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# **GENERALIZED PAIRWISE STAR SEPARATION AXIOM IN BITOPOLOGICAL SPACES**

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## ABSTRACT

In this paper, we introduce the notions of generalized pairwise star separation axioms on bitopological spaces and study some of their properties. The properties of the space  $(X, \tau_1, \tau_2)$  are studied through the study of the space  $(X, \tau_{12})$ which is a supra topology associated to the bitopological space  $(X, \tau_1, \tau_2)$ . The importance of this approach is that we study properties of the bitopological space  $(X, \tau_1, \tau_2)$  via one family  $\tau_{12}$  such that  $(X, \tau_{12})$  is a supra topological space associated to  $(X, \tau_1, \tau_2)$ . Also, we are dealing with one family instead of two families  $\tau_1$  and  $\tau_2$ . Finally this method enable us to study on X more that two topologies. The relation between these approaches has studied.

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**Keywords**: Bitopological space, Supra topology, generalized pairwise star  $T_0$  space, generalized pairwise star  $T_1$  space, generalized pairwise star  $T_2$  space, generalized pairwise star  $R_0$  space generalized pairwise star  $R_1$  space generalized pairwise star regular space and generalized pairwise star normal space, .

# **1. INTRODUCTION**

In 1963, Kelly was first introduced the concept of bitopological space, where is a non-empty set and  $\tau_1, \tau_2$  are topologies on X as method of generalizes topological space  $(X, \tau)$  [1]. Every bitopological space  $(X, \tau_1, \tau_2)$  can be regarded as a topological space  $(X, \tau)$  if  $\tau_1 = \tau_2 = \tau$ . Furthermore, he extended some of the standard results of separation axioms and mappings in a topological space to a bitopological space. In 1983 Mashhor *et al.* [2], [3] introduced supra topological spaces by dropping only the intersection condition. Kandil,[4] generated a supra topological space  $(X, \tau_{12})$  from the bitopological space  $(X, \tau_1, \tau_2)$  and they studied some properties of the space  $(X, \tau_1, \tau_2)$  via properties of the associated space  $(X, \tau_{12})$ . Thereafter, a large number of papers have been written in separation axiom on bitopological space. It has been studied in different versions adopted in the study of the concept of Fukutake by using open and closed collections in the two spaces  $\tau_1, \tau_2$  or using the new space  $(X, \tau_{12})$  resulting from them. [5, 6, 7, 8, 9].

In this paper we introduce the notion of generalized pairwise star  $T_0$  space, generalized pairwise star  $T_1$  space, generalized pairwise star  $T_2$  space, generalized pairwise star  $R_0$  space, generalized pairwise star  $R_1$ space generalized pairwise star regular space and generalized pairwise star normal space by using the concept generalized pairwise open sets and generalized pairwise closed sets in  $(X, \tau_{12})$  and we study some of relation between this spaces.

# 2.PRELIMINARIES

This section contains the basic concept and the proprieties of generalized closed sets, generalized open set and pairwise separation axiom in bitopological spaces.

Corresponding Author: E. A. Shalaby<sup>3\*</sup> <sup>3</sup>Mathematics Department, Faculty of Education, Ain Shams University, Cairo, Egypt. **Definition 2.1:** [1] A bitopological spaces (bts, for short) is a triple  $(X, \tau_1, \tau_2)$  where  $\tau_1$  and  $\tau_2$  are arbitrary topologies on X.

**Definition 2.2:** [10] Let  $(X, \tau_1, \tau_2)$  be a bitopological space. Then,  $A \subseteq X$  is said to be pairwise open (P-open, for short) if  $A = U_1 \cup U_2$ ,  $U_i \in \tau_i$ , (i = 1, 2). A set A is a P-closed if its complement  $A^c$  is P-open.

Note that the notion of P-open sets as well as P-closed sets has studied in [11, 4] under the name of  $P^*$ -open and  $P^*$ -closed.

**Definition 2.3:** [3] A family  $\eta \subseteq P(X)$  is said to be a supra topology on X, if  $\eta$  contains X,  $\phi$  and closed under arbitrary union. The element of  $\eta$  are supra open sets and their complements are said to be supra closed sets.

**Proposition 2.1:** [11] Let  $(X, \tau_1, \tau_2)$  be a bts. The family of all P-open subsets of X, denoted by  $\tau_{12} = \{U_1 \cup U_2 : U_i \in \tau_i, i = 1, 2\}$  is a supratopology on X and  $(X, \tau_{12})$  is the supra topological space associated to the bts  $(X, \tau_1, \tau_2)$ .

**Proposition 2.2:** [11] Let  $(X, \tau_1, \tau_2)$  be a bts. Then, the operator  $cl_{12}: P(X) \rightarrow P(X)$  defined by  $cl_{12}(A) = \overline{A}^1 \cap \overline{A}^2$ , is a supra closure operator such that,  $\tau_{12} = \{A \subseteq X: cl_{12}(A^c) = A^c\}$  where  $\overline{A}^i$  is the closure of A with respect to  $\tau_i$  and i = 1, 2.

**Proposition 2.3:** [11] Let  $(X, \tau_1, \tau_2)$  be a bts. Then the operator,  $int_{12}: P(X) \to P(X)$  defined by,  $int_{12}(A) = A^{o1} \cup A^{o2}$  is a supra interior operator such that  $\tau_{12} = \{A \subseteq X: int_{12}(A) = A\}$ , where  $A^{oi}$ , (i = 1, 2) is the  $\tau_i$ - interior with respect to  $\tau_i$ .

**Definition 2.4:** [12] *Let*  $(X, \tau_1, \tau_2)$  *be a bts,*  $A \subseteq X$  *and*  $(A, \tau_1|_A, \tau_2|_A)$  *be a bitopological subspace of*  $(X, \tau_1, \tau_2)$ . *Then,*  $\tau_{12}(A) = \tau_1|_A \cup \tau_2|_A = \{U_1 \cup U_2 : U_1 \in \tau_1|_A, U_2 \in \tau_2|_A\}$  such that  $\tau_i|_A = \{A \cap U_i : U_i \in \tau_i, i = 1, 2\}$ .

**Theorem 2.1:** [12] *Let*  $(X, \tau_1, \tau_2)$  *be a bts and*  $A \subseteq X$ . *Then,*  $\tau_{12}|_A = \tau_{12}(A)$ , *where*  $\tau_{12}|_A = \{A \cap U : U \in \tau_{12}\}$ .

**Definition 2.5:** [12] Let  $(X, \tau_1, \tau_2)$  be a bts and  $(X, \tau_{12})$  be its associated supra topological space. Then,  $A \subseteq X$  is called a generalized pairwise closed set (gp-closed, for short) if,  $cl_{12}(A) \subseteq 0$  whenever  $A \subseteq 0$ ,  $\Box$  is a P-open. The set of all generalized pairwise closed sets denoted by  $GPC(X, \tau_{12})$ .

## Remark 2.1: [12]

- 1. If A is a P-closed set, then A is a gp-closed set.
- 2. If A and B are gp-closed sets, then  $A \cup B$  and  $A \cap B$  are not necessary gp-closed sets.

**Theorem 2.2:** [12] Let  $(X, \tau_1, \tau_2)$  be a bts and  $A \subseteq X$ . If A is a gp closed set and  $A \subseteq B \subseteq cl_{12}(A)$ . Then, B is a gp-closed set.

**Theorem 2.3:** [12] Let  $(X, \tau_1, \tau_2)$  be a bts,  $\tau_{12}$  be supra topology on X induced by  $\tau_1, \tau_2$ . Then,  $\tau_{12} = \tau_{12}^c$  if and only if every subset of X is a gp- closed set.

**Theorem 2.4:** [12] Let  $(X, \tau_1, \tau_2)$  be a bts. Suppose that  $B \subseteq A \subseteq X$ , B is a gp-closed set relative to A and A is a gp-closed set of X. Then, B is a gp-closed set relative to X.

**Corollary 2.1:** [12] If A is a gp-closed set and F is P-closed set, then  $A \cap F$  is a gp-closed set.

**Theorem 2.5:** [12] Let  $(X, \tau_1, \tau_2)$  be a bts and  $A \subseteq Y \subseteq X$ . Suppose that A is gp-closed in X. Then, A is gp-closed relative to Y.

**Definition 2.6:** [10] Let  $(X, \tau_1, \tau_2)$  be a bts and  $A, B \subseteq X$ . Then A and B are  $P^*$  – separated in X if  $A \cap cl_{12}(B) = \phi$  and  $cl_{12}(A) \cap B = \phi$ .

Not that if A and B are  $P^*$  –separated and  $C \subseteq A, D \subseteq B$ , then CandD are  $P^*$  – separated.

**Theorem 2.6:** [12] Let  $(X, \tau_1, \tau_2)$  be a bts. If A and B are  $P^*$ -separated and gp-open sets, then  $A \cup B$  is a gp-open set.

**Definition 2.7:** [12] A set A is called a generalized pairwise open (for short, gp-open) set if and only if  $A^c$  is a gp-closed set. The set of all generalized closed set is denoted by  $GPO(X, \tau_{12})$ .

**Theorem 2.7:** [12] A space  $(X, \tau_1, \tau_2)$  is a  $P^*$ -symmetric if and only if  $\{x\}$  is a gp-closed  $\forall x \in X$ .

**Theorem 2.8:** [12] If A is a gp-closed set in X and  $f: X \to Y$  is  $P^*$ -cts and is  $P^*$ -closed function, then f(A) is a gp-closed set.

**Theorem 2.9:** [12] If  $f:(X,\tau_1,\tau_2) \to (Y,v_1,v_2)$  is a  $P^*$ -cts and  $P^*$ -closed function, B is a gp-closed(or gp-open) subset of Y, then  $f^{-1}(B)$  is gp-closed (or gp-open) set in X.

**Definition 2.8:** [10] Let  $(X, \tau_1, \tau_2)$  be a bitopological space. X is called a  $P^*T_0$ -space (resp.  $P^*T_1$ -space,  $P^*T_2$ -space,  $P^*T_3$ -space,  $P^*T_4$ -space,  $P^*R_0$ -space,  $P^*R_1$ -space) if X satisfies the following  $P^*T_0$ -separation (resp.  $P^*T_1$ -separation,  $P^*T_2$ -separation,  $P^*T_2$ -separation,  $P^*T_3$ -separation,  $P^*T_4$ -separation,  $P^*R_0$ -separation,  $P^*R_1$ -separation,  $P^$ 

- 1.  $P^*T_0$  -separation: If  $x, y \in X$  and  $x \neq y$ , then  $\exists U \in \tau_{12}$  such that either  $U \cap \{x, y\} = \{x\} \text{or} U \cap \{x, y\} = \{y\}$ .
- 2.  $P^*T_1$ -separation: If  $x, y \in X$  and  $x \neq y$ , then there are  $U_x, U_y \in \tau_{12}$  such that  $U_x \cap \{x, y\} = \{x\}$  and  $U_y \cap \{x, y\} = \{y\}$ .
- 3.  $P^*T_2$ -separation: If  $x, y \in X$  and  $x \neq y$ , then  $\exists U_x \in \tau_{12}$  and  $U_y \in \tau_{12}$  such that  $U_x \cap U_y = \phi$ .
- 4.  $P^*$  regular: If  $x \notin F$  and F is P-closed, then  $\exists U_x \in \tau_{12}$  and  $U_F \in \tau_{12}$  such that  $F \subseteq U_F$  and  $U_x \cap U_F = \phi$ . ( $X, \tau_1, \tau_2$ ) is  $P^*T_3$ -separation if it is  $P^*T_1$  and  $P^*$  regular.
- 5.  $P^*$  normal: If  $F_1$  and  $F_2$  are P-closed and  $F_1 \cap F_2 = \phi$ , then  $\exists U_1, U_2 \in \tau_{12}$  such that  $F_1 \subseteq U_1$  and  $F_2 \subseteq U_2$  such that  $U_1 \cap U_2 = \phi$ .  $(X, \tau_1, \tau_2)$  is  $P^*T_4$ -separation if it is  $P^*T_1$  and  $P^*$ normal.
- 6.  $P^*R_0$ -separation: If  $x \in U \in \tau_{12}$ , then  $cl_{12}(\{x\}) \subseteq U$ .
- 7.  $P^*R_1$ -separation: If  $x \neq y$ , and  $cl_{12}\{x\} \neq cl_{12}\{y\}$  then  $\exists U_1, U_2 \in \tau_{12}$  such that,  $cl_{12}(\{x\}) \subseteq U_1$  and  $cl_{12}(\{y\}) \subseteq U_2$ .

#### 3. GENERALIZED PAIRWISE STAR SEPARATION AXIOMS

In this section we defined the concepts of generalized pairwise star  $T_0$  space, generalized pairwise star  $T_1$  space, generalized pairwise star  $T_2$  space, generalized pairwise star  $T_3$ , generalized pairwise star  $T_4$ , generalized pairwise star  $R_0$  space and generalized pairwise star  $R_1$  space, we also study some of their properties.

**Definition 3.1:** Let  $(X, \tau_1, \tau_2)$  be a bts and  $(X, \tau_{12})$  be its associated supra topological space. Then X is called a generalized pairwise star  $T_0$   $(gp^*T_0, for short)$  space if  $\forall x, y \in X, x \neq y \exists gp$  -open set A such that  $x \in A, y \notin A$ . In another word for every two different elements in X, there exist gp- open set containing one of them but not the other.

**Theorem 3.1:** Let  $(X, \tau_1, \tau_2)$  be a bitopological space,  $(X, \tau_{12})$  its associated supra topological spaces. If  $(X, \tau_{12})$  is a  $gp^*T_0$  space and A is P- closed set in  $(X, \tau_{12})$ , then  $(A, \tau_{12}|_A)$  is a  $gp^*T_0$  space.

**Proof:** Let  $(X, \tau_{12})$  be a  $gp^*T_0$  space and A be a P- closed set in  $(X, \tau_{12})$ . Suppose that  $x, y \in A, x \neq y$ . Thus,  $x, y \in X$  but,  $(X, \tau_{12})$  is a  $gp^*T_0$  this implies that,  $\exists G \in GPO(X, \tau_{12})$  such that  $x \in G, y \notin G$ . Therefore,  $G^c \in GPC(X, \tau_{12})$ . By using corollary2.1 we get,  $G^c \cap A \in GPC(X, \tau_{12})$ . Also we have,  $G^c \cap A \subseteq A \subseteq X$ . Then,  $A \cap G^c \in GPC(A, \tau_{12}|_A)$  by Theorem2.5. This implies that,  $A \cap G \in GPO(A, \tau_{12}|_A)$  and  $x \in A \cap G, y \notin A \cap G$ . Hence, $(A, \tau_{12}|_A)$  is a  $gp^*T_0$  space.

**Theorem 3.2:** Let  $(X, \tau_1, \tau_2)$  and  $(Y, \gamma_1, \gamma_2)$  be a bitopological spaces,  $(X, \tau_{12})$  and  $(Y, \gamma_{12})$  are associated supra topological spaces. Suppose that  $f: X \to Y$  is an injective,  $P^*$  continuous function and  $P^*$  –closed, if  $(Y, \gamma_1, \gamma_2)$  is a  $gp^*T_0$  space, then  $(X, \tau_1, \tau_2)$  is  $gp^*T_0$  space.

**Proof:** Let  $(X, \tau_1, \tau_2)$  and  $(Y, \gamma_1, \gamma_2)$  be a bitopological spaces,  $(X, \tau_{12})$  and  $(Y, \gamma_{12})$  are associated supra topological spaces. Suppose that  $f: X \to Y$  is a injective,  $P^*$  continuous function and  $P^*$  -closed,  $(Y, \gamma_{12})$  is  $gp^*T_0$  space.Let  $x_1, x_2 \in X$  and  $x_1 \neq x_2$  but we have f is injective, this implies that  $f(x_1) \neq f(x_2)$ . But,  $(Y, \gamma_{12})$  is  $gp^*T_0$ . This implies that,  $\exists gp$  -open set V such that,  $f(x_1) \in V, f(x_2) \notin V$ . Therefore,  $x_1 = f^{-1}f(x_1) \in f^{-1}(V), x_2 \notin f^{-1}(V)$ . By using Theorem2.9 we get,  $f^{-1}(V)$  is gp -open sets. Hence,  $(X, \tau_1, \tau_2)$  is  $gp^*T_0$  space.

**Remark 3.1:** Every  $P^*T_0$  space is  $gp^*T_0$  space but the converse is not true. Example 3.1 explain that.

**Example 3.1:** Let  $X = \{a, b, c, d\}$ ,  $\tau_1 = \{X, \phi, \{a, b\}\}$ ,  $\tau_2 = \{X, \phi, \{c, d\}\}$ . Then,  $\tau_{12} = \{X, \phi, \{a, b\}\}, \{c, d\}\}$ . Then  $(X, \tau_{12})$  is  $gp^*T_0$  space but not  $P^*T_0$ . Since, the set of all generalized open set  $(GPO(X, \tau_{12})) = P(X)$ .

**Corollary 3.1:** If  $(X, \tau_1)$  or  $(X, \tau_2)$  is  $T_0$  space, then  $(X, \tau_{12})$  is  $gp^*T_0$ .

**Proof:** The prove is trivially, since  $\tau_1 \subseteq \tau_{12}$  and  $\tau_2 \subseteq \tau_{12}$ .

**Definition 3.2:** Let  $(X, \tau_1, \tau_2)$  be a bts and  $(X, \tau_{12})$  be its associated supra topological space. Then X is called a generalized pairwise star  $T_1$  ( $gp^*T_1$ , for short) space if  $\forall x, y \in X$ ,  $x \neq y$ , then  $\exists gp$  -open sets A and B such that  $x \in A, y \notin A, y \in B$  and  $x \notin B$ . In other word, for every two distant elements in X, there exist two gp- open sets containing one of them and not either.

**Theorem 3.3:** Let  $(X, \tau_1, \tau_2)$  be a bitopological space,  $(X, \tau_{12})$  its associated supra topological space. If  $(X, \tau_{12})$  is a  $gp^*T_1$  space and A is a P-closed set in  $(X, \tau_{12})$ , then  $(A, \tau_{12}|_A)$  is a  $gp^*T_1$  space.

**Proof:** Let  $(X, \tau_{12})$  be a  $gp^*T_1$  space and A be a P- closed set in  $(X, \tau_{12})$ . Suppose that  $x, y \in A, x \neq y$ . Thus,  $x, y \in X$  but,  $(X, \tau_{12})$  is a  $gp^*T_1$ , this implies that,  $\exists G, H \in GPO(X, \tau_{12})$  such that  $x \in G, y \notin G$  and  $y \in H, x \notin H$ . Therefore,  $G^c, H^c \in GPC(X, \tau_{12})$ . By using corollary2.1 we get,  $G^c \cap A, H^c \cap A \in GPC(X, \tau_{12})$ . Also we have,  $G^c \cap A \subseteq A \subseteq X$  and  $H^c \cap A \subseteq A \subseteq X$ . Then,  $A \cap G^c$ ,  $H^c \cap A \in GPC(A, \tau_{12}|_A)$  by Theorem2.5. This implies that,  $A \cap G, A \cap H \in GPO(A, \tau_{12}|_A)$  so,  $x \in A \cap G$ ,  $y \notin A \cap G$  and  $y \in A \cap H$ ,  $x \notin A \cap H$ . Hence,  $(A, \tau_{12}|_A)$  is a  $gp^*T_1$  space.

**Theorem 3.4:** Let  $(X, \tau_1, \tau_2)$  and  $(Y, \gamma_1, \gamma_2)$  be bitopological spaces,  $(X, \tau_{12})$  and  $(Y, \gamma_{12})$  are associated supra topological spaces. Suppose that  $f: X \to Y$  is an injective,  $P^*$  continuous function and  $P^*$ -closed. If  $(Y, \gamma_1, \gamma_2)$  is  $gp^*T_1$  space, then  $(X, \tau_1, \tau_2)$  is  $gp^*T_1$  space.

**Proof:** Let  $(X, \tau_1, \tau_2)$  and  $(Y, \gamma_1, \gamma_2)$  be bitopological spaces,  $(X, \tau_{12})$  and  $(Y, \gamma_{12})$  are associated supra topological spaces. Suppose that  $f: X \to Y$  is an injective,  $P^*$  continuous function and  $P^*$  -closed,  $(Y, \gamma_{12})$  is  $gp^*T_1$  space. Let  $x_1, x_2 \in X$  and  $x_1 \neq x_2$  but we have f is injective, this implies that  $f(x_1) \neq f(x_2)$ . But,  $(Y, \gamma_{12})$  is  $gp^*T_1$ . This implies that  $\exists gp$  - open sets  $V_1, V_2$  such that,  $f(x_1) \in V_1, f(x_2) \notin V_1$  and  $f(x_2) \in V_2, f(x_1) \notin V_2$ . Therefore,  $x_1 = f^{-1}f(x_1) \in f^{-1}(V_1), x_2 \notin V_1$  and  $x_2 = f^{-1}f(x_2) \in f^{-1}(V_2), x_1 \notin V_2$ . By using Theorem2.9 we get,  $f^{-1}(V_1)$  and  $f^{-1}(V_2)$  are gp - open sets. Hence,  $(X, \tau_1, \tau_2)$  is  $gp^*T_1$  space.

#### Remark 3.2:

- 1. Every  $P^*T_1$  space is  $gp^*T_1$  space but the converse is not true. See Example3.1.
- 2. Every  $gp^*T_1$  space is  $gp^*T_0$  space but the converse is not true. Example 3.2 explain that.

**Example 3.2:** Let  $X = \{1,2\}, \tau_1 = \{X,\phi\}, \tau_2 = \{X,\phi,\{1\}\}$ . Then,  $\tau_{12} = \{X,\phi,\{1\}\}$ . Then  $(X,\tau_{12})$  is  $gp^*T_0$  space but not  $gp^*T_1$ . Since, the set of all generalized pairwise open set  $(GPO(X,\tau_{12})) = \{X,\phi,\{1\}\}$ .

**Corollary 3.2:** If  $(X, \tau_1)$  or  $(X, \tau_2)$  is  $T_1$  space, then  $(X, \tau_{12})$  is  $gp^*T_1$ .

**Proof:** The prove is trivially, since  $\tau_1, \tau_2 \subseteq \tau_{12}$ .

**Theorem 3.5:** Every singleton set is gp -closed in  $gp^*T_1$  space.

**Proof:** Let  $(X, \tau_1, \tau_2)$  be a  $gp^*T_1$  space. To prove that  $\{x\}$  is gp -closed set. Let  $\{x\} \subseteq O$ , O is P-open set and we must prove that  $cl_{12}\{x\} \subseteq O$ . Let  $y \notin O$ , then  $y \neq x$  and therefore  $\exists gp$  -open set U such that,  $y \in U$  and  $x \notin U$ . Now,  $x \in U^c$  and  $U^c$  is a gp -closed, then  $cl_{12}(U^c) \subseteq G$  whenever,  $U^c \subseteq G$ , G is P-open. it follows that  $cl_{12}\{x\} \subseteq G$ , whenever  $\{x\} \subseteq G$ , G is P-open. Hence,  $\{x\}$  is gp -closed.

**Definition 3.3:** Let  $(X, \tau_1, \tau_2)$  be a bts and  $(X, \tau_{12})$  be its associated supra topological space. Then X is called a generalized pairwise star  $T_2$   $(gp^*T_2, for short)$ space if  $\forall x, y \in X$  and  $x \neq y$ ,  $\exists$  disjoint gp -open setsA and B such that  $x \in A$  and  $y \in B$ . In other word for every two distinct elements in X, there exist two disjoint gp - open sets containing one of them and not either.

**Theorem 3.6:** Let  $(X, \tau_1, \tau_2)$  be a bitopological space,  $(X, \tau_{12})$  its associated supra topological spaces. If  $(X, \tau_{12})$  is a  $gp^*T_2$  space and A is a P- closed set in  $(X, \tau_{12})$ , then  $(A, \tau_{12}|_A)$  is a  $gp^*T_2$  space.

**Proof:** Let  $(X, \tau_{12})$  be a  $gp^*T_2$  space and A be a P- closed set in  $(X, \tau_{12})$ . Suppose that  $x, y \in A, x \neq y$ . Thus,  $x, y \in X$  but,  $(X, \tau_{12})$  is a  $gp^*T_2$ , this implies that  $\exists G, H \in GPO(X, \tau_{12})$  such that  $x \in G$ ,  $y \in H$  and  $G \cap H = \phi$ . Therefore,  $G^c, H^c \in GPC(X, \tau_{12})$ . By using corollary2.1 we get,  $G^c \cap A, H^c \cap A \in GPC(X, \tau_{12})$ . Also we have,  $G^c \cap A \subseteq A \subseteq X$  and  $H^c \cap A \subseteq A \subseteq X$ . Then,  $A \cap G^c, H^c \cap A \in GPC(A, \tau_{12}|_A)$  by Theorem2.5. This implies that,  $A \cap G, A \cap H \in GPO(A, \tau_{12}|_A)$ ,  $x \in A \cap G$  and  $y \in A \cap H$ . Also,  $(A \cap G) \cap (A \cap H = \phi)$ . Hence,  $(A, \tau_{12}|_A)$  is a  $gp^*T_2$  space.

**Theorem 3.7:** Let  $(X, \tau_1, \tau_2)$  and  $(Y, \gamma_1, \gamma_2)$  be a bitopological spaces,  $(X, \tau_{12})$  and  $(Y, \gamma_{12})$  are associated supra topological spaces. Suppose that  $f: X \to Y$  is an injective,  $P^*$  continuous function and  $P^*$ -closed. If  $(Y, \gamma_1, \gamma_2)$  is  $gp^*T_2$  space, then  $(X, \tau_1, \tau_2)$  is  $gp^*T_2$  space.

**Proof:** Let  $(X, \tau_1, \tau_2)$  and  $(Y, \gamma_1, \gamma_2)$  be a bitopological spaces,  $(X, \tau_{12})$  and  $(Y, \gamma_{12})$  are associated supra topological spaces. Suppose that  $f: X \to Y$  is an injective,  $P^*$  continuous function and  $P^*$  -closed,  $(Y, \gamma_{12})$  is  $gp - T_2$  space. Let  $x_1, x_2 \in X$  and  $x_1 \neq x_2$ , but we have f is injective, this implies that  $f(x_1) \neq f(x_2)$ . But,  $(Y, \gamma_{12})$  is  $gp - T_2$ . This implies that,  $\exists$  disjoint gp -open sets  $V_1, V_2$  such that,  $f(x_1) \in V_1$  and  $f(x_2) \in V_2$ . Therefore,  $x_1 = f^{-1}f(x_1) \in f^{-1}(V_1)$  and  $x_2 = f^{-1}f(x_2) \in f^{-1}(V_2)$ . By using Theorem2.9 we get,  $f^{-1}(V_1)$  and  $f^{-1}(V_2)$  are gp -open sets, also  $f^{-1}(V_1, f^{-1}(V_2)$  are disjoint sets. Hence,  $(X, \tau_1, \tau_2)$  is  $gp^*T_2$  space.

#### Remark 3.3:

- 1. Every  $P^*T_2$  space is  $gp^*T_2$  space but the converse is not true. See Example3.1.
- 2. Every  $gp^*T_2$  space is  $gp^*T_1$  space but the converse is not true. Example 3.3 explain that.

**Example 3.3:** Let  $X = \{1,2,3,4\}, \tau_1 = \{X, \phi, \{1,2,3\}, \{2,3,4\}, \{2,3\}\}, \tau_2 = \{X, \phi, \{1,3,4\}, \{1,2,4\}, \{1,4\}\}$ Then,  $\tau_{12} = \{X, \phi, \{1,2,3\}\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,4\}, \{2,3\}$ . Then  $(X, \tau_1, \tau_2)$  is  $gp^*T_1$  space. Since,  $GPO(X, \tau_{12}) = \tau_{12}$  and

 $1 \neq 2$  we have  $\{1,3,4\}$  and  $\{2,3,4\}$  are gp – open sets.

- $1 \neq 3$  we have  $\{1,4\}$  and  $\{3,4\}$  are gp open sets.
- $1 \neq 4$  we have  $\{1,2,3\}$  and  $\{2,3,4\}$  are gp open sets.
- $2 \neq 3$  we have  $\{1,2,4\}$  and  $\{1,3,4\}$  are gp open sets.
- $2 \neq 4$  we have  $\{2,3\}$  and  $\{1,4\}$  are gp open sets.
- $3 \neq 4$  we have {2,3}and {1,4} are gp open sets.

But  $(X, \tau_1, \tau_2)$  is not  $gp^*T_2$  since,  $1 \neq 4$  and all gp -open containing 1 intersect all gp -open sets containing 4.

**Remark 3.4:** For a bts  $(X, \tau_1, \tau_2)$  we have the following chart.

$$\begin{array}{cccc} P^*T_2 & \to P^*T_1 & \to P^*T_0 \\ \downarrow & \downarrow & \downarrow \\ qp^*T_2 & \to gp^*T_1 & \to gp^*T_0 \end{array}$$

**Definition 3.4:** Let  $(X, \tau_1, \tau_2)$  be a bts and  $(X, \tau_{12})$  its associated supra topological space. Then X is called a generalized pairwise star regular ( $gp^*$  regular, for short) space if for each gp –closed set F and  $x \notin F$ , there exist disjoint gp-open sets G and H such that,  $x \in H$  and  $F \subseteq G$ .

**Theorem 3.8:** Let  $(X, \tau_1, \tau_2)$  be a bitopological space,  $(X, \tau_{12})$  its associated supra topological spaces. If  $(X, \tau_{12})$  is a  $gp^*$  regular space and A is P- closed set in  $(X, \tau_{12})$ , then  $(A, \tau_{12}|_A)$  is a  $gp^*$  regular space.

**Proof:** Let  $(X, \tau_{12})$  be a  $gp^*$  regular space and A be a P- closed set in  $(X, \tau_{12})$ . Suppose that  $x \in A, x \notin F$  and  $F \in GPC(A, \tau_{12}|_A)$ . Thus,  $x \in X$  and  $F \in GPC(X, \tau_{12})$ . But,  $(X, \tau_{12})$  is a  $gp^*$  regular space this implies that,  $\exists G, H \in GPO(X, \tau_{12})$  such that  $x \in G$ ,  $F \subseteq H$  and  $G \cap H = \phi$ . Therefore,  $G^c, H^c \in GPC(X, \tau_{12})$ . By using corollary2.1 we get,  $G^c \cap A, H^c \cap A \in GPC(X, \tau_{12})$ . Also we have,  $G^c \cap A \subseteq A \subseteq X$  and  $H^c \cap A \subseteq A \subseteq X$ . Then,  $A \cap G^c, H^c \cap A \in GPC(A, \tau_{12}|_A)$  by Theorem2.5. This implies that,  $A \cap G, A \cap H \in GPO(A, \tau_{12}|_A)$ ,  $x \in A \cap G$  and  $F \subseteq A \cap H$  and therefore  $(A \cap G) \cap (A \cap H) = \phi$ . Hence,  $(A, \tau_{12}|_A)$  is a  $gp^*$  regular space.

**Definition 3.5:** Let  $(X, \tau_1, \tau_2)$  be a bts and  $(X, \tau_{12})$  be its associated supra topological space. Then  $agp^* - regular$   $gp^*T_1$  space is called a generalized pairwise star  $T_3$   $(gp^*T_3, for short)$  space.

**Theorem 3.9:** Every  $gp^*T_3$  space is  $gp^*T_2$  space.

**Proof:** Let  $(X, \tau_{12})$  be a  $gp^*T_3$  and  $a, b \in X$  such that  $a \neq b$ . Then,  $\{b\}$  is gp-closed set since,  $(X, \tau_{12})$  is  $gp^*T_1$ . This implies that,  $a \notin \{b\}$ . But,  $(X, \tau_{12})$  is  $gp^*$  regular space. Therefore, their exists disjoint gp-open sets G, H such that,  $a \in G, \{b\} \subseteq H$ . Hence,  $(X, \tau_{12})$  is  $gp^*T_2$  space.

**Definition 3.6:** Let  $(X, \tau_1, \tau_2)$  be a bitopological space and  $(X, \tau_{12})$  be its associated supra topological space. Then X is called a generalized pairwise star normal  $(gp^* \text{ normal, for short})$  space if for all disjoint gp –closed sets F and U, their exists disjoint gp-closed sets G and H such that,  $U \subseteq H$  and  $F \subseteq G$ .

**Theorem 3.10:** Let  $(X, \tau_1, \tau_2)$  be a bitopological space,  $(X, \tau_{12})$  is an associated supra topological spaces. If  $(X, \tau_{12})$  is a  $gp^*$  normal space and A is P- closed set in  $(X, \tau_{12})$ , then  $(A, \tau_{12}|_A)$  is a  $gp^*$  normal space.

**Proof:** Let  $(X, \tau_{12})$  be a  $gp^*$  normal space and A be a P- closed set in  $(X, \tau_{12})$ . Suppose that  $F, V \in GPC(A, \tau_{12}|_A)$  and  $F \cap V = \phi$ . Thus,  $F, V \in GPC(X, \tau_{12})$ . But,  $(X, \tau_{12})$  is a gp - normal space, this implies that  $\exists G, H \in GPO(X, \tau_{12})$  such that  $F \subseteq G, V \subseteq H$  and  $G \cap H = \phi$ . Therefore,  $G^c, H^c \in GPC(X, \tau_{12})$ . By using corollary 2.1 we get,  $G^c \cap A, H^c \cap A \in GPC(X, \tau_{12})$ . Also we have,  $G^c \cap A \subseteq A \subseteq X$  and  $H^c \cap A \subseteq A \subseteq X$ . Then,  $A \cap G^c, H^c \cap A \in GPC(A, \tau_{12}|_A)$  by Theorem2.5. This implies that,  $A \cap G, A \cap H \in GPO(A, \tau_{12}|_A)$ ,  $F \subseteq A \cap G$  and  $V \subseteq A \cap H$  also  $(A \cap G) \cap (A \cap H) = \phi$ . Hence, $(A, \tau_{12}|_A)$  is a  $gp^*$  normal space.

**Theorem 3.11:** If  $(X, \tau_1, \tau_2)$  is  $gp^*$  normal, then  $(X, \tau_1, \tau_2)$  is  $P^*$ normal.

**Proof:** Let  $(X, \tau_1, \tau_2)$  be a  $gp^*$  normal, F, H be disjoint P-closed set in  $(X, \tau_1, \tau_2)$ . Then F, H are disjoint gp -closed sets. But,  $(X, \tau_1, \tau_2)$  is  $gp^*$  normal. This implies that, there exist disjoint gp -open sets U, V such that  $F \subseteq U$  and  $H \subseteq V$ . But F, H are P-closed set and U, V gp -open sets. Then,  $F \subseteq int_{12}(U)$  and  $H \subseteq int_{12}(V)$ . Also,  $int_{12}(U) \cap int_{12}(V) = \phi$  and  $int_{12}(U), int_{12}(V) \in \tau_{12}$ . Hence, $(X, \tau_1, \tau_2)$  is  $P^*$  normal.

**Remark 3.5:** A gp<sup>\*</sup> normal space is not gp<sup>\*</sup> regular space in general, Examples 3.4 explain that.

**Example 3.4:** Let  $X = \{a, b, c, d\}, \tau_1 = \{X, \phi, \{a, b\}, \{a, b, c\}\}, \tau_2 = \{X, \phi, \{c\}\}$ . Then  $\tau_{12} = \{X, \phi, \{a, b\}, \{c\}, \{a, b, c\}\}$ and  $\tau_{12}^c = \{X, \phi, \{c, d\}, \{d\}, \{a, b, d\}\}$ . GPC  $(X, \tau_{12}) = \{X, \phi, \{c, d\}, \{d\}, \{a, d\}, \{b, d\}, \{a, b, d\}, \{b, c, dand GPO(X, \tau_{12}) = \{X, \phi, \{a, b\}, \{a, c\}, \{b, c\}, \{a, c\}, \{c\}, \{a\}\}\}$   $(X, \tau_{12})$  is  $gp^*$  normal space but it is not  $gp^*$  regular space. Since,  $a \notin \{d\}$  and there is no disjoint gp -open set cantoning a and  $\{d\}$ .

**Definition 3.8:** Let  $(X, \tau_1, \tau_2)$  be a bts and  $(X, \tau_{12})$  be its associated supra topological space. Then a  $gp^*$  – normal  $gp^*T_1$  space is called a generalized pairwise star  $T_4$  ( $gp^*T_4$ , for short)

**Theorem 3.12:** Every  $gp^*T_4$  space is  $gp^*T_3$  space.

**Proof:** Let  $(X, \tau_{12})$  be a  $gp^*T_4$  and  $a \notin F$  such that F is gp-closed. Then,  $\{a\}$  is gp-closed set since,  $(X, \tau_{12})$  is  $gp^*T_1$ . This implies that,  $\{a\} \cap F = \phi$ . But,  $(X, \tau_{12})$  is  $gp^*$ normal space. Therefore, their exists disjoint gp-open sets G, H such that,  $\{a\} \in G, F \subseteq H$ . Hence,  $(X, \tau_{12})$  is  $gp^*T_3$  space.

**Remark 3.5:** For a bitopological space  $(X, \tau_1, \tau_2)$  we have the following chart.



**Definition 3.8:** Let  $(X, \tau_1, \tau_2)$  be a bts and  $(X, \tau_{12})$  be its associated supra topological space. Then X is called a generalized pairwise star  $R_0$  space  $(gp^*R_0, for short)$  if  $\forall x \in G, G$  is gp-open set, then  $cl_{12}\{x\} \subseteq G$ .

**Theorem 3.13:** Let  $(X, \tau_1, \tau_2)$  be a bitopological space,  $(X, \tau_{12})$  its associated supra topological spaces. If  $(X, \tau_{12})$  is a  $gp^*R_0$  space and A is P- closed set in  $(X, \tau_{12})$ , then  $(A, \tau_{12}|_A)$  is a  $gp^*R_0$  space.

**Proof:** Let  $(X, \tau_{12})$  be a  $gp^*T_2$  space and A be a P- closed set in  $(X, \tau_{12})$ . Suppose that  $x \in G$ ,  $G \in GPO(A, \tau_{12}|_A)$ . Then  $A \setminus G \in GPC(A, \tau_{12}|_A)$ . So,  $A \setminus G = A \cap (X \setminus G)$  and  $X \setminus G \in GPC(X, \tau_{12})$ . This implies that,  $X \setminus A \cup G \in GPO(X, \tau_{12})$  and  $x \in X \setminus A \cup G$  but, X is  $gp^*R_0$ . Then,  $cl_{12}\{x\} \subseteq X \setminus A \cup G$ . Therefore,  $A \cap cl_{12}\{x\} \subseteq A \cap G = G$  consequently,  $cl_{12}|_A\{x\} \subseteq G$ . Hence,  $(A, \tau_{12}|_A)$  is  $gp^*R_0$  space.

**Theorem 3.14:** Let  $(X, \tau_1, \tau_2)$  and  $(Y, \gamma_1, \gamma_2)$  be bitopological spaces,  $(X, \tau_{12})$  and  $(Y, \gamma_{12})$  are associated supra topological spaces. Suppose that  $f: X \to Y$  is an injective,  $P^*$  continuous function and  $P^*$  -closed, if  $(Y, \gamma_{12})$  is  $gp^*R_0$  space, then  $(X, \tau_{12})$  is  $gp^*R_0$  space.

**Proof:** Let  $(X, \tau_1, \tau_2)$  and  $(Y, \gamma_1, \gamma_2)$  be bitopological spaces,  $(X, \tau_{12})$  and  $(Y, \gamma_{12})$  are associated supra topological spaces. Suppose that  $f: X \to Y$  is an injective,  $P^*$  continuous and  $P^*$  -closed,  $(Y, \gamma_{12})$  is  $gp^*R_0$  space. Let  $x \in G$  and G be gp -open set in  $\tau_{12}$ . Then,  $G^c$  be gp -closed, Theorem 2.8,  $f(G^c) = (f(G))^c$  is gp -closed and hence, f(G) is gp - open and  $f(x) \in f(G)$ . We have  $(Y, \gamma_{12})$  is  $gp - R_0$ . Thus,  $cl_{12}|_Y(\{f(x)\}) \subseteq f(G)$ . Then,  $f^{-1}cl_{12}|_Y(\{f(x)\}) \subseteq f^{-1}f(G) = G$ . Then,  $f^{-1}cl_{12}|_Y(\{f(x)\}) \subseteq f^{-1}f(G) = G$ . Then,  $f^{-1}cl_{12}|_Y(\{f(x)\}) \subseteq G$  which means that,  $(X, \tau_1, \tau_2)$  is  $gp^*R_0$ .

**Theorem 3.15:** Let  $(X, \tau_1, \tau_2)$  is  $gp^*R_0$  space. For any  $F \in GPC(X, \tau_{12})$ ,  $x \notin F$ , then  $F \subseteq U$  and  $x \notin U$  for some  $U \in GPO(X, \tau_{12})$ .

**Proof:** Let  $(X, \tau_1, \tau_2)$  be  $gp^*R_0$  space. Suppose that  $F \in GPC(X, \tau_{12})$ ,  $x \notin F$ , then  $cl_{12}\{x\} \subseteq F^c$ . This implies that,  $F \subseteq [cl_{12}\{x\}]^c$ . Take  $U = [cl_{12}\{x\}]^c$  which is gp – open set and containing x.

**Theorem 3.16:** If  $F \cap cl_{12}[\{x\}] = \phi \quad \forall x \notin F$  and  $F \in GPC(X, \tau_{12})$ , then  $(X, \tau_1, \tau_2)gp^*R_0$  space.

**Proof:** Let  $x \in G$  and  $G \in GPO(X, \tau_{12})$ . Then,  $x \notin G^c$  and  $G^c \in GPC(X, \tau_{12})$ . This implies that,  $cl_{12}\{x\} \cap G^c = \phi$ . Thus,  $cl_{12}\{x\} \subseteq G$ . Hence,  $(x, \tau_1, \tau_2)$  is  $gp^*R_0$  space.

**Theorem 3.17:** If  $(X, \tau_1, \tau_2)$  is  $gp^*R_0$  space and  $gp^*T_0$  space then, it is a  $gp^*T_1$  space.

**Proof:** Let  $(X, \tau_1, \tau_2)$  be  $gp^*R_0$  space and  $gp^*T_0$  space. suppose that  $x, y \in X$  and  $x \neq y$ . Then  $\exists G \in GPO(X, \tau_{12})$  such that  $x \in G, y \notin G$ . But we have,  $(X, \tau_1, \tau_2)$  is  $gp^*R_0$ , this implies that,  $cl_{12}(\{x\}) \subseteq G$ . Therefore,  $G^c \subseteq [cl_{12}(\{x\})]^c$  and  $y \in [cl_{12}(\{x\})]^c$ . But  $[cl_{12}(\{x\})]^c$  is gp-open set containing y and not containing x. Hence,  $(X, \tau_1, \tau_2)$  is  $gp^*T_1$ .

**Definition 3.9:** Let  $(X, \tau_1, \tau_2)$  be a bis and  $(X, \tau_{12})$  be its associated supra topological space. Then X is called a generalized pairwise star  $R_1$  space  $(gp^*R_1, for short)$  if  $\forall x, y \in X$ , and  $cl_{12}\{x\} \neq cl_{12}\{y\}$ ,  $\exists gp$ -open sets G and H such that  $cl_{12}\{x\} \subseteq G$ ,  $cl_{12}\{y\} \subseteq H$  and  $G \cap H = \phi$ .

**Remark 3.6:** If  $(X, \tau_1, \tau_2)$  is  $gp^*R_1$  space, then it is  $gp^*R_0$  space by definition. But,  $gp^*R_0$  is not necessary  $gp^*R_1$ . *Examples 3.5 explain that.* 

**Example 3.5:** Take R is real line,  $\tau_1 = \{R, \phi\}$ ,  $\tau_2 = cofinite$  topology. Then  $\tau_{12} = cofinite$  topology, The set of all generalized closed sets is equal to the set of all pairwise closed sets. So, This space is  $gp * R_0$  but not  $gp^*R_1$ . Since  $cl_{12}\{1\} \neq cl_{12}\{2\}$  and there is no disjoint gp-open sets containing 1 and 2.

**Theorem 3.18:** Let  $(X, \tau_1, \tau_2)$  be a bitopological space,  $(X, \tau_{12})$  its associated supra topological spaces. If  $(X, \tau_{12})$  is a  $gp^*R_1$  space and A is P- closed set in  $(X, \tau_{12})$ , then  $(A, \tau_{12}|_A)$  is a  $gp^*R_1$  space.

**Proof:** Let  $(X, \tau_{12})$  be a  $gp^*R_1$  space and A be a P - closed set in  $(X, \tau_{12})$ . Suppose that  $x, y \in A$  and  $cl_{12}|_A\{x\} \neq cl_{12}|_A\{y\}$ . Then,  $cl_{12}\{x\} \neq cl_{12}\{y\}$ . But,  $(X, \tau_{12})$  is a  $gp^*R_1$  space. This implies that,  $\exists G, H \in GPO(X, \tau_{12})$  such that  $cl_{12}\{x\} \subseteq G$ ,  $cl_{12}\{y\} \subseteq H$  and  $G \cap H = \phi$ . By using corollary2.1and Theorem2.5 we get,  $G \cap A, F \cap A \in GPO(A, \tau_{12}|_A)$  and  $(G \cap A) \cap (F \cap A) = \phi$ . Also,  $cl_{12}|_A\{x\} \subseteq A \cap G$ ,  $cl_{12}|_A\{y\} \subseteq A \cap H$ . Hence,  $(A, \tau_{12}|_A)$  is a  $gp^*R_1$  space.

**Theorem 3.19:** Let  $(X, \tau_1, \tau_2)$  and  $(Y, \gamma_1, \gamma_2)$  be a bitopological spaces,  $(X, \tau_{12})$  and  $(Y, \gamma_{12})$  are associated supra topological spaces. Suppose that  $f: X \to Y$  is an injective,  $P^*$  continuous function and  $P^*$ -closed. If  $(Y, \gamma_1, \gamma_2)$  is  $gp^*R_1$  space, then  $(X, \tau_1, \tau_2)$  is  $gp^*R_1$  space.

**Proof:** Let  $(X, \tau_1, \tau_2)$  and  $(Y, \gamma_1, \gamma_2)$  be bitopological spaces,  $(X, \tau_{12})$  and  $(Y, \gamma_{12})$  are associated supra topological spaces. Suppose that  $f: X \to Y$  is an injective,  $P^*$  continuous function and  $P^*$  -closed,  $(Y, \gamma_{12})$  is  $gp^*R_1$  space. Let  $x_1, x_2 \in X$  and  $x_1 \neq x_2$  such that  $cl_{12}|_X(\{x_1\}) \neq cl_{12}|_X(\{x_2\})$ . Then, we have f is injective, this implies that,  $f(x_1) \neq f(x_2)$ . But, $(Y, \gamma_{12})$  is  $gp^*R_1 f(cl_{12}|_X(\{x_1\})) \neq f(cl_{12}|_X(\{x_2\}))$ . Thus,  $cl_{12}|_Y(f(\{x_1\})) \neq cl_{12}|_Y(f(\{x_2\}))$ . Then,  $\exists gp$  -open sets  $H_1, H_2$  in  $(Y, \gamma_{12})$  such that  $cl_{12}|_Y(f(\{\{x_1\}\})) \subseteq H_1, cl_{12}|_Y(f(\{x_2\})) \subseteq H_2$  and  $H_1 \cap H_2 = \phi$ . But, f is an injective  $P^*$  - continuous function. Then,  $f^{-1}(H_1), f^{-1}(H_2)$  are gp -open sets in  $(X, \tau_{12})$  say  $f^{-1}(H_1) = G_1, f^{-1}(H_2) = G_2$ . Also,  $f^{-1}cl_{12}|_Y(f(\{x_1\})) = cl_{12}|_X(x_1)$  and  $f^{-1}cl_{12}|_Y(f(\{x_2\})) = cl_{12}|_X(x_2)$ . This implies that,  $cl_{12}|_X(\{x_1\}) \subseteq G_1$  and  $cl_{12}|_X(\{x_2\}) \subseteq G_2$ . Hence,  $(X, \tau_{12})$  is  $gp^*R_1$ .

**Theorem 3.20:** If  $(X, \tau_1, \tau_2)$  is  $gp^*R_0$  and  $gp^*T_1$  space then, it is  $gp^*T_2$   $(X, \tau_1, \tau_2)$ .

**Proof:** Let  $(X, \tau_1, \tau_2)$  be  $gp^*R_0$  space and  $gp^*T_1$  space. suppose that  $x, y \in X$  and  $x \neq y$ . Then  $\exists G, H \in GPO(X, \tau_{12})$  such that  $x \in G, y \notin G$  and  $y \in H, x \notin H$ . But we have,  $(X, \tau_1, \tau_2)$  is  $gp^*R_0$ , this implies that,  $cl_{12}(\{x\}) \subseteq G$ . Therefore,  $G^c \subseteq [cl_{12}(\{x\})]^c \ni y$ . But  $[cl_{12}(\{x\})]^c$  is gp-open set containing y and not containing x. Hence,  $(X, \tau_1, \tau_2)$  is  $gp^*T_2$ .

**Theorem 3.21:** For a bitopological space  $(X, \tau_1, \tau_2)$ , the following statements are equivalent:

- 1.  $(X, \tau_1, \tau_2)$  is  $gp^*R_1$  space.
- 2. For all  $x, y \in X$  one of the following holds:
  - (i) If  $x \in U \in GPO(X, \tau_{12})$ , then  $y \in U$ .
  - (ii) There exist disjoint sets  $G, H \in GPO(X, \tau_{12})$  such that  $x \in G$  and  $y \in H$ .
- 3. If  $x, y \in X$  such that  $cl_{12}\{x\} \neq cl_{12}\{y\}$ , then  $\exists F_1, F_2GPC(X, \tau_{12})$  such that,  $x \notin F_1$ ,  $y \notin F_2$  and  $F_1 \cup F_2 = X$ .

**Proof:** (1)  $\Rightarrow$  (2).Let  $(X, \tau_1, \tau_2)$  be  $gp^*R_1$  space and  $cl_{12}\{x\} = cl_{12}\{y\}$ . Suppose that,  $x \in U \in GPO(X, \tau_{12})$  such that  $cl_{12}\{x\} \subseteq U$ . Since, every  $gp^*R_1$  space is  $gp^*R_0$  space. But,  $cl_{12}\{x\} = cl_{12}\{y\}$ . Then,  $y \in cl_{12}\{x\} \subseteq U$ . If  $cl_{12}\{x\} \neq cl_{12}\{y\}$  then, by definition of  $gp^*R_1$  space we get  $\exists gp$ -open sets G and H such that  $cl_{12}\{x\} \subseteq G$ ,  $cl_{12}\{y\} \subseteq H$  and  $G \cap H = \phi$ .

(2)  $\Rightarrow$  (3): Suppose that,  $x, y \in X$  such that  $cl_{12}\{x\} \neq cl_{12}\{y\}$ . Then, by (2) there exist disjoint sets  $G, H \in GPO(X, \tau_{12})$  such that  $x \in G$  and  $y \in H$ . Thus,  $F_1 = X \setminus U$  and  $F_2 = X \setminus V$  which are gp - closed sets and There exist disjoint sets  $F_1, F_2 \in GPC(X, \tau_{12})$  such that  $x \notin F_1$  and  $y \notin F_2$  and  $F_1 \cup F_2 = X$ .

(3)  $\Rightarrow$  (1): Let  $x, y \in X$  such that  $cl_{12}\{x\} \neq cl_{12}\{y\}$ . Then,  $\exists F_1, F_2GPC(X, \tau_{12})$  such that,  $x \notin F_1$ ,  $y \notin F_2$  and  $F_1 \cup F_2 = X$ . So,  $F_1^c, F_2^cGPO(X, \tau_{12})$  such that,  $x \in F_1^c$ ,  $y \in F_2^c$  and  $F_1^c \cap F_2^c = \phi$ . Hence,  $(X, \tau_1, \tau_2)$  is  $gp^*R_1$  space.

## 4. CONCLUSION

- 1. The notation of  $P^*$  -regular and  $gp^*$  regular are not equal, when we tried to given examples for one not the other, we have two cases: (i)  $\tau_{12} = GPC(X, \tau_{12})$  and (ii)  $GPC(X, \tau_{12}) = P(X)$ .
- 2. Also, the notation of  $P^*$  –normal and  $gp^*$  normal are not equal, when we tried to given examples for one not the other, we have the above two cases.
- 3. Also, we tried to find an example to show the relation between  $gp^*$  –regular and  $gp^*$  normal, we see that  $GPO(X, \tau_{12}) = GPC(X, \tau_{12}) = P(X)$ . These examples remain missing and this is an open area of research.

## REFERENCES

- 1. J.C.Kelly, Bitopological spaces, Proc Lond Math Soc. 13 (1963), 71-79.
- 2. A. S. Mashhour, A. A. Allam, F. S. Mahmoud and F. H. Khedr, On supra topological space, *Indian J. Pure Appl. Math.* 14 (1983), 502-510.
- 3. M. E. Abd-El-Monsef and E. F Lashine, Some operation on supratopological spaces, *Dirasat*, XIII (7) (1986), 43-50.
- 4. A. Kandil, S. A. El-Sheikh, On bitopological spaces, first Intern.conf.on Mathematics and Statistics, Assuit University 3 (1990), 73-96.
- 5. Fukutake On Generalized Closed Sets in Bitopological spaces. Bull Fukuoka Univ Ed. Part III 35 (1986), 19-28.
- 6. H. S. Al-Saadi and A. H. Zakari, On some separation axioms and strongly generalized closed sets in bitopological spaces, *International Mathematical Forum*, 21(3) (2008), 1039 1054.
- 7. R. Suganya and N. Rajesh, Separation axioms in bitopological spaces, *International Journal of Mathematical Archive*, 11(2) (2015), 857-866.
- 8. P. Padma, S. Udayakumar and K. Chandrasekhararao, Pairwise Q\* separation axioms in bitopological spaces, *International Journal of Mathematical Archive*, 3(12) (2012), 4959-4971.
- 9. N. Aziz, F. Ahmad and M. Salih. New types of separation axioms in bitopological spaces, *Zanco Journal of Pure and Applied Sciences*, 27(4) (2015), 20-26.
- 10. S.A.EL-Sheikh, Dimension Theory of Bitopological Spaces, Master Thesis, Ain Shams University, Cairo, Egypt(1987).
- 11. S.A.EL-Sheikh, Some bitopological properties via grills, *International Mathematical Forum*. 13(7) (2014), 335-353.
- A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh and E. A. Shalaby, Generalized closed sets in bitopological spaces, South Asian J. Math. 6(2) (2016), 72-81.