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A COMMON FIXED POINT THEOREM FOR A BANACH OPERATOR PAIR OF MAPPINGS IN A CONE METRIC SPACE

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ABSTRACT

In this paper, we prove a common fixed theorem for a Banach operator pair of mappings satisfying a contraction condition given by Singh [17] in cone metric spaces. Our result generalizes the results of Bhatt et al. [5] and some well known previous results in cone metric spaces.

Mathematical Subject Classification: *47H10, 54H25.*

Keywords: *Cone metric space, Banach operator pair, common fixed point.*

1. INTRODUCTION

In 2007, Huang and Zhang [7] introduced the concept of cone metric space by replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions. The category of cone metric spaces is larger than metric spaces and there are different types of cones. Subsequently, many authors Abbas and Jungck [1], Abbas and Rhoades [2], Ilic and Rakocevic [9], [10], Jungck *et al.* [11], Kadelburg *et al.* [12], Raja and Vezapour [15] have generalized the results of Huang and Zhang [7] and studied the existence of common fixed points of a pair of self mappings satisfying a contractive type condition in the framework of normal cone metric spaces. Rhoades [16] made a comparison of various different types of contraction mappings. A new generalization of contraction mappings acting on complete metric spaces is introduced by Beiranvand *et al.* [4] called T- contraction mappings which are depending on another function. Morales and Rojas [14] have extended the concept for T- contraction mappings to cone metric space by proving fixed point theorems of Kannan [13], Zamfirescu, weakly contraction mappings.

In 1975, Subrahmanyam [18] obtained the fixed point of a continuous Banach operator in complete metric space. In 2007, Chen and Li [6] extended the concept of Banach operator to Banach operator pair and proved various best approximation results using common fixed point theorems for f- nonexpansive mappings. Where f is a self-mapping of the subset M of a metric space X. Hussain [8], Al- Thagafi and Shahzad [3] generalized the results of Chen and Li [6] proved various common fixed point theorems and invariant approximation results for generalized non-expansive Banach operator pair of mappings. The Purpose of this paper is to prove a fixed point theorem for a T-Singh type contraction mapping in a cone metric space. If the pair of mappings is a Banach pair, then we have obtained a common fixed point. Our result generalizes the results of Bhatt *et al.* [5] and some well known previous results in cone metric spaces.

2. DEFINITIONS AND PRELIMINARIES

Definition 2.1: A self-mapping T of a metric space (X, d) is said to be a contraction mapping, if there exists a real number $0 \leq k < 1$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq kd(x, y) \tag{2.1}$$

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Definition 2.2: Let T and f be two self-mappings of a metric space (X, d). The self-mapping f of X is said to be T-contraction, if there exists a real number $0 \leq k < 1$ such that for all $x, y \in X$,

$$d(Tfx, Tfy) \leq kd(Tx, Ty) \tag{2.2}$$

If $T = I$, the identity mapping, then the Definition (2.2) reduces to Banach contraction mapping.
It is obvious that a T-contraction mapping need not be contraction mapping.

Example 2.1: Let $X = [1, \infty)$ be with the usual metric. Define two mappings $T, f: X \to X$ as $Tx = \frac{1}{2x} + 6$ and $fx = 2x$ obviously, f is not contraction but f is T- contraction which is seen from the following;

$$|Tfx - Tfy| = \left|\frac{1}{4x} - \frac{1}{4y}\right| = \frac{1}{2}|Tx - Ty|$$

Definition 2.3: Let T and f be two self-mappings of a metric space (X, d). The self mapping f of X is said to be T-contractive, if for every $x, y \in X$ such that $Tx \neq Ty$ and

$$d(Tfx, Tfy) < d(Tx, Ty).$$

It is obvious that every T- contraction mapping is T- contractive but the converse need not be true.

Example 2.2: Let $X = [1, \infty)$ be with usual metric. Define two mappings $T, f : X \to X$ as $Tx = \frac{1}{1000}x + 2$ and $fx = sinx$. Obviously, f is not T- contraction but f is T- contractive.

Definition 2.4: Let T be a self-mapping of a metric space (X, d). Then
- (i) The mapping T is said to be sequentially convergent, if the sequence $\{y_n\}$ in X is convergent whenever $\{Ty_n\}$ is convergent.
- (ii) The mapping T is said to be subsequentially convergent, if $\{y_n\}$ has a convergent subsequence whenever $\{Ty_n\}$ is convergent.

Theorem 2.1: Let (X, d) be a complete metric space and $T: X \to X$ be a one to one, continuous and subsequentially convergent mapping. Then every T- contraction and continuous self mapping $f: X \to X$ has a unique fixed point in X. Also if T is sequentially convergent, then for each $x_0 \in X$, the sequence of iterates $\{f^n x_0\}$ converges to the fixed point.

Definition 2.5: Let T be a self-mapping of a normal space X. Then T is called a Banach operator of type k if

$$||T^2x - Tx|| \leq k||Tx - x||$$

For some $k \geq 0$ and for all $x \in X$.

Definition 2.6: Let T and f be two self-mappings of a nonempty subset M of a normed linear space X. Then (T, f) is a Banach operator pair, if any one of the following conditions is satisfied:
1. $T[F(f)] \subseteq F(f)$ is T-invariant.
2. $fTx = Tx$ for each $x \in F(f)$.
3. $fTx = Tfx$ for each $x \in F(f)$.
4. $\left||Tfx - fx|\right| \leq k\left||fx - x|\right|$ forsome $k \geq 0$.

Definition 2.7: Let E be a real Banach space. A subset P of E is called a cone if and only if
1. P is nonempty, closed and $P \neq \{0\}$;
2. $\alpha, \beta \in R, \alpha, \beta \geq 0$ and $x, y \in P \Rightarrow \alpha x + \beta y \in P$.
3. $x \in P$ and $-x \in P$ $i.e$ $P \cap -P = \{0\}$.

For a given cone $P \subseteq E$, a partial ordering is defined as $\leq$ on E with respect to P by $x \leq$ y if and only if $y - x \in P$. It is denoted as $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$, where int$P$ denotes the interior of P. The cone $P \subset E$ is called normal, if there is a number $K > 0$ such that for all $x, y \in E, 0 \leq x \leq y$ implies

$$\left||x|\right| \leq K\left||y|\right| \tag{2.3}$$

The least positive number K satisfying (2.3) is called the normal constant of P.

Definition 2.8: Let X be a non-empty set. Suppose E is a real Banach space, P is a cone with int$P \neq \phi$ and $\leq$ is a partial ordering with respect to P. If the mapping $d: X \times X \to E$ satisfies:
1. $0 \leq d(x, y)$for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \leq d(x, z) + d(z, y)$for all $x, y, z \in X$;

   Then d is called a cone metric on X and (X, d) is called a cone metric space.

Example 2.3: Let $E = \mathbb{R}^2, P = \{(x,y) \in E: x,y \geq 0\} \subset \mathbb{R}^2, X = \mathbb{R}$ and $d: X \times X \to E$ such that $d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant. Then (X,d) is a cone metric space.

Definition 2.9: Let (X,d) be a cone metric space and $\{x_n\}$ be a sequence in X. Then,

(i) $\{x_n\}$ converges to $x \epsilon X$, if for every c ϵE with $0 \ll c$, there is $n_0 \epsilon \mathbb{N}$, the set of all natural numbers such that for all $n \geq n_0$, $d(x_n, x) \ll c$.
It is denoted by $\lim_{n\to\infty} x_n = x$.

(ii) If for $c \epsilon E$, there is a number $n_0 \epsilon N$ such that for all $n \geq n_0$ $d(x_n, x) \ll c$. Then $\{x_n\}$ is called a Cauchy sequence in X.

(iii) (X,d) is a complete cone metric space, if every Cauchy sequence in X is convergent.

(iv) A self-mapping $T: X \to X$ is said to be continuous at a point $x \in X$, if $\lim_{n\to\infty} x_n = x$ implies $\lim_{n\to\infty} T\, x_n = Tx$ for every $\{x_n\}$ in X.

Lemma 2.1: Let (X,d) be a cone metric space and P be a normal cone with normal constant K. A sequence $\{x_n\}$ in X is a Cauchy sequence if and only if $d(x_n, x) \to 0\ as\ n \to \infty$

Lemma 2.2: Let (X,d) be a cone metric space and P be a normal cone with normal constant K. A sequence $\{x_n\}$ in X is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$

Corollary 2.1: $Let\ a, b, c, u\ \in E$, the real Banach space.

(1) If $a \leq b$ and $b \ll c$, then $a \leq c$;
(2) If $a \ll b$ and $b \ll c$, then $a \ll c$;
(3) If $0 \leq u \ll c$ for each $c \in \text{int}P$, then $u = 0$;

Remark 2.1: If $c \in \text{int}P, 0 \leq a_n\ and\ a_n \to 0$, then there exists n_0 such that for all $n > n_0$, it follows that $a_n \ll c$.

3. MAIN RESULTS

Theorem 3.1: Let T and f be two continuous self-mappings of a complete cone metric space (X,d). Assume that T is an injective mapping and P is a normal cone with normal constant. If the mappings T and f satisfying

$$d(T^p fx, T^p fy) \leq a[d(T^p x, T^p fx) + d(T^p y, T^p fy)] \tag{3.1}$$

for all x, y ∈ X, where p is a positive integer and $a \in (0, 1/2)$. Then f has a fixed point in X. Moreover, if (T,f) is a Banach pair, then T and f have unique common fixed point in X.

Proof: Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X such that $x_{n+1} = fx_n \Rightarrow x_n = fx_{n-1}$ for each $n = \mathbb{N} \cup \{0\}$ consider,

$$d(T^p x_n, T^p x_{n+1}) = d(T^p fx_{n-1}, T^p fx_n) \tag{3.2}$$

$$\leq a[d(T^p x_{n-1}, T^p fx_{n-1}) + d(T^p x_n, T^p fx_n)$$

$$d(T^p x_n, T^p x_{n+1}) \leq a[d(T^p x_{n-1}, T^p x_n) + d(T^p x_n, T^p x_{n+1})$$

$$d(T^p x_n, T^p x_{n+1}) \leq \frac{a}{1-a} d(T^p x_{n-1}, T^p x_n),$$

$$\leq \ldots$$

in this way, we get.

$$d(T^p x_n, T^p x_{n+1}) \leq \frac{a^n}{1-a} d(T^p x_0, T^p x_1) \tag{3.3}$$

Let $\frac{a^n}{1-a} = k^n$, then (3.2) reduces to,

$$d(T^p x_n, T^p x_{n+1}) \leq k^n d(T^p x_0, T^p x_1) \tag{3.4}$$

Next to claim that $\{T^p x_n\}$ is a Cauchy sequence consider $m, n \in \mathbb{N}$ such that $m > n$. Now

$$d(T^p x_n, T^p x_m) \leq d(T^p x_n, T^p x_{n+1}) + d(T^p x_{n+1}, T^p x_{n+2}) + \ldots + d(T^p x_{m-1}, T^p x_m)$$

$$\leq (k^n + k^{n+1} + \cdots k^{m-1})\, d(T^p x_0, T^p x_1)$$

$$\leq (k^n + k^{n+1} + \cdots)\, d(T^p x_0, T^p x_1)$$

$$d(T^p x_n, T^p x_m) \leq \frac{k^n}{1-k} d(T^p x_0, T^p x_1) \tag{3.5}$$

From (2.3); it follows that

$$||d(T^p x_n, T^p x_m)|| \leq \frac{k^n}{1-k} ||d(T^p x_0, T^p x_1)||$$

Since $k \in (0,1) \Rightarrow \quad k \to 0\ as\ n \to \infty$. Therefore $\left||d(T^p x_n, T^p x_m)\right|| \to 0$ as $m, n \to \infty$. Thus $\{T^p x_n\}$ is a Cauchy sequence in X. As X is a complete cone metric space, there exists $z \in X$ such that

$$\lim_{n\to\infty} x_m = z.$$

Since T^p is subsequentially convergent, $\{x_n\}$ has a convergent subsequence $\{x_m\}$ such that $\lim_{n\to\infty} x_{\mathrm{m}} = u$. As T is continuous

$$\lim_{m\to\infty} T^p x_m = T^p u. \tag{3.6}$$

By uniqueness of the limit, $z = Tu$, since f is continuous, we have

$$\lim_{m\to\infty} f x_m = fu. \tag{3.7}$$

Again T is continuous, $\lim_{m\to\infty} T^p f x_m = T^p fu$. (3.8)

Therefore $\lim_{m\to\infty} T^p x_{m+1} = T^p fu$. (3.9)

Now consider,

$$\begin{aligned} d(T^p fu, T^p u) &\leq d(T^p fu, T^p x_m) + d(T^p x_m, T^p fu) \\ &= d(T^p fu, T^p f x_{m-1}) + d(T^p x_m, T^p u) \end{aligned} \tag{3.10}$$

Now using condition (3.1) from (3.10), we get.

$$\begin{aligned} d(T^p fu, T^p u) &\leq a d(T^p u, T^p fu) + d(T^p x_{m-1}, T^p f x_{m-1}) + d(T^p x_m, T^p u) \\ &= a d(T^p u, T^p fu) + d(T^p x_{m-1}, T^p x_m) + d(T^p x_m, T^p u) \\ d(T^p fu, T^p u) &\leq \frac{a}{1-a} d(T^p x_{m-1}, T^p x_m) + \frac{1}{1-a} d(T^p x_m, T^p u) \end{aligned} \tag{3.11}$$

Let $0 \leq c$ be arbitrary then by (3.5); $d(T^p x_m, T^p u) \leq c\,\frac{(1-a)}{2}$ similarly by (3.7) $d(T^p x_{m-1}, T^p x_m) \leq c\,\frac{1-a}{2a}$, then (3.11) becomes;

$$d(T^p fu, T^p u) \leq \frac{c}{2} + \frac{c}{2} = c$$

Thus $d(T^p fu, T^p u) \leq c$ for each $c \in \mathrm{int}P$. Now using Corollary 2.1, it follows that $d(T^p fu, T^p u) = 0$. Which implies that $T^p u = T^p fu$, as T is injective then $u = fu$. Thus $u\ is$ fixed point of f.

To prove uniqueness, suppose that w is another fixed point point of f then $fw = w$. Now consider,

$$d(T^p u, T^p w) = d(T^p fu, T^p fw)$$

Using (3.1) in above equation, we obtain

$$d(T^p u, T^p w) = d(T^p fu, T^p fw) \leq a[d(T^p u, T^p fu) + d(T^p w, T^p fw)].$$

Now by $fu = u$ and $fw = w$, we have $d(T^p u, T^p w) \leq 0$.

Therefore $d(T^p u, T^p w) = 0$.

Thus $T^p u = T^p w$. Since T is injective then $u = w$. Hence f has unique fixed point.

As (T, f) is a Banach pair, T and f commutes at the fixed point of f which implies that $Tfu = fTu$ for $u \in F(f)$ $i.e\ Tu = fTu$; which implies that Tu is another fixed point of f. By uniqueness of fixed point of f, $u = Tu$. Hence $u = fu = Tu\ is$ unique common fixed point of f and T in X.

Example 3.2: Let $X = [0,1] \cup \{2\}$ be a set with usual partial metric. Define functions such that

$$f(x) = \begin{cases} 0, when\ x \in [0,1] \\ 1, when\ x = 2 \end{cases}$$

$$T(x) = \begin{cases} 0, when\ x \in [0,1] \\ \dfrac{1}{2}, when\ x = 2 \end{cases}$$

Here $T[0,1] = 0, T(2) = \frac{1}{2}$, we get $T^2(x) = T^3(x) = \cdots = T^p(x) = 0$; for all x. We have $Tfx = T[f[0,1]] = T[0] = 0, T[f[2]] = T(1) = 0.$ Therefore, $T^2 fx = T^3 fx = T^4 fx = \cdots = T^p fx = 0$, for every x. Let $x = 0$ and $y = 1$. Applying (3.1), we get

$$\begin{aligned} p(T^p f(0), T^p f(1)) &\leq a[p(T^p(0), T^p f(0)) + p(T^p(1), T^p f(1))], \\ p(0,0) &\leq a[p(0,0) + p(0,0)] \\ 0 &\leq 0. \end{aligned}$$

So, Theorem (3.1) is verified and we get f and T have a unique common fixed point 0.

Corollary 3.1: Let f be self-mapping of a complete cone metric space. (X, d) satisfying

$$d(f^p x, f^p y) \leq a[d(x, f^p x) + d(y, f^p y)]$$

where p is a positive integer and a such that $0 < a < \frac{1}{2}$ for every $x, y \in X$. Then f has a unique fixed point in X.

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