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SOLUTION OF EIGHTH ORDER BOUNDARY VALUE PROBLEMS BY ONE DIMENSIONAL DIFFERENTIAL TRANSFORM METHOD

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ABSTRACT

One dimensional Differential Transformation Method (1-dim DTM) is used to find the solution of the eighth order boundary value problems. The approximate solution of the problem is obtained in the form of a rapid convergent series. To illustrate the ability, efficiency and reliability of the method, some examples are given, revealing its effectiveness and simplicity and the calculated results are compared with exact solution.

Keywords: *Eighth order boundary value problem, one dimensional differential transform method, Series solution.*

1. INTRODUCTION

Generally, the eighth-order boundary value problems (BVPs) are recognized to occur in Mathematics, Physics and Engineering Sciences [1, 2]. In the book written by Chandrasekhar [3], we find therein that whenever horizontal infinite layers of fluid are heated from below and those are then under the rotation-action, and hence instability develops. When the instability develops as ordinary convection, then that ordinary differential equation (ODE) is a sixth-order ODE. When the said instability sets as over stability, then it is to be modelled by eighth-order ODE. Bishop investigated eighth-order D.E for uniform beams for their torsional vibration. Over the years, some authors used different methods and worked on such types (eighth-order) boundary value problems [4, 5, 6, 7, 8, 9, 10, 11, 12]. Differential quadrature method [9], finite difference method (FDM) [4], spline method and modified decomposition method [10] are applicable to find the solution of eighth-order BVPs. But, these methods involve huge calculations and the solution is found at grid points only.

In 1986, Zhou and Pukhov have developed a so-called differential transformation method (DTM) for electrical circuit's problems [13]. The DTM is a technique that uses Taylor series for the solution of differential equations in the form of a polynomial. The Taylor series method is computationally unexciting for high order equations. The DTM leads to an iterative procedure for obtaining an analytic series solutions of functional equations. In recent years, many papers were devoted to the problem of approximate solution of system of differential equations. The implementation of the Differential Transform Method (DTM) [14, 15, 16] amongst others has shown reliable results to solve ordinary differential equations, partial differential equations, integral-differential equations, the Schrödinger equations, Analytic solution for Telegraph equation, Systems of Volterra Integral Equations of the First Kind, non-linear fractional differential equations and delay differential equations.

2. ONE-DIMENSIONAL DIFFERENTIAL TRANSFORMS METHOD

Definition:
One dimensional Differential transformation of function $y(x)$ is defined as follows [14]

$$Y(K) = \frac{1}{K!}\left[\frac{d^K y(x)}{dx^K}\right]_{x=0} \tag{1}$$

In equation (1), $y(x)$ is the original function and $Y(K)$ is the transformed function. One dimensional Differential inverse transform of $Y(K)$ is defined as

$$y(x) = \sum_{K=0}^{\infty} x^K\, Y(K) \tag{2}$$

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In fact, from (1) and (2), we obtain

$$y(x) = \sum_{K=0}^{\infty} \frac{x^K}{K!}\left[\frac{d^K y(x)}{dx^K}\right]_{x=0} \tag{3}$$

Equation (3) implies that the concept of differential transformation is derived from the Taylor series expansion. The differential transform's operators are shown in the table 1.

Table-1: Differential transform operators

Sr. No	Original function	Transform function
1	$u(x) \pm v(x)$	$U(K) \pm V(K)$
2	$\alpha u(x)$ Where α is a constant	$\alpha U(K)$
3	$\frac{du(x)}{dx}$	$(K+1)\,U(K+1)$
4	$\frac{d^n u(x)}{dx^n}$	$(K+1)(K+2)\cdots(K+n)U(K+n)$
6	x^m	$\delta(K-m)$ where $\delta(K-m) = \begin{cases} 1, & \text{if } K = m \\ 0, & \text{if } K \neq m \end{cases}$
7	$e^{\alpha x}$	$\frac{\alpha^K}{K!}$
8	$\sin(\omega x + \alpha)$	$\frac{\omega^K}{K!}\sin\left(\frac{\pi K}{2} + \alpha\right)$
9	$\cos(\omega x + \alpha)$	$\frac{\omega^K}{K!}\cos\left(\frac{\pi K}{2} + \alpha\right)$

3. NUMERICAL APPLICATION

Example 1: [12]

$$y^{(\mathbf{viii})}(x) = e^{-x}y^2 \qquad 0 < x < 1$$

Initial conditions are

$$y^i(0) = 1, \quad i = 0,1,2,\ldots,7$$

The exact solution is $y(x) = e^x$

Solution:

Applying 1-dim D.T.M. on initial condition we get

$$Y(0) = 1, \qquad Y(1) = 1, \; Y(2) = \tfrac{1}{2}, \qquad Y(3) = \tfrac{1}{3!}$$
$$Y(4) = \tfrac{1}{4!}, \qquad Y(5) = \tfrac{1}{5!}, \; Y(6) = \tfrac{1}{6!}, \qquad Y(7) = \tfrac{1}{7!}$$

Using inverse 1-dim differential transform we have

$y(x) = \sum_{k=0}^{\infty} Y(k)x^k$

$y(x) = Y(0) + Y(1)x + Y(2)x^2 + Y(3)x^3 + Y(4)x^4 + \cdots$

$$y(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$$

Table-2: Exact and approximate values of solution, and error of solution

x	Exact $y(x)$	1-Dim DTM $y^*(x)$	$\lvert y(x) - y^*(x)\rvert$
0	1.000000000000000	1.000000000000000	0
0.1	1.105170918075648	1.105170918075397	$2.511324000\text{x}10^{-13}$
0.2	1.221402758160170	1.221402758095238	$6.49318377\text{x}10^{-11}$
0.3	1.349858807576003	1.349858805892857	$1.6831460670\text{x}10^{-9}$
0.4	1.491824697641270	1.491824680634921	$1.70063498839\text{x}10^{-8}$
0.5	1.648721270700128	1.648721168154762	$1.025453664205\text{x}10^{-7}$
0.6	1.822118800390509	1.822118354285714	$4.461047944382\text{x}10^{-7}$
0.7	2.013752707470477	2.013751158194445	$1.5492760319091\text{x}10^{-6}$
0.8	2.225540928492468	2.225536365714286	$4.5627781819491\text{x}10^{-6}$
0.9	2.459603111156950	2.459591262678571	$1.18484783784290\text{x}10^{-5}$
1.0	2.718281828459046	2.718253968253968	$2.78602050771681\text{x}10^{-5}$

$y^*(x)$ is the approximate solution of $y(x)$

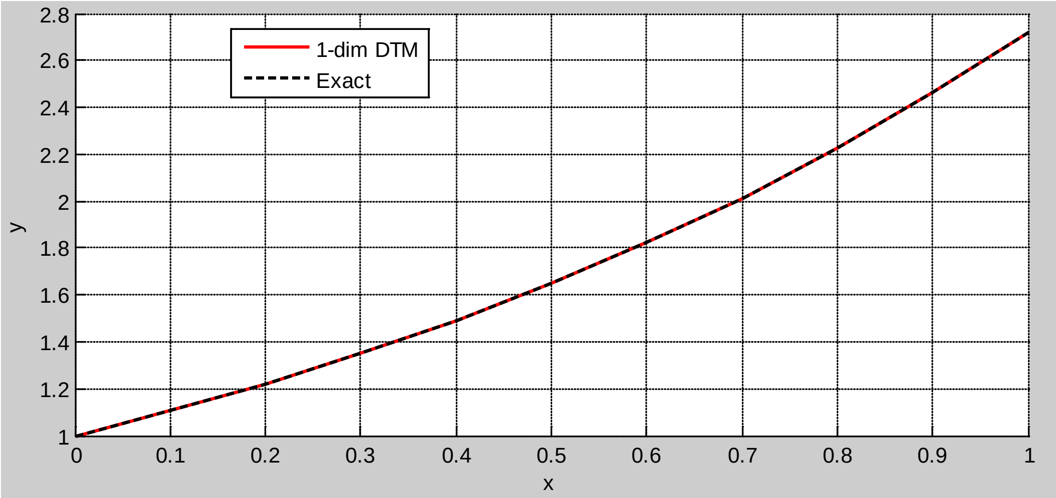

Figure-1: Comparative analysis of exact and 1-dim DTM solution results

Example 2: [11]

$$y^{(viii)} + y^{(vii)} + 2y^{(vi)} + 2y^{(v)} + 2y^{(iv)} + 2y''' + 2y'' + y' + y = 14\cos x - 16\sin x - 4x\sin x$$
$$0 < x < 1$$

Boundary conditions are

$y(0) = 0,\ y'(0) = -1, y''(0) = 0, y'''(0) = 7$

$y(1) = 0, y'(1) = 2\sin 1\ , y''(1) = 4\cos 1 + 2\sin 1, y'''(1) = 6\cos 1 - 6\sin 1$ (4)

The exact solution is $y(x) = (x^2 - 1)\sin x$

Solution:

Applying 1-dim D.T.M. on (3) we get

$$Y(0) = 0,\ Y(0) = -1,\ Y(2) = 0,\ Y(3) = \frac{7}{6}$$

Using inverse 1-dim D.T.M. We have

$y(x) = \sum_{k=0}^{\infty} Y(k)x^k$

$y(x) = Y(0) + Y(1)x + Y(2)x^2 + \cdots$

Let $Y(4) = A,\ \ Y(5) = B,\ Y(6) = C,\ Y(7) = D$

$y(x) = -x + \frac{7}{6}x^3 + Ax^4 + Bx^5 + Cx^6 + Dx^7 + \cdots$ (5)

As $y(1) = 0$ in (4) we get

$A + B + C + D = -\dfrac{1}{6}$ (6)

As $y'(1) = 2\sin 1$ in (4) we get

$4A + 5B + 6C + 7D = -0.8170580304$ (7)

As $y''(1) = 4\cos 1 + 2\sin 1$ in (4) we get

$12A + 20B + 30C + 42D = -3.155848807$ (8)

As $y'''(1) = 6\cos 1 - 6\sin 1$ in (4) we get

$24A + 60B + 120C + 210D = -8.84867874$ (9)

From equations (6) to (9), we get the values

$A = 0.00075045083330$

$B = -0.177827573600001$

$C = 0.003795158433337$

$D = 0.00661529766666$

Putting the values of A, B, C and D in (5) we get

$$y(x) = -x + \frac{7}{6}x^3 + 0.00075045083330x^4 - 0.177827573600001x^5 + 0.003795158433337x^6 + 0.00661529766666x^7 + \cdots$$

Table-3: Exact and approximate values of solution, and error of solution

x	Exact $y(x)$	1-Dim DTM $y^*(x)$	$\|y(x) - y^*(x)\|$
0	0	0	0
0.1	-0.098835082480360	-0.098835032107298	$5.0373062054910\text{x}10^{-08}$
0.2	-0.190722557563259	-0.190722043202935	$5.1436032327823\text{x}10^{-07}$
0.3	-0.268923388061819	-0.268921828916000	$1.5591458186192 7\text{x}10^{-6}$
0.4	-0.327111407539266	-0.327108692673024	$2.71486624231487\text{x}10^{-6}$
0.5	-0.359569153953152	-0.359565893801042	$3.26015211038477\text{x}10^{-6}$
0.6	-0.361371182972823	-0.361368360786509	$2.82218631347808\text{x}10^{-6}$
0.7	-0.328551020491222	-0.328549335580049	$1.68491117297220\text{x}10^{-6}$
0.8	-0.258248192723828	-0.258247614837010	$5.77886818231570\text{x}10^{-7}$
0.9	-0.148832112829222	-0.148832053983802	$5.8845419786960\text{x}10^{-8}$
1.0	0	-0.000000000000001	$1.267220000000\text{x}10^{-15}$

$y^*(x)$ is the approximate solution of $y(x)$

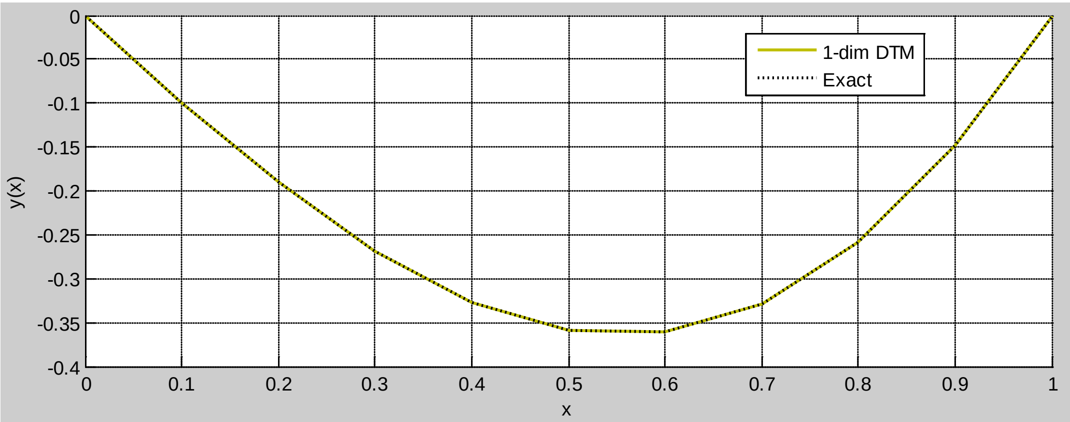

Figure-2: Comparative analysis of exact and 1-dim DTM solution results

Example 3: [11]

$y^{(viii)}(x) + xy = -(48 + 15x + x^3)e^x \qquad 0 < x < 1$

Boundary conditions are

$y(0) = 0,\ \ y'(0) = 1,\ y''(0) = 0,\ \ y'''(0) = -3$ (10)

$y(1) = 0,\ \ y'(1) = -e,\ y''(1) = -4e,\ \ y'''(1) = -9e$ (11)

The exact solution is $y(x) = x(1 - x)e^x$

Solution:

Applying 1-dim D.T.M. on (10)

$$Y(0) = 0, \qquad Y(1) = 1, \quad Y(2) = 0, \qquad Y(3) = -\frac{1}{2}$$

Using Inverse 1-dim Differential Transform we have

$y(x) = Y(0) + Y(1)x + Y(2)x^2 + Y(3)x^3 + Y(4)x^4 + \cdots$

$y(x) = x - \dfrac{x^3}{2} + Y(4)x^4 + Y(5)x^5 + Y(6)x^6 + Y(7)x^7 + \cdots$

Let $\quad Y(4) = A,\ \ Y(5) = B,\ \ Y(6) = C,\ \ Y(7) = D$

$y(x) = x - \dfrac{x^3}{2} + Ax^4 + Bx^5 + Cx^6 + Dx^7 + \cdots$ (12)

As $y(1) = 0$ gives

$A + B + C + D = -\dfrac{1}{2}$ (13)

As $y'(1) = -e$ gives

$4A + 5B + 6C + 7D = -e + \dfrac{1}{2}$ (14)

As $y''(1) = -4e$ gives
$$12A + 20B + 30C + 42D = -4e + 3 \tag{15}$$

As $y'''(1) = -9e$ gives
$$24A + 60B + 120C + 210D = -9e + 3 \tag{16}$$

From equation (13), (14), (15) and (16) we get

$$A = -0.331168115016212$$
$$B = -0.133368341115217$$
$$C = -0.021477144261922$$
$$D = -0.013986399606659$$

Putting A, B, C and D into (12) we get

$$y(x) = x - \frac{1}{2}x^3 - 0.331168115016212x^4 - 0.133368341115217x^5 - 0.021477144261922x^6 - 0.013986399606659x^7 + \cdots$$

Table-4: Exact and approximate values of solution, and error of solution

x	Exact $y(x)$	1-Dim DTM $y^*(x)$	$\lvert y(x) - y^*(x)\rvert$
0	0	0	0
0.1	0.099465382626808	0.099465526629303	$1.4400249472024\text{x}10^{-7}$
0.2	0.195424441305627	0.195425899583669	$1.45827804226562\text{x}10^{-6}$
0.3	0.283470349590961	0.283474737535698	$4.38794473711646\text{x}10^{-6}$
0.4	0.358037927433905	0.358045518742553	$7.59130864785851\text{x}10^{-6}$
0.5	0.412180317675032	0.412189383025617	$9.06535058464719\text{x}10^{-6}$
0.6	0.437308512093722	0.437316322770066	$7.81067634431088\text{x}10^{-6}$
0.7	0.422888068568800	0.422892713798975	$4.64523017534857\text{x}10^{-6}$
0.8	0.356086548558795	0.356088136976538	$1.58841774272611\text{x}10^{-6}$
0.9	0.221364280004125	0.221364441395012	$1.61390886932460\text{x}10^{-7}$
1.0	0	-0.000000000000010	$1.0023230000000\text{x}10^{-14}$

$y^*(x)$ is the approximate solution of $y(x)$

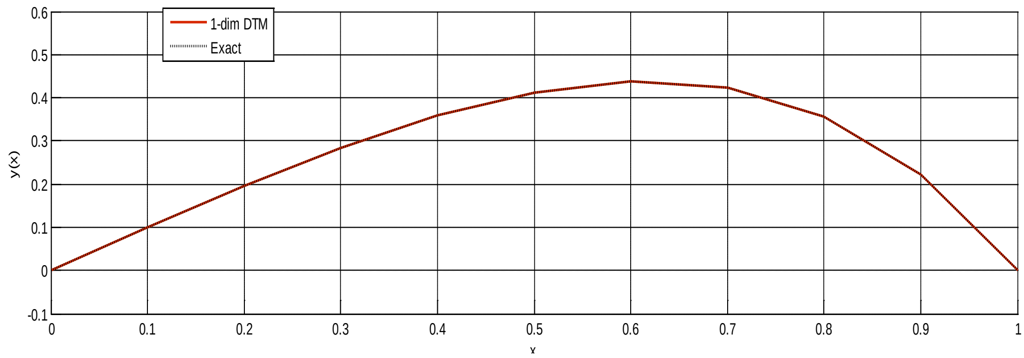

Figure-3: Comparative analysis of exact and 1-dim DTM solution results

Example 4: [12]

$y^{(viii)}(x) + 8e^x - y(x) = 0 \qquad 0 < x < 1$

Boundary conditions are

$$y(0) = 0,\ y'(0) = 0,\ y''(0) = -1,\ y'''(0) = -2,\ y^{(iv)}(0) = -3,\ y^{(v)}(0) = -4 \tag{17}$$
$$y'(1) = -e,\ y''(1) = -2e \tag{18}$$

The exact solution is $y(x) = (1 - x)e^x$

Solution:

Applying 1-dim D.T.M. on (17) we get

$$Y(0) = 1,\ \ Y(1) = 0,\ \ Y(2) = -\tfrac{1}{2},\ \ Y(3) = -\tfrac{1}{3},\ \ Y(4) = -\tfrac{1}{8},\ \ Y(5) = -\tfrac{1}{30}$$

Using Inverse 1-dim Differential Transform we have
$y(x) = Y(0) + Y(1)x + Y(2)x^2 + Y(3)x^3 + Y(4)x^4 + Y(5)x^5 + Y(6)x^6 + Y(7)x^7 + \cdots$

$$y(x) = 1 - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{8} - \frac{x^5}{30} + Y(6)x^6 + Y(7)x^7 + \cdots$$

Let $Y(6) = A,\ Y(7) = B$

$$y(x) = 1 - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{8} - \frac{x^5}{30} + Ax^6 + Bx^7 + \cdots \tag{19}$$

As $y'(1) = -e$ gives

$$6A + 7B = -e + \frac{8}{3} \tag{20}$$

As $y''(1) = -2e$ Gives

$$30A + 42B = -2e + \frac{31}{6} \tag{21}$$

From equation (20) and (21) we get

$$A = -0.006632330083808$$
$$B = -0.001688740184218$$

Putting A, and B into (19) we get

$$y(x) = 1 - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{8} - \frac{x^5}{30} - 0.006632330083808x^6 - 0.001688740184218x^7 + \cdots$$

Table-5: Exact and approximate values of solution, and error of solution

x	Exact $y(x)$	1-Dim DTM $y^*(x)$	$\lvert y(x) - y^*(x)\rvert$
0	1.000000000000000	1.000000000000000	0
0.1	0.994653826268083	0.994653826532129	$2.6404622930\text{x}10^{-10}$
0.2	0.977122206528136	0.977122220581667	$1.40535308946\text{x}10^{-8}$
0.3	0.944901165303202	0.944901295703891	$1.304006884695\text{x}10^{-7}$
0.4	0.895094818584762	0.895095400477392	$5.818926300805\text{x}10^{-7}$
0.5	0.824360635350064	0.824362343226418	$1.7078763537848\text{x}10^{-6}$
0.6	0.728847520156204	0.728851288090389	$3.7679341854391\text{x}10^{-6}$
0.7	0.604125812241143	0.604132471315550	$6.6590744072226\text{x}10^{-6}$
0.8	0.445108185698493	0.445117886643695	$9.7009452019181\text{x}10^{-6}$
0.9	0.245960311115695	0.245972088672914	$1.17775572190670\text{x}10^{-5}$
1.0	0	0.000012263065307	$1.22630653073522\text{x}10^{-5}$

$y^*(x)$ is the approximate solution of $y(x)$

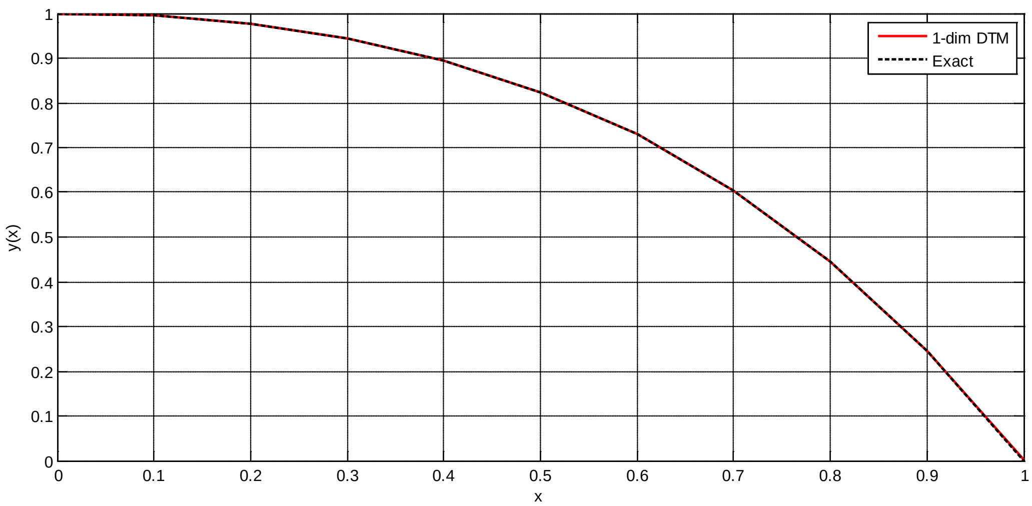

Figure-4: Comparative analysis of exact and 1-dim DTM solution results

4. CONCLUSION

In this work, one dimensional Differential Transformation Method has been applied to obtain the numerical solution of linear and nonlinear eighth order boundary value problems. The present method has been applied in a direct way without using linearization, discretization, or perturbation. The numerical results obtained by this method are in good agreement with the exact solutions available in the literature. This technique is fast converging to the exact solution and requires much less computational work than other methods. The objective of this paper is to present a simple method to solve an eighth order boundary value problem and its easiness for implementation.

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