e$. Thus $y = ex$ for some $x \in G$. Therefore there is an edge from $x$ to $y$ in the induced subgraph with vertex set $xE(T_X)$ of the Cayley graph $Cay(T_X, E(T_X))$.

**Proposition 3.7:** Let $T_X$ be the full transformation semigroup on a set $X$ with group of units $G$. Then there is a path from $x \in G$ to any elements in the induced subgraph with vertex set $xE(T_X)$ of the Cayley graph $Cay(T_X, E(T_X))$, where $E(T_X)$ is the set of all idempotents in $T_X$.

**Proof:** Let $x \in G$ and $y \in E(T_x)$ with $x \neq y$. Then we have $y = xe$ for some $e \in E(T_X)$. Thus $yx = (xe)e$ for some $x \in G$. Hence $yxR_G xe$. Since $R = R_G$ on $T_X$, we have $yxRy$. Then by Proposition 3.6, there is an edge from $x$ to $y$ in the induced subgraph with vertex set $xE(T_X)$ of the Cayley graph $Cay(T_X, E(T_X))$.

**Proposition 3.8:** Let $T_X$ be the full transformation semigroup on a set $X$ with group of units $G$ and set of idempotents $E(T_X)$. Then for $x, y \in gE(T_X)$ with $x \neq y$ and $g \in G$, there is an edge between $x$ and $y$ in the induced subgraph with vertex set $gE(T_X)$ of the Cayley graph $Cay(T_X, E(T_X))$ if and only if $x\, L\, y$.



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**Proof:** Let $x, y \in gE(T_X)$ with $x \neq y$ and $g \in G$. Suppose that there is an edge between $x$ and $y$ in the induced subgraph with vertex set $gE(T_X)$ of the Cayley graph $Cay(T_X, E(T_X))$, then by Definition 2.17, there exist some $e, e' \in E(T_x)$ such that $ex = y$ and $e'y = x.$ Hence by Lemma 2.5, we have $x\,L\,y.$

Conversely assume that $x\,L\,y.$ Then by Lemma 2.5, there exist $u, v \in T_X$ such that $ux = y$ and $vy = x$. As $x, y \in gE(T_X)$ with $x \neq y$, we have $x = ge$, $y = ge'$ for some $e, e' \in E(T_x)$. Hence by Definition 2.10, we see that $xL_G e$ and $yL_G e'$. $xLy$ implies $eLe'$ which in turn implies $ee' = e$ and $e'e = e'$. Since $eLe'$, we can find some $f, f' \in E(T_X)$ with $f\,L\,f'$, $f\,R\,y$ and $f'\,R\,x$. We know $R = R_G$ on $T_X$, hence by Definition 2.10, we have $f\,g_1 = y$ and $f'\,g_2 = x$ for some $g_1, g_2 \in G$. Also $f\,L\,f'$ gives $fg = y$ and $f'g = x$ for $g \in G.$ Thus $f\,x = (fg)e = ye = g(e'e) = g\,e' = y$ and $f'\,y = (f'g)e' = xe' = g(ee') = ge = x.$ Hence by Definition 2.17 there is anedge between $x$ and $y$ in the induced subgraph with vertex set $gE(T_X)$ of the Cayley graph $Cay(T_X, E(T_X))$.

**Proposition 3.9:** Let $T_X$ be the full transformation semigroup on a set $X$ with group of units $G$ and set of idempotents $E(T_X)$ .Then $Cay(T_X, E(T_X))$ is the union of induced subgraphs with vertex set $gE(T_X)$ of $Cay(T_X, E(T_X))$, $g \in G.$

**Proof:** Since $T_X$ is a semigroup and $G$ is a subgroup of $T_X$ and $E(T_X)$ is the set of idempotents of $T_X$ by Theorem 2.8 we have $T_X = G\,E(T_X)$. Hence the Cayley graph $Cay(T_X, E(T_X))$ is the union of induced subgraphs with vertex set $gE(T_X)$ of $Cay(T_X, E(T_X))$.

**Proposition 3.10:** Let $T_X$ be the full transformation semigroup on a set $X$ with group of units $G$ and set of idempotents $E(T_X)$. Then for $x, y \in T_X$ with $x \neq y,$ there is an edge between $x$ and $y$ in the Cayley graph $Cay(T_X, E(T_X))$ if and only if $x\,L\,y$ provided $x = ge$ and $y = ge'$ for some $e, e' \in E(T_X)$.

**Proof:** Let $x, y \in T_X$ with $x = ge$ and $y = ge'$ for some $e, e' \in E(T_X)$. Then by Proposition 3.8, there is an edge between $x$ and $y$ in the induced subgraph with vertex set $gE(T_X)$ of the Cayley graph $Cay(T_X, E(T_X))$ if and only if $x\,L\,y.$ Also by Proposition 3.9, we have the Cayley graph $Cay(T_X, E(T_X))$ is the union of induced subgraphs with vertex set $gE(T_X)$ of $Cay(T_X, E(T_X))$. Hence there is an edge between $x$ and $y$ in the Cayley graph $Cay(T_X, E(T_X))$ if and only if $x\,L\,y$ .

**Proposition 3.11:** Let $T_X$ be the full transformation semigroup on a set $X$ with group of units $G$ and $L$ be any $L-$classes of $T_X$ .Then the induced subgraph with vertex set $L$ other than $G$ of the Cayley graph $Cay(T_X, E(T_X))$ is Hamiltonian.

**Proof:** Since $L$ be any $L-$classes of $T_X$ other than $G$, we have either $L = \{e\}$ or $L = Ge \cup Ge'$ for some $e, e' \in E(T_X)$. If $L = \{e\}$, it is trivial. Let $x, y \in L$ with $x \neq y.$ Then $xLy.$ Suppose $x = ge$ and $y = ge'$ for some $g \in G.$ Then by Proposition 3.10, there exist an edge between $x$ and $y$ in the Cayley graph $Cay(T_X, E(T_X))$ .Therefore there is an edge between $x$ and $y$ in the induced subgraph with vertex set $L$ .Since $Ge \cap Ge' = \varphi$ and $g \in G$ is arbitrary ,it follows that there exist an edge between any $x = ge \in Ge$ to $y = ge' \in Ge'.$

 

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Let $G = G_1 \cup G_2$ where $G_1 =< g >$. Then we have $G_1e = \{e, ge, g^2e, g^3e, ..., g^{n-2}e, g^{n-1}e\}$ and $G_1e' = \{e', ge', g^2e', g^3e', ..., g^{n-2}e', g^{n-1}e'\}$. Therefore we get $n$ distinct edges in the induced subgraph with vertex set $L$ of $Cay(T_X, E(T_X))$ in which one end vertex in $G_1e$ and other in $G_1e'$ as $e \to e', ge \to ge', g^2e \to g^2e', ....., g^{n-1}e \to g^{n-1}e'$. Also we have $G_2e = \{e, ge, g^2e, g^3e, ..., g^{n-2}e, g^{n-1}e\}$ and $G_2e' = \{ge', g^2e', g^3e', g^4e', ..., g^{n-1}e', e'\}$.

As above we get $n$ distinct edges in the induced subgraph with vertex set $L$ of $Cay(T_X, E(T_X))$ in which one end vertex in $G_2e$ and other in $G_2e'$ as $e \Leftrightarrow ge', ge \Leftrightarrow g^2e', g^2e \Leftrightarrow g^3e', ....., g^{n-1}e \Leftrightarrow e'$. Thus we get a Hamiltonian cycle $e \to ge' \to ge \to g^2e' \to g^2e \to ....... \to g^{n-1}e \to e' \to e$ in the induced subgraph with vertex set $L$ of $Cay(T_X, E(T_X))$.

**Remark 3.12:** Let $T_X$ be the full transformation semigroup on a set $X$ with group of units $G$ and $L$ be any $L-$ classes of $T_X$ .Then the induced subgraph with vertex set $L$ other than $G$ of the Cayley graph $Cay(T_X, E(T_X))$ is bipartite.

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