International Journal of Mathematical Archive-9(5), 2018, 203-207
Available online through www.ijma.info ISSN 2229 – 5046

CONSTRUCTING THE CANTOR SET: AN ALTERNATIVE APPROACH

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(Received On: 16-03-18; Revised & Accepted On: 17-04-18)

ABSTRACT

This approach is used for constructing the Cantor set. The numerators of the upper bounds of Cantor sets are used to generate other Cantor sets. It's about the fastest approach and technique in Cantor set construction. The idea stems from the fact that lower bounds could be used to generate the Cantor set. This approach guarantees accuracy, effectiveness, efficiency and speed.

Keywords: *Upper bound, closed set, Cantor set, Perfect set, Compact set, Open set.*

1. INTRODUCTION

The Cantor set has a lot of deep and wonderful properties. Henry John Stephen Smith discovered the Cantor set [5]. Georg Ferdinand Ludwig Philipp Cantor, a German Mathematician, came up with the Cantor Ternary set by trisecting the closed interval [0, 1] and then deleting the middle third [1]. The Cantor set was therefore introduced by Georg Cantor in 1883 [9]. The Cantor set is perfect [8]. There are several approaches for generating or constructing the Cantor set. The most popular approach for obtaining the set is by deleting or removing the middle thirds of a line segment [0, 1] [2].

The Cantor set shows that closed subsets of real line can be more complicated than intuition might at first suggest [4].

The Cantor is often used to construct difficult, counter-intuitive objects in analysis [4]. Some other applications of the set are in set theory, fractal geometry etc. Cantor sets are types or forms of fractal. Other Mathematicians have contributed to this idea in the past, and in the past few years, quite a number of works has been done in this area of mathematics.

Over the years, one popular question which has remained unanswered in relation to the Cantor set has been; is there a formula to generate or construct the Cantor set without breaking existing intervals into two or depending on previously constructed segments. This work attempts to provide an answer to this question which has lasted decades.

2. PRELIMINARIES

Definition 2.1: A set S is perfect if it is closed and every point of S is an accumulation point [4]

Definition 2.2: Let S be a subset of R. If $m \leq s$, for all $s \in S$, then m is a lower bound for s which implies S is bounded below. Any set that has both an upper bound and lower bound is said to be bounded [10]

Definition 2.3: Let S be a subset of R. If there exist a real number m such that $m \geq s$ for all $s \in S$, then m is called an upper bound for S, hence S is bounded above [10].

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Definition 2.4: A set *S* is compact if and only if every open cover contains a finite sub cover [10].

3. METHODOLOGY AND RESULTS

Different approaches have been used over the years since the discovery of the Cantor set. Some are popular whiles others are lesser known. The conditions that has been key in describing which approach is the best includes: the amount of work done in the process, the speed and ease of construction, the margin of errors or accuracy, convenience etc.

3.1The Upper Bound Numerator Generator Approach (UBNG)

This approach is used to construct the Cantor sets. Here, segments can be computed independent of each other. This approach uses the numerators of the upper bound. The Upper Bound Numerator Generator Approach (UBNG) approach is derived in relation to the numerators of the Cantor set without reducing them to their least form. The Lower bound numerator generator approach from which this approach is derived can generate the Cantor set easily [6]. It is thus far one of the few if not the only formula where higher segment sets are constructed without necessarily having to construct the immediate segment before it [6]. This approached compared to other existing approaches uses about one tenth computational time in constructing the Cantor set. The steps and examples for the Upper Bound Numerator Approach are outline below.

Steps: Some key notations that will be used consistently in this approach are defined.

Let C_n represent a Cantor set of segment *n*

Let $C_n = \left[\frac{L_n}{3^n}, \frac{U_n}{3^n}\right]$

Also $\frac{L_n}{3^n}$ and $\frac{U_n}{3^n}$ are the lower bound and upper bound term respectively of the Cantor set.

We call L_n and U_n the Lower bound numerator and upper bound numerator respectively.

The steps for the Upper Bound Numerator approach is outline as follows
Given the interval [0,1] and with $n \geq 2$

1. Find the interval of the Cantor set using $\frac{1}{3^n}$ and the number of segments for the Cantor set using 2^n
2. The first segment $C_1 = \left[0, \frac{1}{3^n}\right]$
3. The second segment of every Cantor set is derived by $C_1 = \left[\frac{2}{3^n}, \frac{3}{3^n}\right]$
4. The last term of every Cantor set is derived by $C_{2^n} = \left[\frac{3^n - 1}{3^n}, 1\right]$
5. There are now $2^n - 3$ segments to be computed. This is done using the formula below to generate the remaining set in odd and even pairs a
   a. $C_{2n-1} = \left[\frac{3U_n - 3}{3^m}, \frac{3U_n - 2}{3^m}\right]$
   b. $C_{2n} = \left[\frac{3U_n - 1}{3^m}, \frac{3U_n}{3^m}\right]$, for $n \geq 2 \ldots 2^n - 1$

Note: Now due to the fact that the construction will also introduce another variable *n*, we dummy the *n* variable associated with the interval 3^n to become 3^m. Here the *m* is the *n* used to generate the interval and the number of segments, whiles the new *n* will be used for the recursive construction of the cantor set

3.1.1 Examples: In the examples that follow, we do the construction of the Cantor set exhausting every step outlined in the previous session.

1. Construct the Cantor set with *n* = 2 using the Upper bound numerator generator approach

Step-1: Find the interval of the Cantor set using $\frac{1}{3^n}$ and the number of segments for the Cantor set using 2^n
For *n* = 2 , the interval is $\frac{1}{3^n} = \frac{1}{3^2} = \frac{1}{9}$
For *n* = 2, the number of segments is $2^n = 2^2 = 4$

Step-2: The first segment $C_1 = \left[0, \frac{1}{3^n}\right]$, $C_1 = \left[0, \frac{1}{3^n}\right] = \left[0, \frac{1}{3^2}\right] = \left[0, \frac{1}{9}\right]$

Step-3: The second segment of every Cantor set is derived by $C_2 = \left[\frac{2}{3^n}, \frac{3}{3^n}\right]$
$C_2 = \left[\frac{2}{3^n}, \frac{3}{3^n}\right] = \left[\frac{2}{3^2}, \frac{3}{3^2}\right] = \left[\frac{2}{9}, \frac{3}{9}\right]$

Step-4: The last term of every Cantor set is derived by $C_{2^n} = \left[\frac{3^n - 1}{3^n}, 1\right]$

$$C_{2^n} = C_4 = \left[\frac{3^n - 1}{3^n}, 1\right] = \left[\frac{3^2 - 1}{3^2}, 1\right] = \left[\frac{8}{9}, 1\right]$$

After Step 4, the Cantor set for *n* = 2 looks like this

$$\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup C_3 \cup \left[\frac{8}{9}, 1\right]$$

We complete the construction process using step 5

Step-5: There are now $2^n - 3$ segments to be computed. This is done using the formula below to generate the remaining Cantor set in odd and even pairs as follows;

a. $C_{2n-1} = \left[\frac{3U_n - 3}{3^m}, \frac{3U_n - 2}{3^m}\right]$

b. $C_{2n} = \left[\frac{3U_n - 1}{3^m}, \frac{3U_n}{3^m}\right]$, for $n \geq 2 \ldots 2^n$

For *n* = 2 and *m* = 2
Also $U_n = U_2 = 3$

$$C_{2n-1} = C_3 = \left[\frac{3U_n - 3}{3^m}, \frac{3U_n - 2}{3^m}\right] = \left[\frac{(3 \times 3) - 3}{9}, \frac{(3 \times 3) - 2}{9}\right] = \left[\frac{6}{9}, \frac{7}{9}\right]$$

The Cantor set for $n = 2$ is a 4-segment with an interval of $\frac{1}{9}$ and is given as

$$\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$

2. Construct the Cantor set with *n* = 2 using the Upper bound numerator generator approach

Step-1: Find the interval of the Cantor set using $\frac{1}{3^n}$ and the number of segments for the Cantor set using 2^n.
For $n = 4$, the interval is $\frac{1}{3^n} = \frac{1}{3^4} = \frac{1}{81}$

For $n = 4$, the number of segments is $2^n = 2^4 = 16$

Step-2: The first segment $C_1 = \left[0, \frac{1}{3^n}\right]$, $C_1 = \left[0, \frac{1}{3^n}\right] = \left[0, \frac{1}{3^4}\right] = \left[0, \frac{1}{81}\right]$

Step-3: The second segment of every Cantor set is derived by $C_2 = \left[\frac{2}{3^n}, \frac{3}{3^n}\right]$

$$C_2 = \left[\frac{2}{3^n}, \frac{3}{3^n}\right] = \left[\frac{2}{3^4}, \frac{3}{3^4}\right] = \left[\frac{2}{81}, \frac{3}{81}\right]$$

Step-4: The last term of every Cantor set is derived by $C_{2^n} = \left[\frac{3^n - 1}{3^n}, 1\right]$

$$C_{2^n} = C_{16} = \left[\frac{3^n - 1}{3^4}, 1\right] = \left[\frac{3^4 - 1}{3^4}, 1\right] = \left[\frac{80}{81}, 1\right]$$

After Step 4, the Cantor set for $n = 4$ looks like this

$$\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup C_3 \cup C_4 \cup C_5 \cup C_3 \cup C_6 \cup C_7 \cup C_8 \cup C_9 \cup C_{10} \cup C_{11} \cup C_{12} \cup C_{13} \cup C_{14} \cup C_{15} \cup \left[\frac{80}{81}, 1\right]$$

We complete the construction process using step 5

Step-5: There are now $2^n - 3$ segments to be computed. This is done using the formula below to generate the remaining Cantor set in odd and even pair

a. $C_{2n-1} = \left[\frac{3U_n - 3}{3^m}, \frac{3U_n - 2}{3^m}\right]$

b. $C_{2n} = \left[\frac{3U_n - 1}{3^m}, \frac{3U_n}{3^m}\right]$, for $n \geq 2 \ldots 2^n$

For $n = 2$ and $m = 4$
$U_n = U_2 = 3$

$$C_{2n-1} = C_3 = \left[\frac{3U_n - 3}{3^m}, \frac{3U_n - 2}{3^m}\right] = \left[\frac{3U_2 - 3}{3^m}, \frac{3U_2 - 2}{3^m}\right] = \left[\frac{(3 \times 3) - 3}{81}, \frac{(3 \times 3) - 2}{81}\right] = \left[\frac{6}{81}, \frac{7}{81}\right]$$

$$C_{2n} = C_4 = \left[\frac{3U_n - 1}{3^m}, \frac{3U_n}{3^m}\right] = \left[\frac{3U_2 - 1}{3^m}, \frac{3U_2}{3^m}\right] = \left[\frac{(3 \times 3) - 1}{81}, \frac{(3 \times 3)}{81}\right] = \left[\frac{8}{81}, \frac{9}{81}\right]$$

For $n = 3$ and $m = 4$
$U_n = U_3 = 7$

$$C_{2n-1} = C_5 = \left[\frac{3U_n - 3}{3^m}, \frac{3U_n - 2}{3^m}\right] = \left[\frac{3U_3 - 3}{3^m}, \frac{3U_3 - 2}{3^m}\right] = \left[\frac{(3\times 7) - 3}{81}, \frac{(3\times 7) - 2}{81}\right] = \left[\frac{18}{81}, \frac{19}{81}\right]$$

$$C_{2n} = C_6 = \left[\frac{3U_n - 1}{3^m}, \frac{3U_n}{3^m}\right] = \left[\frac{3U_3 - 1}{3^m}, \frac{3U_3}{3^m}\right] = \left[\frac{(3\times 7) - 1}{81}, \frac{(3\times 7)}{81}\right] = \left[\frac{20}{81}, \frac{21}{81}\right]$$

For $n = 4$ and $m = 4$
$U_n = U_2 = 9$

$$C_{2n-1} = C_7 = \left[\frac{3U_n - 3}{3^m}, \frac{3U_n - 2}{3^m}\right] = \left[\frac{3U_4 - 3}{3^4}, \frac{3U_4 - 2}{3^4}\right] = \left[\frac{(3\times 9) - 3}{81}, \frac{(3\times 9) - 2}{81}\right] = \left[\frac{23}{81}, \frac{24}{81}\right]$$

$$C_{2n} = C_8 = \left[\frac{3U_n - 1}{3^m}, \frac{3U_n}{3^m}\right] = \left[\frac{3U_3 - 1}{3^m}, \frac{3U_3}{3^m}\right] = \left[\frac{(3\times 9) - 1}{81}, \frac{(3\times 9)}{81}\right] = \left[\frac{26}{81}, \frac{27}{81}\right]$$

For $n = 5$ and $m = 4$
$U_n = U_5 = 19$

$$C_{2n-1} = C_9 = \left[\frac{3U_n - 3}{3^m}, \frac{3U_n - 2}{3^m}\right] = \left[\frac{3U_5 - 3}{3^4}, \frac{3U_5 - 2}{3^4}\right] = \left[\frac{(3\times 19) - 3}{81}, \frac{(3\times 19) - 2}{81}\right] = \left[\frac{54}{81}, \frac{55}{81}\right]$$

$$C_{2n} = C_{10} = \left[\frac{3U_n - 1}{3^m}, \frac{3U_n}{3^m}\right] = \left[\frac{3U_5 - 1}{3^4}, \frac{3U_5}{3^4}\right] = \left[\frac{(3\times 19) - 1}{81}, \frac{(3\times 19)}{81}\right] = \left[\frac{56}{81}, \frac{57}{81}\right]$$

For $n = 6$ and $m = 4$
$U_n = U_6 = 21$

$$C_{2n-1} = C_{11} = \left[\frac{3U_n - 3}{3^m}, \frac{3U_n - 2}{3^m}\right] = \left[\frac{3U_6 - 3}{3^4}, \frac{3U_6 - 2}{3^4}\right] = \left[\frac{(3\times 21) - 3}{81}, \frac{(3\times 21) - 2}{81}\right] = \left[\frac{60}{81}, \frac{61}{81}\right]$$

$$C_{2n} = C_{12} = \left[\frac{3U_n - 1}{3^m}, \frac{3U_n}{3^m}\right] = \left[\frac{3U_6 - 1}{3^4}, \frac{3U_6}{3^4}\right] = \left[\frac{(3\times 21) - 1}{81}, \frac{(3\times 21)}{81}\right] = \left[\frac{62}{81}, \frac{63}{81}\right]$$

For $n = 7$ and $m = 4$
$U_n = U_7 = 25$

$$C_{2n-1} = C_3 = \left[\frac{3U_n - 3}{3^m}, \frac{3U_n - 2}{3^m}\right] = \left[\frac{3U_7 - 3}{3^4}, \frac{3U_7 - 2}{3^4}\right] = \left[\frac{(3\times 25) - 3}{81}, \frac{(3\times 25) - 2}{81}\right] = \left[\frac{72}{81}, \frac{73}{81}\right]$$

$$C_{2n} = \left[\frac{3U_n - 1}{3^m}, \frac{3U_n}{3^m}\right] = \left[\frac{3U_7 - 1}{3^4}, \frac{3U_7}{3^4}\right] = \left[\frac{(3\times 25) - 1}{81}, \frac{(3\times 25)}{81}\right] = \left[\frac{74}{81}, \frac{75}{81}\right]$$

For $n = 8$ and $m = 4$
$U_n = U_2 = 27$

$$C_{2n-1} = C_{15} = \left[\frac{3U_n - 3}{3^m}, \frac{3U_n - 2}{3^m}\right] = \left[\frac{3U_8 - 3}{3^4}, \frac{3U_8 - 2}{3^4}\right] = \left[\frac{(3\times 27) - 3}{81}, \frac{(3\times 27) - 2}{81}\right] = \left[\frac{78}{81}, \frac{79}{81}\right]$$

$$C_{2n} = C_{16} = \left[\frac{3U_n - 1}{3^m}, \frac{3U_n}{3^m}\right] = \left[\frac{3U_8 - 1}{3^4}, \frac{3U_8}{3^4}\right] = \left[\frac{(3\times 27) - 1}{81}, \frac{(3\times 27)}{81}\right] = \left[\frac{80}{81}, 1\right]$$

The Cantor set for $n = 4$ is a 16-segment with an interval of $\frac{1}{81}$ and is given as

$$\left[0, \frac{1}{81}\right] \cup \left[\frac{2}{81}, \frac{3}{81}\right] \cup \left[\frac{6}{81}, \frac{7}{81}\right] \cup \left[\frac{8}{81}, \frac{9}{81}\right] \cup \left[\frac{18}{81}, \frac{19}{81}\right] \cup \left[\frac{20}{81}, \frac{21}{81}\right] \cup \left[\frac{24}{81}, \frac{25}{81}\right] \cup \left[\frac{26}{81}, \frac{27}{81}\right] \cup \left[\frac{54}{81}, \frac{55}{81}\right] \cup \left[\frac{56}{81}, \frac{57}{81}\right] \cup$$
$$\left[\frac{60}{81}, \frac{61}{81}\right] \cup \left[\frac{62}{81}, \frac{63}{81}\right] \cup \left[\frac{72}{81}, \frac{73}{81}\right] \cup \left[\frac{72}{81}, \frac{73}{81}\right] \cup \left[\frac{74}{81}, \frac{75}{81}\right] \cup \left[\frac{78}{81}, \frac{79}{81}\right] \cup \left[\frac{80}{81}, 1\right]$$

Which is simplified

$$\left[0, \frac{1}{81}\right] \cup \left[\frac{2}{81}, \frac{3}{81}\right] \cup \left[\frac{6}{81}, \frac{7}{81}\right] \cup \left[\frac{8}{81}, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{19}{81}\right] \cup \left[\frac{20}{81}, \frac{7}{27}\right] \cup$$
$$\left[\frac{8}{27}, \frac{25}{81}\right] \cup \left[\frac{26}{81}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{55}{81}\right] \cup \left[\frac{56}{81}, \frac{19}{27}\right] \cup \left[\frac{20}{27}, \frac{61}{81}\right] \cup \left[\frac{62}{81}, \frac{7}{9}\right] \cup$$
$$\left[\frac{72}{81}, \frac{73}{81}\right] \cup \left[\frac{74}{81}, \frac{75}{81}\right] \cup \left[\frac{78}{81}, \frac{79}{81}\right] \cup \left[\frac{80}{81}, 1\right]$$

CONCLUSION

Many different approaches have been proposed for the construction of the Cantor set. Higher segment Cantors can be constructed easily and within a short time since there is no need to construct preceding segments before the required one. In the Upper bound numerator approaches, the level of work done is minimal and not as complicated compare to the other approaches. The approach or method is less complicated, accurate and concise. It's one of the fastest approach in construction of the Cantor set. The approach is similar to the Lower Bound Numerator Generator approach.

COMPETING INTEREST

The authors declare that there is no competing interest regarding the publication of this paper

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