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THE EVOLUTION OF LINEARIZED PERTURBATIONS IN A TWO LAYERED STRATIFIED SHEAR FLOW

S. S. BELLAD[1] AND VIJAYALAKSHMI A.R.*[2]

[1]Department of Physics,
Maharani's Science College for Women, Palace Road, Bangalore – 560 001. India.

[2]Department of Mathematics,
Maharani's Science College for Women, Palace Road, Bangalore – 560 001. India.

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ABSTRACT

The evolution of linearized disturbances in a two-layered stratified shear flow is studied by making use of the initial value problem approach. The resulting equation in time posed by using Squire transform and Fourier transform is solved for the Fourier amplitudes. The initial distributions that are considered are a point source of the field of transverse velocity and density. For small values of Brunt Väisälä frequency, the perturbation solutions are obtained.

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Key words: *Two-layered stratified shear flow, perturbations, initial value problem, Fourier transform, Brunt Väisälä frequency, point source of the field of transverse velocity and density.*

1. INTRODUCTION

Atmosphere and ocean are stably stratified, i.e. their density increases in the direction of gravity. A direct consequence of such density stratification is that, beyond a certain threshold stratification, internal gravity waves can be supported in the medium. If there is a sheaflow in this environment, the interaction of stable stratification and background shear leads to a range of phenomena. This interaction plays a crucial role in a wide range of situations of engineering and geophysical interest. The destabilization of stratified layer occurs when the pressure gradients in the flow can overcome gravity and overturn the fluid.

The stability of stratified shear flow has been investigated by many researchers. [5] made a first attempt to solve an initial-value problem for a stratified shear flow. They considered the flow between two parallel walls, by solving it as an initial-value problem they showed that a disturbance originating from an arbitrary initial conditions would behave asymptotically like $t^{\frac{1}{2}(\gamma-1)}, \gamma = (1-4J)^{\frac{1}{2}}$ for $-\frac{3}{4} < J_0 < \frac{1}{4}$ but would be exponentially unstable for Richardson number, $J_0 < -\frac{3}{4}$, for the semi-infinite case. [3] gave a rather complete stability analysis for the problem of a semi – infinite exponential atmosphere. [9] established the conjecture that a sufficient condition for stability in a parallel stratified, inviscid flow is that the local Richardson number J_0 should everywhere exceed $\frac{1}{4}$. [8] has studied small perturbations of plane Couette flow in stably and unstably stratified fluids. He found that the system to be more unstable when it is bounded both above and below than when its depth is infinite.[4] studied the stability of stratified shear flow and concluded that the flow will be unstable if the local Richardson number falls below $\frac{1}{4}$ anywhere in the flow. [2] have resolved the controversy surrounding the decay rate in favour of original results of [5]. [1] studied numerical simulations of stratified shear flow instabilities in 2-dimensions in the Boussinesq limit and concluded that although mixing is suppressed at large Richardson numbers it is not negligible and turbulent mixing processes in strongly stratified environments cannot be excluded. [7] studied density stratification and suggested that the total

Corresponding Author: Vijayalakshmi A.R.[2], Department of Mathematics, Maharani's Science College for Women, Palace Road, Bangalore – 560 001. India.*

perturbation energy, the sum of kinetic and potential energies is the relevant measure. [6] investigated analytically the short-time response of disturbances in a density-varying Couetteflow without viscous and diffusive effects. The complete inviscid problem is also solved as an initial value problem with a density perturbation. They showed that the kinetic energy of the disturbances grows algebraically at early times, contrary to the wellknown algebraic decay at time tending to infinity.

In this paper, we have investigated two-layered stratified shear flow, with unit pulse of velocity and density as initial conditions. The essence of the approach is as follows: Taking a multilayered basic flow with piecewise linear velocity profile, complete general solutions to the linearized equations of motion are obtained as functions of all space variables and time, when posed as initial-value problems. The distributions are resolved into two components, rotational and irrotational. The solution for the hypothetical initial-value problem for which the basic flow is unbounded but coincides with the actual flow in the layer is the rotational solution. The irrotational solution in each layer is specified uniquely by satisfying the interfacial conditions and boundary conditions.

2. MATHEMATICAL FORMULATION

We consider an inviscid, incompressible fluid of density ρ moving with velocity $\vec{q}$ under the influence of gravity $\vec{g}$ directed in the negative y-direction. The fluid is assumed to be inviscid, incompressible, stratified and is Boussinesq for which motion is governed by the equations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{2.1}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}, \tag{2.2}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x} \tag{2.3}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z} \tag{2.4}$$

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}=0 \tag{2.5}$$

where p is the pressure.

In the basic unperturbed equilibrium state,

$$\vec{q}_0 = \left(U(y)=\sigma y,0,0\right), p=p_0(y),\ \rho=\rho_0(y) \tag{2.6}$$

where σ, the intensity of the shear is constant.

For linear stability analysis, we superimpose a small perturbation upon the mean flow i,e.,

$$\vec{q}=\vec{q}_0+\vec{q}\,',\ p=p_0(y)+p',\ \rho=\rho_0(y)+\rho', \tag{2.7}$$

where $\vec{q}\,'$, p' and ρ' are the perturbed quantities of velocity, pressure and density respectively.

To study the evolution of linearized perturbations in a stratified shear flow, we linearize equations (2.1)–(2.5) using (2.7). The linearized differential equations of motion (neglecting the primes) by employing (i) moving co-ordinates transformation by defining the transformation of co-ordinates of the form

$$T=t,\ \xi=x-\sigma y t,\ \eta=y,\ \zeta=z \tag{2.8}$$

(ii) three - dimensional Fourier transformation of the form

$$\hat{u}(\alpha;\beta;\gamma;T)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u(\xi;\eta;\zeta;T)\,e^{i(\alpha\xi+\beta\eta+\gamma\zeta)}\,d\xi\,d\eta\,d\zeta, \tag{2.9}$$

and with similar expressions for $\hat{v}$, $\hat{w}$, $\hat{p}$ and $\hat{\rho}$,

(iii) Squire transformation by defining the velocity components in the $\overline{\alpha}$ and φ directions as

$$\overline{u}=\frac{\alpha\hat{u}+\gamma\hat{w}}{\overline{\alpha}},\ \overline{w}=\frac{-\gamma\hat{u}+\alpha\hat{w}}{\overline{\alpha}} \tag{2.10}$$

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and eliminating $\hat{p}$, the resulting differential equation is of the form

$$\frac{d^2}{dT^2}\left(K^2\hat{v}\right)+N^2\bar{\alpha}^2\hat{v}=0 \tag{2.11}$$

where $K^2=\bar{\alpha}^2+(\beta-\sigma\alpha T)^2$ and $\bar{\alpha}^2=\left(\alpha^2+\gamma^2\right)$, $N^2=-\frac{g}{\rho_0}\frac{d\rho_0}{dy}$, ρ_0 is the equilibrium density, N is the Brunt Väisälä frequency.

Equation (2.11) is solved with appropriate initial conditions for $\hat{v}$ and $\hat{\rho}$, The pressure $\hat{p}$ is obtained by taking divergence of the momentum equations. and it is found that

$$\hat{p}=\frac{-i\left(2\sigma\alpha\rho_0\hat{v}-g(\beta-\sigma\alpha T)\hat{\rho}\right)}{K^2}, \text{ for } K^2\neq 0$$

Two sets of solutions exist for equation (2.11), when $K^2\neq 0$, the disturbance is rotational and for $K^2=0$, the disturbance is irrotational. The vanishing of the product $K^2\hat{v}$ corresponds to Laplace equation $\nabla^2\hat{v}=0$ in real space. We denote $\hat{v}$ as $\hat{v}_R$ when $K^2\neq 0$ which is called the rotational solution and $\hat{v}$ as $\hat{v}_I$ when $K^2=0$ and is called the irrotational solution. Therefore $\hat{v}$ can be resolved into two components and thus $\hat{v}=\hat{v}_R+\hat{v}_I$.

Now considering the case $K^2\neq 0$, we assume the regular perturbation expansion of $\hat{v}$ in terms of the parameter N^2 in the form

$$\hat{v}_R(\alpha,\beta,\gamma,T)=\hat{v}_0(\alpha,\beta,\gamma,T)+N^2\hat{v}_1(\alpha,\beta,\gamma,T)+\left(N^2\right)^2\hat{v}_2(\alpha,\beta,\gamma,T)+\ldots \tag{2.12}$$

where $\hat{v}_R$ is the rotational component of $\hat{v}$.

We find that,

$$\hat{v}_0=\frac{T\,\hat{\Omega}_0(\alpha,\beta,\gamma)+\hat{\Omega}_1(\alpha,\beta,\gamma)}{\bar{\alpha}^2+(\beta-\sigma\alpha T)^2} \tag{2.13}$$

$$\hat{v}_1=-\bar{\alpha}^2\left[\left(\frac{\beta\hat{\Omega}_0}{\sigma^2\alpha^2\bar{\alpha}}-\frac{\hat{\Omega}_1}{\sigma\alpha\bar{\alpha}}\right)\left(\frac{\beta-\sigma\alpha T}{\bar{\alpha}}\right)\tan^{-1}\left(\frac{\beta-\sigma\alpha T}{\bar{\alpha}}\right)+\frac{\bar{\alpha}}{2\sigma\alpha}\log\left(\frac{\bar{\alpha}^2+(\beta-\sigma\alpha T)^2}{\bar{\alpha}^2}\right)+\frac{1}{2\sigma}\left(\left(\frac{\beta-\sigma\alpha T}{\bar{\alpha}}\right)\right.\right.$$

$$\left.\left.\log\left(\frac{\bar{\alpha}^2+(\beta-\sigma\alpha T)^2}{\bar{\alpha}^2}\right)-2\left(\left(\frac{\beta-\sigma\alpha T}{\bar{\alpha}}\right)-\tan^{-1}\left(\frac{\beta-\sigma\alpha T}{\bar{\alpha}}\right)\right)\right)\left(\frac{\hat{\Omega}_0}{\sigma^2\alpha^2\bar{\alpha}^2}\right)\right]\frac{1}{\bar{\alpha}^2+(\beta-\sigma\alpha T)^2}. \tag{2.14}$$

$$\tan^{-1}\left(\frac{\beta-\sigma\alpha T}{\bar{\alpha}}\right)-\frac{1}{2}\left(\log\left(\frac{\bar{\alpha}^2+(\beta-\sigma\alpha T)^2}{\bar{\alpha}^2}\right)\right)^2+\log\left(\frac{\bar{\alpha}^2+(\beta-\sigma\alpha T)^2}{\bar{\alpha}^2}\right)+\frac{(\beta-\sigma\alpha T)^4}{12\sigma2\bar{\alpha}^2}\left(\frac{1}{\bar{\alpha}}+\frac{1}{\sigma\alpha\bar{\alpha}}\right)\right]$$

$$\hat{v}_2=\frac{\bar{\alpha}^3}{\sigma\alpha}\left[\left(\frac{\hat{\Omega}_0}{\sigma^2\alpha^2\bar{\alpha}}-\frac{\hat{\Omega}_1}{\sigma\alpha\bar{\alpha}}\right)\left[\frac{1}{2\sigma^2\alpha^2}\left(\left(\frac{\beta-\sigma\alpha T}{\bar{\alpha}}\right)\tan^{-1}\left(\frac{\beta-\sigma\alpha T}{\bar{\alpha}}\right)\log\left(\frac{\bar{\alpha}^2+(\beta-\sigma\alpha T)^2}{\bar{\alpha}^2}\right)\right)-2\left(\frac{\beta-\sigma\alpha T}{\bar{\alpha}}\right)\right.\right.$$

$$-\frac{\hat{\Omega}_0}{8\sigma^4\alpha^4\bar{\alpha}}\left[\left(\frac{\beta-\sigma\alpha T}{\bar{\alpha}}\right)\left(\log\left(\frac{\bar{\alpha}^2+(\beta-\sigma\alpha T)^2}{\bar{\alpha}^2}\right)\right)^2-2\left(\left(\frac{\beta-\sigma\alpha T}{\bar{\alpha}}\right)-\tan^{-1}\left(\frac{\beta-\sigma\alpha T}{\bar{\alpha}}\right)\right)\right]$$

$$\left(\log\left(\frac{\bar{\alpha}^2+(\beta-\sigma\alpha T)^2}{\bar{\alpha}^2}\right)\right)-2\left(\left(\frac{\beta-\sigma\alpha T}{\bar{\alpha}}\right)-\tan^{-1}\left(\frac{\beta-\sigma\alpha T}{\bar{\alpha}}\right)\log\left(\frac{\bar{\alpha}^2+(\beta-\sigma\alpha T)^2}{\bar{\alpha}^2}\right)\right)-\frac{1}{3}\left(\frac{\beta-\sigma\alpha T}{\bar{\alpha}}\right)^3\Bigg]$$

$$-\frac{\hat{\Omega}_0}{2\sigma^4\alpha^3\bar{\alpha}}\left(\frac{\beta-\sigma\alpha T}{\bar{\alpha}}\right)\log\left(\frac{\bar{\alpha}^2+(\beta-\sigma\alpha T)^2}{\bar{\alpha}^2}\right)-2\left(\left(\frac{\beta-\sigma\alpha T}{\bar{\alpha}}\right)-\tan^{-1}\left(\frac{\beta-\sigma\alpha T}{\bar{\alpha}}\right)\right)+\frac{2}{\sigma^2\alpha^2}\Bigg[\tan^{-1}\left(\frac{\beta-\sigma\alpha T}{\bar{\alpha}}\right)$$

$$\cos\left(\tan^{-1}\left(\frac{\beta-\sigma\alpha T}{\bar{\alpha}}\right)\right)-\sin\left(\tan^{-1}\left(\frac{\beta-\sigma\alpha T}{\bar{\alpha}}\right)\right)\Bigg]\Bigg]\frac{1}{\bar{\alpha}^2+(\beta-\sigma\alpha T)^2} \tag{2.15}$$

From the linearized equations using (2.8)-(2.10), we obtain the expression for $\hat{\rho}$.

The solution for $K^2=0$ which corresponds to irrotational motion is obtained by considering the two–dimensional Fourier transform of the perturbation equations instead of the full three–dimensional decomposition. Using moving co-ordinate transformation given by equation (2.8), $K^2\hat{v}=0$ corresponds to

$$\frac{\partial^2\breve{v}_I}{\partial\eta^2}+2i\sigma\alpha T\frac{\partial\breve{v}_I}{\partial\eta}-\left(\bar{\alpha}^2+\sigma^2\alpha^2T^2\right)\breve{v}_I = 0, \tag{2.16}$$

Where

$$\breve{v}_I=\breve{v}_I(\alpha,\eta,\gamma;T) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_I(\xi,\eta,\varsigma,T)e^{i(\alpha\xi+\gamma\zeta)}\,d\xi\,d\zeta \tag{2.17}$$

is the irrotational part of $\hat{v}$. The solution of equation (2.16) is found to be

$$\breve{v}_I=A(T)\,e^{-\bar{\alpha}\eta-i\sigma\alpha T\eta}+B(T)\,e^{\bar{\alpha}\eta-i\sigma\alpha T\eta}, \tag{2.18}$$

where A(T) and B(T) are constants of integration .

In order to combine $\hat{v}_R$ and $\breve{v}_I$ to obtain the complete the solution and satisfy the matching condition $\hat{v}_R$ must be inverted once to obtain $\breve{v}_R(\alpha,\eta,\gamma;T)$i,e.,

$$\breve{v}_R(\alpha,\eta,\gamma;T)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{v}_R(\alpha,\beta,\gamma;T)\,e^{-i\beta\eta}\,d\beta. \tag{2.19}$$

With initial velocity and initial density as unit pulse, the initial conditions are expressed as

$$v(x,y,z,0)=V_0\,\delta(x-x_0)\delta(y-y_0)\delta(z-z_0). \tag{2.20}$$

$$\rho(x,y,z,0)=\tilde{\rho}_0\,\delta(x-x_0)\delta(y-y_0)\delta(z-z_0) \tag{2.21}$$

In terms of moving co-ordinates and three-dimensional Fourier transform , equations (2.20) and (2.21) becomes

$$\breve{v}_0(\alpha,\beta,\gamma)=\Omega_0(\alpha,\beta,\gamma)=V_0e^{i(\alpha x_0+\beta y_0+\gamma z_0)}. \tag{2.22}$$

$$\tilde{\rho}_0(\alpha,\beta,\gamma)=\Omega_1(\alpha,\beta,\gamma)=\tilde{\rho}_0e^{i(\alpha x_0+\beta y_0+\gamma z_0)} \tag{2.23}$$

$\breve{v}_R$ is found to be

$$\breve{v}_R=e^{i(\alpha x_0+\gamma z_0-\sigma\alpha T\bar{\eta})}\Bigg\{e^{-\bar{\alpha}|\bar{\eta}|}\left(TV_0+\tilde{\rho}_0+\frac{N^4\bar{\alpha}^2}{12\sigma^2\alpha^2}\left(\frac{V_0}{\sigma^2\alpha^2\bar{\alpha}}-\frac{\tilde{\rho}_0}{\sigma\alpha\bar{\alpha}}\right)\left(1+\frac{1}{\sigma\alpha\bar{\alpha}}\right)\right)$$

$$+\left[\left(\frac{V_0}{\sigma^2\alpha^2\bar{\alpha}}-\frac{\tilde{\rho}_0}{\sigma\alpha\bar{\alpha}}\right)\left(-N^2\alpha+\frac{2\bar{\alpha}^4N^4}{\sigma}\right)+\frac{i\,N^4\bar{\alpha}V_0}{2\sigma^4\alpha^3\bar{\alpha}^2}\right]\int_{-\infty}^{\infty}\frac{\eta' e^{-\bar{\alpha}|\bar{\eta}-\eta'|-\bar{\alpha}|\bar{\eta}|}}{(\bar{\eta}-\eta')}d\eta'$$

$$+\left(\frac{V_0}{\sigma^2\alpha^2\bar{\alpha}}-\frac{\tilde{\rho}_0}{\sigma\alpha\bar{\alpha}}\right)\left(\frac{N^2}{2\sigma}\right)\int_{-\infty}^{\infty}\frac{(\bar{\eta}-\eta')e^{-\bar{\alpha}|\bar{\eta}-\eta'|-\bar{\alpha}|\bar{\eta}|}}{\eta'}d\eta'-iV_0\bar{\eta}e^{-\bar{\alpha}|\bar{\eta}|}$$

$$\left(\frac{N^2}{\sigma^2\alpha^2\bar{\alpha}^2}-\frac{11\bar{\alpha}^3N^4}{48\sigma^5\alpha^5}-\frac{\bar{\alpha}N^4}{2\sigma^4\alpha^3\bar{\alpha}^2}\right)-\left(\frac{V_0}{\sigma^2\alpha^2\bar{\alpha}}-\frac{\tilde{\rho}_0}{\sigma\alpha\bar{\alpha}}\right)\left(\left(\frac{N^4\bar{\alpha}^3}{2\sigma^3\alpha^3}-\frac{N^4\bar{\alpha}^3}{2\sigma\alpha}\right)\right.$$

$$\left.-\left(\frac{V_0N^4}{8\sigma^4\alpha^4\bar{\alpha}}-\frac{2iV_0\bar{\alpha}^2N^4}{8\sigma^5\alpha^5}\right)\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-\bar{\alpha}|\bar{\eta}-\eta'-\eta''|-\bar{\alpha}|\eta'-\eta''|-\bar{\alpha}|\eta''|}}{(\eta'-\eta'')\,\eta''}d\eta' d\eta''$$

$$\left.+\left(\frac{iV_0N^4}{\sigma^4\alpha^3\bar{\alpha}}\left(1+\frac{1}{\sigma^2\alpha^2}\right)\right)\int_{-\infty}^{\infty}\frac{e^{-\bar{\alpha}|\bar{\eta}-\eta'|-\bar{\alpha}|\eta'|}}{(\bar{\eta}-\eta')}d\eta'\right\} \tag{2.24}$$

Here $\bar{\eta}=\eta-y_0$. Now the complete solution will be

$$\breve{v}=\breve{v}_R+\breve{v}_I. \tag{2.25}$$

$\breve{v}_R$ and $\breve{v}_I$ given by equations (2.24) and (2.18).

3. TWO LAYERED STRATIFIED SHEAR FLOW

Here we consider two shear layers with constant density ρ_0 and shears σ_1 and σ_2 respectively which are separated by an interface at $\eta=0$ (Fig. 1) The solutions for the perturbation in the regions I $(\eta>0)$ are given by

$$U(y)=\begin{cases}\sigma_1 y & \text{for } y>0\\ \sigma_2 y & \text{for } y<0\end{cases}$$

$$\bar{\rho}=\rho_0 e^{-\lambda y}$$

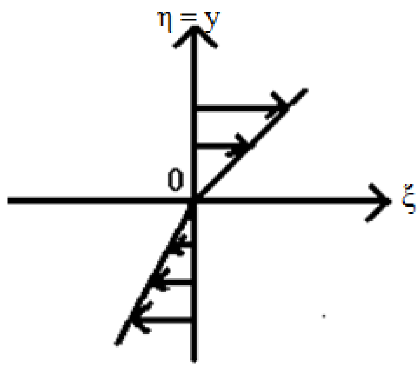

Figure-1: Sketch of two–layered unbounded stratified shear flow.

$$\breve{v}_1 = A(T)e^{-\bar{\alpha}\eta-i\sigma_1\alpha T\eta}+\breve{v}_{R_1}(\alpha;\eta;\gamma;T) \tag{3.1}$$

$$\breve{v}_1 = A(T)e^{-\bar{\alpha}\eta-i\sigma_1\alpha T\eta}+e^{i\left(\alpha x_0+\gamma z_0-\sigma_1\alpha T\bar{\eta}\right)}\left(TV_0+\tilde{\rho}_0\right)e^{-\bar{\alpha}\bar{\eta}} \tag{3.2}$$

The solution in region II i,e., for $\eta < 0$ the solutions for the perturbations are

$$\breve{v}_2 = B(T)e^{\bar{\alpha}\eta - i\sigma_2\alpha T\eta} + \breve{v}_{R_2}(\alpha,\eta,\gamma,T), \tag{3.3}$$

$$\breve{v}_2 = B(T)e^{\bar{\alpha}\eta - i\sigma_2\alpha T\eta} + e^{i\left(\alpha x_0 + \gamma z_0 - \sigma_2\alpha T\bar{\eta}\right)}\left(TV_0 + \tilde{\rho}_0\right)e^{\bar{\alpha}\,\bar{\eta}}. \tag{3.4}$$

The solutions for $\breve{v}_1$ and $\breve{v}_2$ given by (3.1) and (3.3) satisfy the required boundary condition at $\eta = \pm\infty$.

Using the first matching condition i,e., the continuity of $\breve{v}_1$ and $\breve{v}_2$ at the interface i,e., $\breve{v}_1 = \breve{v}_2$ at $\eta = 0$ yields

$$A - B = \breve{v}_{R_2}(\alpha,0,\gamma,T) - \breve{v}_{R_1}(\alpha,0,\gamma,T), \tag{3.5}$$

$$A - B = e^{i\left(\alpha x_0 + \gamma z_0\right)}\left(TV_0 + \tilde{\rho}_0\right)\left(e^{i\sigma_2\alpha T y_0 - \bar{\alpha} y_0} - e^{i\sigma_1\alpha T y_0 - \bar{\alpha} y_0}\right). \tag{3.6}$$

The pressure matching condition is that the pressure is continuous across the interface which means that the difference in pressure across the interface is zero. The continuity of the pressure can be obtained by considering the Fourier transforms of the linearized equations, Solving for $\breve{p}(\alpha,\gamma,\eta,T)$ we obtain

$$\frac{\partial^2 \breve{v}_1}{\partial\eta\partial T} + i\sigma_1\alpha T\frac{\partial \breve{v}_1}{\partial T} + 2i\sigma_1\alpha\breve{v}_1 = \frac{\partial^2 \breve{v}_2}{\partial\eta\partial T} + i\sigma_2\alpha T\frac{\partial \breve{v}_2}{\partial T} + 2i\sigma_2\alpha\breve{v}_2, \tag{3.7}$$

for $\breve{p}_1 = \breve{p}_2$ at $\eta = 0$.

Using (3.1) and (3.3) in (3.7) we obtain

$$-\bar{\alpha}\left(\dot{A} + \dot{B}\right) + i\alpha\left(\sigma_1 A - \sigma_2 B\right) = \left\{\frac{\partial^2 \breve{v}_{R_2}}{\partial\eta\partial T} - \frac{\partial^2 \breve{v}_{R_1}}{\partial\eta\partial T}\right\}_{\eta=0} + i\alpha\, T\left\{\sigma_2\frac{\partial \breve{v}_{R_2}}{\partial T} - \sigma_1\frac{\partial \breve{v}_{R_1}}{\partial T}\right\}_{\eta=0}$$

$$+2i\alpha\left\{\sigma_2\breve{v}_{R_2} - \sigma_1\breve{v}_{R_1}\right\}_{\eta=0}. \tag{3.8}$$

Let

$$G_0 = \left\{\left\{\frac{\partial^2 \breve{v}_{R_2}}{\partial\eta\partial T} - \frac{\partial^2 \breve{v}_{R_1}}{\partial\eta\partial T}\right\} + i\alpha\, T\left\{\sigma_2\frac{\partial \breve{v}_{R_2}}{\partial T} - \sigma_1\frac{\partial \breve{v}_{R_1}}{\partial T}\right\} + 2i\alpha\left\{\sigma_2\breve{v}_{R_2} - \sigma_1\breve{v}_{R_1}\right\}\right. \tag{3.9}$$

$$F_0 = \left\{\breve{v}_{R_2} - \breve{v}_{R_1}\right\}_{\eta=0}. \tag{3.10}$$

Equations (3.6)-(3.10) yields

$$2\bar{\alpha}\dot{A} - i\alpha\left(\sigma_1 - \sigma_2\right)A = -G_0 + i\alpha\,\sigma_2 F_0 + \bar{\alpha}\dot{F}_0 \tag{3.11}$$

$$2\bar{\alpha}\dot{B} - i\alpha\left(\sigma_1 - \sigma_2\right)B = -G_0 + i\alpha\,\sigma_1 F_0 - \bar{\alpha}\dot{F}_0. \tag{3.12}$$

with

$$G_0 = e^{i\left(\alpha x_0 + \gamma z_0\right)}\left[\left(TV_0 + \tilde{\rho}_0\right)i\alpha\left(\sigma_2\left(1 + y_0\right)e^{-\bar{\alpha} y_0 + i\sigma_2\alpha T y_0} - \sigma_1\left(1 - y_0\right)e^{\bar{\alpha} y_0 + i\sigma_1\alpha T y_0}\right)\right.$$

$$\left. + \bar{\alpha}V_0\left(e^{-\bar{\alpha} y_0 + i\sigma_2\alpha T y_0} - e^{\bar{\alpha} y_0 + i\sigma_1\alpha T y_0}\right)\right]. \tag{3.13}$$

From equations (3.10), (3.11), (3.12) and (3.13), A and B are found to be

$$A = e^{i\left(\alpha x_0+\gamma z_0-\left(\frac{\alpha}{2\bar{\alpha}}(\sigma_1-\sigma_2)-\sigma_1\alpha y_0\right)T\right)+\bar{\alpha}y_0}\left\{\frac{V_0 i\alpha\left(\sigma_2-\sigma_1+2\bar{\alpha}\sigma_1 y_0\right)}{-\left(\sigma_1\alpha y_0-\frac{\alpha(\sigma_1-\sigma_2)}{2\bar{\alpha}}\right)^2}-\left(\left(TV_0+\tilde{\rho}_0\right)\left(\sigma_2-\sigma_1\right)+\sigma_1\alpha\bar{\alpha}y_0-2i\bar{\alpha}V_0\right)\frac{1}{\left(\sigma_1\alpha y_0-\frac{\alpha(\sigma_1-\sigma_2)}{2\bar{\alpha}}\right)}\right\}+C_1 e^{-\frac{\alpha}{2\bar{\alpha}}(\sigma_1-\sigma_2)T}. \tag{3.14}$$

$$B = e^{i\left(\alpha x_0+\gamma z_0-\left(\frac{\alpha}{2\bar{\alpha}}(\sigma_1-\sigma_2)\right)T\right)}\left\{e^{i\sigma_2\alpha y_0 T-\bar{\alpha}y_0}\left[-\frac{\left(TV_0+\tilde{\rho}_0\right)\left(\sigma_2-\sigma_1+2\bar{\alpha}\sigma_1 y_0\right)}{\left(\sigma_2\alpha y_0-\frac{\alpha}{2\bar{\alpha}}(\sigma_1-\sigma_2)\right)}-\frac{V_0\left(\sigma_2\alpha-i\sigma_1\alpha\right)}{\left(\sigma_2\alpha y_0-\frac{\alpha}{2\bar{\alpha}}(\sigma_1-\sigma_2)\right)^2}\right]+e^{i\sigma_1\alpha y_0 T+\bar{\alpha}y_0}\left[-\frac{V_0\left(2i\alpha\bar{\alpha}\sigma_1 y_0\right)}{\left(\sigma_1\alpha y_0-\frac{\alpha}{2\bar{\alpha}}(\sigma_1-\sigma_2)\right)^2}\left[\frac{2\bar{\alpha}V_0}{\left(\sigma_1\alpha y_0-\frac{\alpha}{2\bar{\alpha}}(\sigma_1-\sigma_2)\right)}\right]\right]\right\}+C_2 e^{-\frac{\alpha}{2\bar{\alpha}}(\sigma_1-\sigma_2)T}. \tag{3.15}$$

Using A(0) = B(0) = 0, it is found that

$$C_1 = -e^{i\left(\alpha x_0+\gamma z_0\right)+\bar{\alpha}y_0}\left\{\frac{V_0 i\alpha\left(\sigma_2-\sigma_1+2\bar{\alpha}\sigma_1 y_0\right)}{-\left(\sigma_1\alpha y_0-\frac{\alpha(\sigma_1-\sigma_2)}{2\bar{\alpha}}\right)^2}-\frac{\tilde{\rho}_0\left(\left(\sigma_2-\sigma_1\right)+\sigma_1\alpha\bar{\alpha}y_0-2i\bar{\alpha}V_0\right)}{\left(\sigma_1\alpha y_0-\frac{\alpha(\sigma_1-\sigma_2)}{2\bar{\alpha}}\right)}\right\} \tag{3.16}$$

$$C_2 = e^{i\left(\alpha x_0+\gamma z_0\right)+\bar{\alpha}y_0}\left\{\left[-\frac{\tilde{\rho}_0\left(\sigma_2-\sigma_1+2\bar{\alpha}\sigma_1 y_0\right)}{\left(\sigma_2\alpha y_0-\frac{\alpha}{2\bar{\alpha}}(\sigma_1-\sigma_2)\right)}-\frac{V_0\left(\sigma_2\alpha-i\sigma_1\alpha\right)}{\left(\sigma_2\alpha y_0-\frac{\alpha}{2\bar{\alpha}}(\sigma_1-\sigma_2)\right)^2}\right]\right.$$

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$$-\left(\frac{V_0\,2i\sigma_1\alpha\bar{\alpha}y_0}{\left(\sigma_1\alpha y_0-\frac{\alpha}{2\bar{\alpha}}\left(\sigma_{1-}\sigma_2\right)\right)^2}-\frac{2\bar{\alpha}V_0}{\left(\sigma_1\alpha y_0-\frac{\alpha}{2\bar{\alpha}}\left(\sigma_{1-}\sigma_2\right)\right)}\right)\Bigg\}. \tag{3.17}$$

4. RESULTS AND DISCUSSIONS

In this problem, we have studied the evolution of linearized perturbations of a basic flow of a two layered inviscid stratified shear flow using piecewise linear velocity profiles. The initial disturbances used are unit pulse for velocity and density. In these broken line (piecewise linear) profiles, we have resolved the perturbations into rotational and irrotational components. Figs. 2(a)–(b) are plots of $\breve{v}_R$ versus T for different values of Brunt Vaisala frequency N (N = 0, 0.2, 0.5) and ϕ $(\phi = 0^0, 45^0, 180^0)$. It is observed that as the value of N increases $\breve{v}_R$ decays at a faster rate for large time. It is seen that that for $\phi = 180^0$, $\breve{v}_R$ decays early. Figs. 3(a)–(b) are plots of $\hat{\rho}$ versus T for different values of Brunt Vaisala frequency N (N = 0, 0.2, 0.5) and ϕ $(\phi = 0^0, 45^0, 180^0)$. As time increases there is growth in $\hat{\rho}$ for $\phi = 0^0, 45^0$. But for $\phi = 180^0$, $\hat{\rho}$ is constant. Hence we can conclude that stratification stabilizes the flow velocity but there is growth in the perturbation density.

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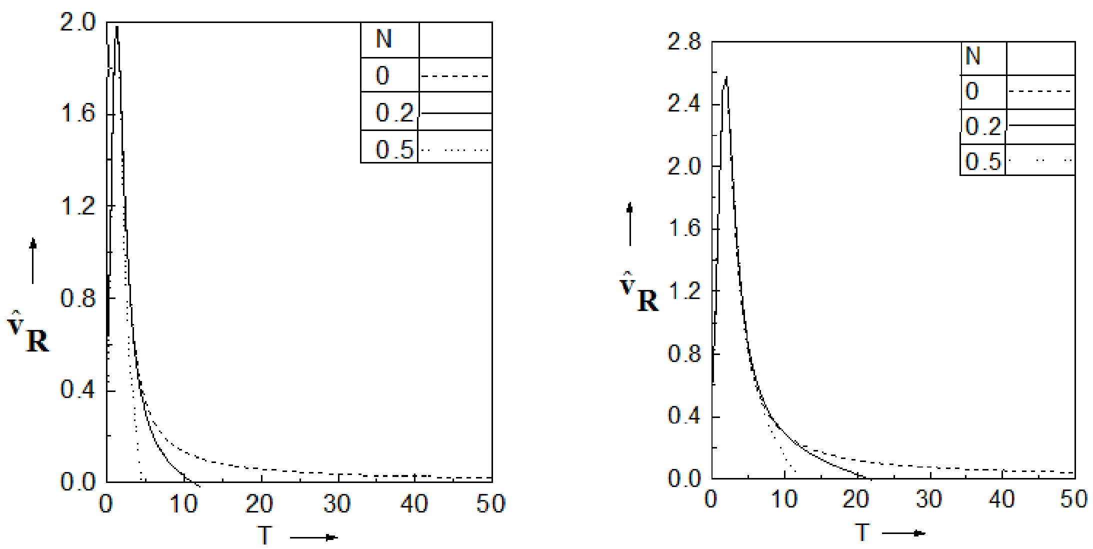

(a) (b)

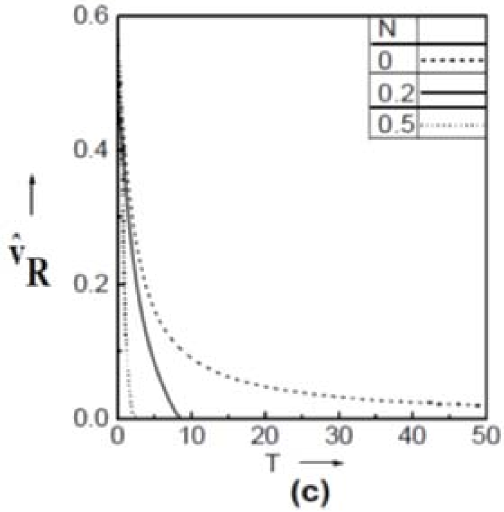

Figure-2: Plots of $\hat{v}R$ versus T for (a) $\varphi = 0^0$ and (b) $\varphi = 45^0$,(c) $\varphi = 180^0$ for different values of N.

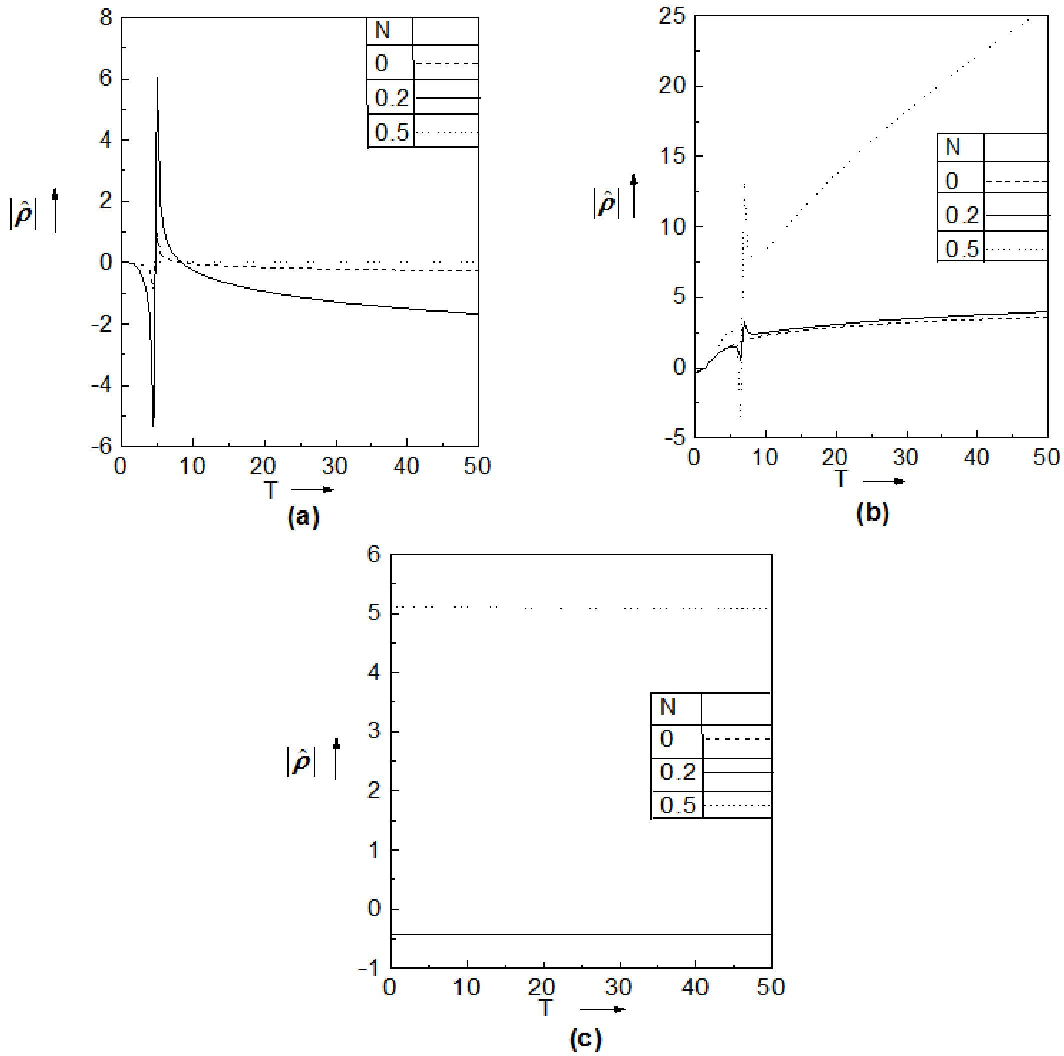

Figure-3: Curves of $|\hat{\rho}|$ versus T for (a) $\varphi = 0^0$, (b) $\varphi = 45^0$,(c) $\varphi = 180^0$ for different values of N.

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