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GENERALIZED COMMON FIXED POINT THEOREM OF COMPATIBLE MAPPING OF TYPE (R) IN COMPLETE METRIC SPACE

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ABSTRACT

The purpose of this paper is to present a common fixed theorem for compatible mapping of type (R) in complete metric space satisfying a generalized inequality. we also present a example that shows the applicability and validity of our result.

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***Keywords:** Common Fixed Point, Cauchy Sequence, Compatible Mapping, Complete Metric Space.*

1. INTRODUCTION AND PRELIMINARIES

Initially Jungck in 1976 [2] proved a common fixed point theorem for commuting maps, this result was extended and generalized in various ways by many authors. Recently Jungck in 1986 [3] introduced the generalized concept of weak commutativity which is called compatibility. In 1994 Pathak, Chang and Cho [5] gave the idea of compatible mapping of type (P). Rohen, Singh and Shambu [9] in 2004 gave the idea of compatible mapping of type (R) by combining the definition of compatible mapping and compatible mapping type (P).

The aim of this paper is to present a common fixed point theorem of compatible mapping type (R) in complete metric space by in view of four maps. This result revise the result of Bijendra and Chauhan [1] and others.

Before starting our main result following definitions and propositions are required in the sequel.

Definition 1.1: Let P and Q be self maps of a complete metric space (X, d) are said to be compatible on X if $\lim_{n\to\infty} d(PQx_n, QPx_n) = 0$ when $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Px_n = t = \lim_{n\to\infty} Qx_n$ for some $t \in X$.

Definition 1.2: Let P and Q be self maps of a complete metric space (X, d) are said to be compatible of type (P) on X if $\lim_{n\to\infty} d(PPx_n, QQx_n) = 0$ when when $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Px_n = t = \lim_{n\to\infty} Qx_n$ for some $t \in X$.

Definition 1.3: Let P and Q be self maps of a complete metric space (X, d) are said to be compatible of type (R) on X if $\lim_{n\to\infty} d(PQx_n, QPx_n) = 0$ when when $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Px_n = t = \lim_{n\to\infty} Qx_n$ for some $t \in X$.

2. MAIN RESULT

We need the following proposition for our main result.

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Proposition 2.1: Let P and Q be self maps of a complete metric space (X, d). if a pair (P,Q) is compatible type R on X and Pt = Qt for t ϵ X. then PQt =QPt = PPt = QQt .

Proposition 2.2: P and Q be self maps of a complete metric space (X, d). if a pair (P,Q) is compatible type R on X and and $\lim_{n\to\infty} Px_n = t = \lim_{n\to\infty} Qx_n$ for some t ϵ X. then

(i) $\lim_{n\to\infty} d(PQx_n, Qt)=0$ if Q is continuous
(ii) $\lim_{n\to\infty} d(QPx_n, Pt)=0$ if P is continuous
(iii) PQt = QPt and Pt = Qt If P and Q are continuous at t.

Theorem 2.3: Let P, Q, R, S be self maps of a complete metric space (X, d) satisfying the following conditions .

(1) $P(X) \subseteq S(X)$ and $Q(X) \subseteq R(X)$

(2) $[d(Px, Qy)]^2 \leq k_1[d(Rx, Qy)d(Rx, Sy) + d(Rx, Sy)d(Px, Rx)] + k_2[d(Px, Qy)d(Px, Rx) + d(Px, Qy)d(Px, Sy]$
Where $0 \leq k_1 + 3k_2 < 1$ and $k_1, k_2 \geq 0$

(3) One of P, Q, R, S is continuous

(4) [P, R] and [Q, S] are compatible type (R) on X.

Then P, Q, R, S have a unique common fixed point in X.

Proof: let $x_0 \epsilon X$ then by (1) $\exists\, x_1 \epsilon X$ such that $Sx_1 = Px_0$ and for $x_1 \epsilon X \;\exists\, x_2 \epsilon X$ Such that $Rx_2 = Qx_1$ and so on.
Continuing this way we can define a sequence $\{y_n\}$ in X such that

$Y_{2n+1} = Sx_{2n+1} = Px_{2n}$ and $y_{2n} = Rx_{2n} = Qx_{2n-1}$

$$\begin{aligned}
d[(y_{2n+1},y_{2n})]^2 &= [d(Px_{2n}, Qx_{2n-1})]^2 \leq K_1[d(Rx_{2n}, Qx_{2n-1})d(Rx_{2n}, Sx_{2n-1}) + d(Rx_{2n}, Sx_{2n-1})d(Px_{2n}, Rx_{2n})] \\
&\quad + K_2[d(Px_{2n}, Qx_{2n-1})d(Px_{2n}, Rx_{2n}) + d(Px_{2n}, Qx_{2n-1})d(Px_{2n}, Sx_{2n-1})] \\
&= K_1[d(y_{2n},y_{2n})d(y_{2n},y_{2n-1})+d(y_{2n},y_{2n-1})d(y_{2n+1},y_{2n})] \\
&\quad + K_2[d(y_{2n+1}, y_{2n})d(y_{2n+1}, y_{2n}) + d(y_{2n+1}, y_{2n})d(y_{2n+1}, y_{2n-1})] \\
&= K_1[d(y_{2n}, y_{2n-1})d(y_{2n+1}, y_{2n})] + K_2[d(y_{2n+1}, y_{2n})^2 + d(y_{2n}, y_{2n+1})d(y_{2n+1}, y_{2n-1})]
\end{aligned}$$

$(1-k_2)d[(y_{2n+1},y_{2n})]^2 \leq d[(y_{2n+1},y_{2n})]\,[K_1d(y_{2n},y_{2n-1})+K_2d(y_{2n+1},y_{2n-1})]$

$(1-k_2)d[(y_{2n+1},y_{2n})] \leq K_1d(y_{2n},y_{2n-1})+K_2d(y_{2n+1},y_{2n-1})]$

$(1-k_2)d[(y_{2n},y_{2n+1})] \leq K_1d(y_{2n},y_{2n-1})+K_2d(y_{2n+1},y_{2n-1})]$

$(1-k_2)d[(y_{2n},y_{2n+1})] \leq K_1d(y_{2n},y_{2n-1})+K_2[d(y_{2n+1},y_{2n}) +d(y_{2n},y_{2n-1})]$

$(1-2k_2)d[(y_{2n},y_{2n+1})] \leq (K_1+K_2)\, d(y_{2n},y_{2n-1})]$

$d[(y_{2n},y_{2n+1})] \leq p\,d(y_{2n},y_{2n-1})]$

where $p = \dfrac{k_1 + k_2}{1-2k_2} < 1$

$\{y_n\}$ is a Cauchy sequence.

Since $\{y_n\}$ is a Cauchy sequence and since X is a complete metric then $\exists$ a point z ϵ X such that $\lim y_n = z$ as $n\to\infty$ consequently subsequences Px_{2n}, Rx_{2n}, Qx_{2n-1} and Sx_{2n+1} converges to z .

Let R be a continuous, since P and R are compatible type (R) on X, then by proposition (2.2) we have $R^2 x_{2n}\to Rz$ and $PRx_{2n}\to Rz$ as $n\to\infty$.

Now by condition (2) of theorem, we have

$$\begin{aligned}
[d(PRx_{2n},Qx_{2n-1})]^2 &\leq K_1[d(R^2x_{2n}, Qx_{2n-1})d(R^2x_{2n}, Sx_{2n-1}) + d(R^2x_{2n}, Sx_{2n-1})d(PRx_{2n}, R^2x_{2n})] \\
&\quad + K_2[d(PRx_{2n}, Qx_{2n-1})d(PRx_{2n}, R^2x_{2n}) + d(PRx_{2n}, Bx_{2n-1})d(PRx_{2n}, Sx_{2n-1})]
\end{aligned}$$

as $n\to\infty$ we have

$[d(Rz, z)]^2 \leq K_1[d(Rz, z)d(Rz, z) + d(Rz, z)d(Rz, Rz)] + K_2[d(Rz, z)d(Rz, Rz) + d(Rz, z)d(Rz, z)]$

$[d(Rz, z)]^2 \leq K_1[d(Rz, z)^2] + K_2[d(Rz, z)^2)]$

$[d(Rz, z)]^2 \leq (K_1+K_2)[d(Rz, z)^2)]$

This is a contradiction.

Than $d(Rz, z) = 0$

hence $Rz = z$

Now

$$\begin{aligned}
[d(Pz,Qx_{2n-1})]^2 &\leq K_1[d(Rz, Qx_{2n-1})d(Rz, Sx_{2n-1}) + d(Rz, Sx_{2n-1})d(Pz, Rz)] \\
&\quad + K_2[d(Pz,Qx_{2n-1})d(Pz, Rz) + d(Pz, Qx_{2n-1})d(Pz, Sx_{2n-1})]
\end{aligned}$$

Taking limas $n\to\infty$ we have

$$\begin{aligned}
[d(Pz, z)]^2 &\leq K_1[d(z, z)d(z, z) + d(z,, z)d(Pz, z)] + K_2[d(Pz, z)d(Pz, z) + d(Pz, z)d(Pz, z)] \\
&= 2k_2[d(Pz, z)]^2
\end{aligned}$$

$[d(Pz, z)]^2 \leq 2k_2[\, d(Pz, z)]^2$

Which is a contradiction.
Then d(Pz, z) = 0
Hence Pz = z

Since by condition (1) $z\epsilon$ S(X) also S is a self map of X so $\exists$ a point $u\epsilon$ X such that z = P(z) = S(u) more over by condition (2), we have
$[d(z, Qu)]^2 = [d(Pz, Qu)]^2 \leq K_1[d(Rz, Qu)d(Rz, Su) + d(Rz,, Su)d(Pz, Rz)] + K_2[d(Pz, Qu)d(Pz, Rz) + d(Pz, Qu)d(Pz, Su)]$

Taking lim as $n\to\infty$.
$[d(z, Qu)]^2 \leq K_1[d(z, Qu)d(z, z)+d(z, z)d(z, z)] +K_2[d(z, Qu)d(z, z)+ d(z, Qu)d(z, z)]$
$[d(z, Qu)]^2 \leq 0$
$[d(z, Qu)] = 0$

i.e Qu = z

by condition (4) we have

[d(SQu, QSu)] = 0

Hence [d(Sz, Qz)]= 0
Sz = Qz

Now,
$[d(z, Sz)]^2 = [d(Pz, Sz)]^2 \leq K_1[d(Rz,Qz)d(Rz,Sz) + d(Rz, Sz)d(Pz, Rz)] + K_2[d(Pz, Qz)d(Pz, Rz) + d(Pz, Qz)d(Pz, z)]$

Taking lima $sn\to\infty$.

$[d(z, Sz)]^2 = \leq K_1[d(z, Qz)d(z, z) + d(z, z)d(z, z)] +K_2[d(z, Qz)d(z, z) + d(z, Qz)d(z, z)]$

$[d(z, Sz)]^2 \leq 0$

d (z, Sz) = 0
z = Sz

Hence Qz = Sz = z

Hence z is a common fixed of P, Q, R, S.

Uniqueness of z: let w is another common fixed point of P, Q, R, S, then we have

$$\begin{aligned}[d(z, w)]^2 &= [d(Pz, Qw)]^2 \leq K_1[d(Rz, Qw)d(Rz, Sw) + d(Rz,, Sw)d(Pz,Rz)] + K_2[d(Pz, Qw)d(Pz, Rz) + d(Pz, Qw)d(Pz, Sw)] \\ &\leq K_1[d(z, w)d(z, w) + d(z,, w)d(z, z)] + K_2[d(z, w)d(z, z) + d(z, w)d(z, w)] \\ &\leq (k_1+k_2)[d(z, w)]^2\end{aligned}$$
$[d(z, w)]^2 \leq (k_1+ k_2) [d(z, w)]^2$
Hence z = S(w) = w
z = w

Example 2.4: Let X=[0,∞) be endowed with a complete metric space (X, d) with metric $d(x,y) = \left|x-y\right|^2 = (x-y)^2$, define P, Q, R, S on X by $P(x) = \log\left(1+\frac{x}{4}\right)$,$Q(x) = \log\left(1+\frac{x}{6}\right)$
$R(x) = e^{3x} - 1,\ S(x) = e^{2x} - 1$.

Obviously P(x) = Q(x) = R(X) = S(X) = [0, ∞) .

We show that the pair (P, R) is compatible

Let $\{x_n\}$ be a sequence in X such that for some $t\epsilon X$ lim $n\to\infty$ d (Px_n, t) =0 and $t\epsilon X$
sslim $n\to\infty$ d $(Rx_n, t) = 0$

i.e lim n→∞ $\left|Px_n - t\right|$=0 , lim n→∞ $\left|Rx_n - t\right|$=0. Since P and R are continuous, we have

lim n→∞ d(PRx$_n$, RPx$_n$) = lim n→∞ $\left|PRx_n - RPx_n\right|^2 = \left|Pt - Rt\right|^2 \left|\log(1+\frac{t}{4}) - (e^{3t}-1)\right|^2 = 0 \Leftrightarrow t = 0$

then (P, R) are compatible .

Similarly {Q, S} are compatible.
for each x, y ϵ X

$$[d(Px, Qy)]^2 = [(Px - Qy)^2]^2 = \left[\left\{\log\left(1+\frac{x}{4}\right) - \log\left(1+\frac{y}{6}\right)\right\}^2\right]^2 \leq \left[\left(\frac{x}{4} - \frac{y}{6}\right)^2\right]^2 \leq \frac{1}{(12)^4}[(3x-2y)^2]^2$$

$$\leq \frac{1}{(12)^4}[e^{3x} - e^{2y})^2]^2$$

$$= \frac{1}{(12)^4}[d(Rx, Sy)]^2$$

$$\leq \frac{10}{(12)^5}[d(Rx,Sy)d(Rx,Sy) + d(Rx,Sy)d(Rx,Rx)]$$

$$+ \frac{2}{(12)^5}[d(Rx,Sy)d(Rx,Rx) + d(Rx,Sy)d(Rx,Sy]$$

Where $k_1 = \frac{10}{(12)^5} \geq 0$ and $k_2 = \frac{2}{(12)^5} \geq 0$ and $k_1 + 3k_2 = \frac{10}{(12)^5} + 3\frac{2}{(12)^5} < 1$

Thus P, Q, R, S satisfy all condition of theorem (2.3) .moreover 0 is the unique common fixed point of P, Q, R, S.

This complete the proof.

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