(ψ*g*)*- CLOSED SETS IN TOPOLOGICAL SPACES

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ABSTRACT

The aim of this paper is to introduce and study a new class of generalized closed sets called (ψ*g*)*-closed sets in topological spaces. This class was obtained by generalizing closed sets via ψ*g* open sets which was introduced by N. Balamani, A. Parvathi [3]. This new class falls strictly between the class of closed sets and ψg*-closed sets. Also some of their properties have been investigated.

Keywords: (ψ*g*)*-closed sets and (ψ*g*)*-open sets

I. INTRODUCTION


The purpose of this paper is to introduce a new class of generalized closed sets called (ψ*g*)*-closed sets in topological spaces and study some properties.

II. PRELIMINARIES

Throughout this paper (X, τ) represents non-empty topological space on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A) and int(A) denote the closure of A and the interior of A respectively.

Definition 2.1: A subset A of a topological space (X, τ) is called
(i) Regular open set [20] if A = int(cl(A))
(ii) Semi-open set [7] if A ⊆ cl(int(A))
(iii) α-open set [16] if A ⊆ int(cl(int(A)))
(iv) Pre-open set [13] if A ⊆ int(cl(A))
(v) semi pre-open set [2] if A ⊆ cl(int(clA)))

The complements of the above mentioned sets are called regular closed, semi closed, α-closed, pre-closed and semi pre-closed sets respectively.

The intersection of all regular closed subsets of (X, τ) containing A is called the regular closure of A and is denoted by rcl(A). Similarly scl(A)-semi closure of A, ucl(A) - α closure of A, pcl(A)-pre closure of A and spcl(A)-semi pre closure of A are defined.

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Definition 2.2: A subset $A$ of a topological space $(X, \tau)$ is called
(a) generalized closed set (g-closed) \[8\] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.
(b) semi-generalized closed set (sg-closed) \[4\] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $(X, \tau)$.
(c) generalized $\alpha$-closed set (g$\alpha$-closed) \[9\] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open in $(X, \tau)$.
(d) $\alpha$-generalized closed set (arg-$\alpha$) \[10\] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.
(e) generalized semi-pre-closed set (gsp-closed) \[6\] if $\text{spcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.
(f) g$^*$-closed set \[23\] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is g$^*$-open in $(X, \tau)$.
(g) g$^*$-closed set \[24\] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is g$^*$-closed in $(X, \tau)$.
(h) gsp-closed set \[17\] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is gsp-closed in $(X, \tau)$.

The complements of the above mentioned sets are called their respective open-sets.

Remark 2.3: r-closed(r-open) $\implies$ closed (open) $\implies$ $\alpha$-closed (open) $\implies$ semi-closed(semi-open) $\implies$ $\psi$-closed($\psi$-open) $\implies$ semi pre-closed(semi-pre-open).

Remark 2.4: r-closed $\implies$ closed $\implies$ $\psi$-closed $\implies$ g$^*$-$\psi$-closed $\implies$ g$^*$-$\psi$-closed $\implies$ g$^*$-$\psi$-open $\implies$ gsp-closed

III. BASIC PROPERTIES OF ($\psi$g$^*$)-CLOSED SETS

Definition 3.1: A subset $A$ of a topological space $(X, \tau)$ is said to be ($\psi$g$^*$)-closed set if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\psi$g$^*$-open in $(X, \tau)$. The class of all ($\psi$g$^*$)-closed sets of $(X, \tau)$ is denoted by $\psi$g$^*$C$(X, \tau)$.

Proposition 3.2: Every closed set in $(X, \tau)$ is ($\psi$g$^*$)-closed but not conversely.

Proof: Let $A$ be a closed set of $(X, \tau)$. Let $U$ be any $\psi$g$^*$-open set containing $A$. Since $A$ is closed set. Therefore $\text{cl}(A) = A$

Hence $\text{cl}(A) \subseteq U$. Therefore $A$ is ($\psi$g$^*$)-closed.

Example 3.3: Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. Then the subset $\{b\}$ is ($\psi$g$^*$)-closed but not closed in $(X, \tau)$.

Proposition 3.4: Every regular closed set in $(X, \tau)$ is ($\psi$g$^*$)-closed but not conversely.

Proof: As every regular closed set is closed and by proposition 3.2 the proof follows.

Example 3.5: Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Then the subset $\{c\}$ is ($\psi$g$^*$)-closed but not regular closed in $(X, \tau)$.

Proposition 3.6: Every ($\psi$g$^*$)-closed set in $(X, \tau)$ is $\psi$g$^*$-closed but not conversely.

Proof: Let $A$ be a ($\psi$g$^*$)-closed set and $U$ be any open set containing $A$ in $X$. Since every open set is $\psi$g$^*$-open, Therefore $\text{wcl}(A) \subseteq \text{cl}(A) \subseteq U$. Hence $A$ is $\psi$g$^*$-closed.
Example 3.7: Let \( X = \{a, b, c\} \) with \( \tau = \{\emptyset, X, \{a\}, \{a, c\}\} \). Then the subset \( \{a, b\} \) is \( \psi g - \text{closed} \) but not \( (\psi g^*)^* - \text{closed} \) in \( (X, \tau) \).

**Proposition 3.8:** Every \( (\psi g^*)^* - \text{closed} \) set in \( (X, \tau) \) is \( \text{gsp} - \text{closed} \) but not conversely.

**Proof:** Let \( A \) be a \( (\psi g^*)^* - \text{closed} \) set and \( U \) be any open set containing \( A \) in \( X \). Since every open set is \( \psi g^* - \text{open} \) and \( \text{spcl}(A) \subseteq \text{cl}(A) \subseteq U \). Hence \( A \) is \( \text{gsp} - \text{closed} \).

Example 3.9: Let \( X = \{a, b, c\} \) with \( \tau = \{\emptyset, X, \{a\}, \{b, c\}\} \). Then the subset \( \{a\} \) is \( \text{gsp} - \text{closed} \) but not \( (\psi g^*)^* - \text{closed} \) in \( (X, \tau) \).

**Proposition 3.10:** Every \( (\psi g^*)^* - \text{closed} \) set in \( (X, \tau) \) is \( \text{gsp} - \text{closed} \) but not conversely.

**Proof:** Let \( A \) be a \( (\psi g^*)^* - \text{closed} \) set and \( U \) be any regular open set containing \( A \) in \( X \). Since every regular open set is \( \psi g^* - \text{open} \) and \( \text{cl}(\text{int}(A)) \subseteq \text{cl}(A) \subseteq U \). Therefore \( A \) is \( (\psi g^*)^* - \text{closed} \).

Example 3.11: Let \( X = \{a, b, c\} \) with \( \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \). Then the subset \( \{a, b\} \) is \( (\psi g^*)^* - \text{closed} \) but not \( (\psi g^*)^* - \text{closed} \) in \( (X, \tau) \).

Example 3.12: Let \( X = \{a, b, c\} \) with \( \tau = \{\emptyset, X, \{a\}, \{b, c\}\} \). Then the subset \( \{a, b\} \) is \( \text{gsp} - \text{closed} \) but not \( (\psi g^*)^* - \text{closed} \) in \( (X, \tau) \).

**Proposition 3.14:** Every \( (\psi g^*)^* - \text{closed} \) set in \( (X, \tau) \) is \( \text{rg} - \text{closed} \) but not conversely.

**Proof:** Let \( A \) be a \( (\psi g^*)^* - \text{closed} \) set and \( U \) be any regular open set containing \( A \) in \( X \). Since every regular open set is \( \psi g^* - \text{open} \) and \( \text{cl}(A) \subseteq U \). Hence \( A \) is \( \text{rg} - \text{closed} \).

Example 3.15: Let \( X = \{a, b, c\} \) with \( \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \). Then the subset \( \{a, b\} \) is \( \text{rg} - \text{closed} \) but not \( (\psi g^*)^* - \text{closed} \) in \( (X, \tau) \).

Example 3.16: Let \( X = \{a, b, c\} \) with \( \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \). Then the subset \( \{a, b\} \) is \( \text{rg} - \text{closed} \) but not \( (\psi g^*)^* - \text{closed} \) in \( (X, \tau) \).

Example 3.17: Let \( X = \{a, b, c\} \) with \( \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \). Then the subset \( \{a, b\} \) is \( \text{rg} - \text{closed} \) but not \( (\psi g^*)^* - \text{closed} \) in \( (X, \tau) \).

**Proposition 3.18:** Every \( (\psi g^*)^* - \text{closed} \) set in \( (X, \tau) \) is \( \text{rg} - \text{closed} \) but not conversely.

**Proof:** Let \( A \) be a \( (\psi g^*)^* - \text{closed} \) set and \( U \) be any regular open set containing \( A \) in \( X \). Since every regular open set is \( \psi g^* - \text{open} \) and \( \text{cl}(A) \subseteq U \). Hence \( A \) is \( \text{rg} - \text{closed} \).

Example 3.19: Let \( X = \{a, b, c\} \) with \( \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \). Then the subset \( \{a, b\} \) is \( \text{rg} - \text{closed} \) but not \( (\psi g^*)^* - \text{closed} \) in \( (X, \tau) \).

**Proposition 3.20:** \( (\psi g^*)^* - \text{closedness} \) is independent from \( \psi g^* - \text{closedness}, \psi - \text{closedness, semi closedness, g}^*\psi - \text{closedness, g}^* - \text{closedness and } \alpha - \text{closedness.} \)

Example 3.21: Let \( X = \{a, b, c\} \) with \( \tau = \{\emptyset, X, \{a\}, \{b, c\}\} \). Then the subset \( \{a, c\} \) is \( (\psi g^*)^* - \text{closed but not } \psi g^* - \text{closed, semi closed, g}^*\psi - \text{closed, g}^* - \text{closed, } \psi g^* - \text{closed and } \alpha - \text{closed.} \)

Example 3.22: Let \( X = \{a, b, c\} \) with \( \tau = \{\emptyset, X, \{a\}, \{a, c\}\} \). Then the subset \( \{c\} \) is \( \psi g^* - \text{closed, semi closed, g}^*\psi - \text{closed, } \psi g^* - \text{closed and } \alpha - \text{closed but not } (\psi g^*)^* - \text{closed in } (X, \tau). \) The subset \( \{a, b\} \) is \( g^* - \text{closed but not } (\psi g^*)^* - \text{closed in } (X, \tau). \)
IV. \((\psi g^*)^*\)-OPEN SET

**Definition 4.1:** A subset \(A\) of a topological space \((X, \tau)\), is called \((\psi g^*)^*\)-open set if and only if \(A^c\) is \((\psi g^*)^*\)-closed in \(X\). We denote the family of all \((\psi g^*)^*\)-open sets in \(X\) by \((\psi g^*)^*\)-O(X).

**Theorem 4.2:**

(i) Every open set is \((\psi g^*)^*\)-open.
(ii) Every regular open set is \((\psi g^*)^*\)-open set.
(iii) Every \((\text{gsp})^*\)-open set is \((\psi g^*)^*\)-open set.
(iv) Every \((\psi g^*)^*\)-open set is \(\text{gsp}\)-open set.
(v) Every \((\psi g^*)^*\)-open set is strongly \(\psi g\)-open set.
(vi) Every \((\psi g^*)^*\)-open set is \(\text{gpr}\)-open set.
(vii) Every \((\psi g^*)^*\)-open set is \(\text{rwg}\)-open set.
(viii) Every \((\psi g^*)^*\)-open set is \(r^g\)-open set.
(ix) Every \((\psi g^*)^*\)-open set is \(\text{rg}\)-open set.

A \(\rightarrow\) B represents A implies B. But not converse and A \(\leftrightarrow\) B represents A and B are independent of each other.

V. CHARACTERISTICS OF \((\psi g^*)^*\)-CLOSED AND \((\psi g^*)^*\)-OPEN SETS

**Theorem 5.1:** If \(A\) and \(B\) are \((\psi g^*)^*\)-closed sets in \(X\) then \(A \cup B\) is \((\psi g^*)^*\)-closed set in \(X\).

**Proof:** Let \(A\) and \(B\) are \((\psi g^*)^*\)-closed sets in \(X\) and \(U\) be any \(\psi g^*\)-open set containing \(A\) and \(B\). Therefore \(\text{cl}(A) \subseteq U, \text{cl}(B) \subseteq U\). Since \(A \subseteq U\), \(B \subseteq U\) then \(A \cup B \subseteq U\). Hence \(\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B) \subseteq U\). Therefore \(A \cup B\) is \((\psi g^*)^*\)-closed set in \(X\).

**Theorem 5.2:** If a set \(A\) is \((\psi g^*)^*\)-closed set iff \(\text{cl}(A)-A\) contains no non empty \(\psi g^*\)-closed set.

**Proof:**

**Necessity:** Let \(F\) be a \(\psi g^*\)-closed set in \(X\) such that \(F \subseteq \text{cl}(A)-A\). Then \(A \subseteq X - F\). Since \(A\) is \((\psi g^*)^*\)-closed set and \(X - F\) is \(\psi g^*\)-open, then \(\text{cl}(A) \subseteq X - F\). (i.e.) \(F \subseteq X - \text{cl}(A)\). So \(F \subseteq (X - \text{cl}(A)) \cap (\text{cl}(A)-A)\). Therefore \(F = \phi\).

**Sufficiency:** Let us assume that \(\text{cl}(A) \subseteq \text{cl}(A)-A\) contains no non empty \(\psi g^*\)-closed set. Let \(A \subseteq U\), \(U\) is \(\psi g^*\)-open set. Suppose that \(\text{cl}(A)\) is not contained in \(U\), \(\text{cl}(A) \cap U\) is a nonempty \(\psi g^*\)-closed set of \(\text{cl}(A)-A\) which is contradiction. Therefore \(\text{cl}(A) \subseteq U\). Hence \(A\) is \((\psi g^*)^*\)-closed.

**Theorem 5.3:** The Intersection of any two subsets of \((\psi g^*)^*\)-closed sets in \(X\) is \((\psi g^*)^*\)-closed set in \(X\).

**Proof:** Let \(A\) and \(B\) are any two sub sets of \((\psi g^*)^*\)-closed sets. \(A \subseteq U\), \(U\) is any \(\psi g^*\)-open and \(B \subseteq U\), \(U\) is \(\psi g^*\)-open. Then \(\text{cl}(A) \subseteq U\), \(\text{cl}(B) \subseteq U\), therefore \(\text{cl}(A \cap B) \subseteq U\), \(U\) is \(\psi g^*\)-open in \(X\). Since \(A\) and \(B\) are \((\psi g^*)^*\)-closed set, Hence \(A \cap B\) is a \((\psi g^*)^*\)-closed set.
Theorem 5.4: If $A$ is $(\psi^*g^*)^*$-closed set in $X$ and $A \subseteq B \subseteq \text{cl}(A)$, then $B$ is $(\psi^*g^*)^*$-closed set in $X$.

**Proof:** Since $B \subseteq \text{cl}(A)$, we have $\text{cl}(B) \subseteq \text{cl}(A)$ then $\text{cl}(B) - B \subseteq \text{cl}(A) - A$. By theorem 5.2, $\text{cl}(A) - A$ contains no non empty $\psi^*g^*$-closed set. Hence $\text{cl}(B) - B$ contains no non empty $\psi^*g^*$-closed set. Therefore $B$ is $(\psi^*g^*)^*$-closed set in $X$.

Theorem 5.5: If $A \subseteq Y \subseteq X$ and suppose that $A$ is $(\psi^*g^*)^*$-closed set in $X$ then $A$ is $(\psi^*g^*)^*$-closed set relative to $Y$.

**Proof:**

Given that $A \subseteq Y \subseteq X$ and $A$ is $(\psi^*g^*)^*$-closed set in $X$. To prove that $A$ is $(\psi^*g^*)^*$-closed set relative to $Y$.

Let us assume that $A \subseteq Y \cap U$, where $U$ is $\psi^*g^*$-open in $X$. Since $A$ is $(\psi^*g^*)^*$-closed set, $A \subseteq U$ implies $\text{cl}(A) \subseteq U$. It follows that $Y \cap \text{cl}(A) \subseteq Y \cap U$. That is $A$ is $(\psi^*g^*)^*$-closed set relative to $Y$.

Theorem 5.6: If $A$ is both $\psi^*g^*$-open and $(\psi^*g^*)^*$-closed set in $X$, then $A$ is $\psi^*g^*$-open.

**Proof:**

Since $A$ is $\psi^*g^*$-open and $(\psi^*g^*)^*$-closed set in $X$. To prove that $A$ is $(\psi^*g^*)^*$-open set in $X$.

Let us assume that $A \subseteq Y \cap U$, where $U$ is $\psi^*g^*$-open in $X$. Since $A$ is $(\psi^*g^*)^*$-closed set, $A \subseteq U$ implies $\text{cl}(A) \subseteq U$. It follows that $Y \cap \text{cl}(A) \subseteq Y \cap U$. That is $A$ is $(\psi^*g^*)^*$-open in $X$.

Theorem 5.7: For $x \in X$, then the set $X - \{x\}$ is a $(\psi^*g^*)^*$-closed set or $\psi^*g^*$-open.

**Proof:**

Suppose that $X - \{x\}$ is not $\psi^*g^*$-open, then $X$ is the only $\psi^*g^*$-open set containing $X - \{x\}$.

(i.e.) $\text{cl}(X - \{x\}) \subseteq X$. Then $X - \{x\}$ is $(\psi^*g^*)^*$-closed in $X$.

Theorem 5.8: If $A$ and $B$ are $(\psi^*g^*)^*$-open sets in a space $X$. Then $A \cap B$ is also $(\psi^*g^*)^*$-open set in $X$.

**Proof:**

If $A$ and $B$ are $(\psi^*g^*)^*$-open sets in a space $X$. Then $A^c$ and $B^c$ are $(\psi^*g^*)^*$-closed sets in a space $X$. By theorem 5.1 $A^c \cup B^c$ is also $(\psi^*g^*)^*$-closed set in $X$. (i.e.) $A^c \cup B^c = (A \cap B)^c$ is a $(\psi^*g^*)^*$-closed set in $X$. Therefore $A \cap B$ is $(\psi^*g^*)^*$-open set in $X$.

Theorem 5.9: If $A$ and $B$ are $(\psi^*g^*)^*$-open sets in $X$ then $A \cup B$ is $(\psi^*g^*)^*$-open set in $X$.

**Proof:**

If $A$ and $B$ are $(\psi^*g^*)^*$-open sets in a space $X$. Then $A^c$ and $B^c$ are $(\psi^*g^*)^*$-closed sets in a space $X$. By theorem 5.4 $A^c \cup B^c$ is also $(\psi^*g^*)^*$-closed set in $X$. (i.e.) $A^c \cup B^c = (A \cup B)^c$ is a $(\psi^*g^*)^*$-closed set in $X$. Therefore $A \cup B$ is $(\psi^*g^*)^*$-open set in $X$.

Theorem 4.15: If $\text{Int}(B) \subseteq B \subseteq A$ and if $A$ is $(\psi^*g^*)^*$-open in $X$, then $B$ is $(\psi^*g^*)^*$-open in $X$.

**Proof:**

Suppose that $\text{Int}(B) \subseteq B \subseteq A$ and $A$ is $(\psi^*g^*)^*$-open in $X$ then $A \subseteq B \subseteq \text{cl}(A)$. Since $A$ is $(\psi^*g^*)^*$-closed in $X$, by Theorem 5.4 $B$ is $(\psi^*g^*)^*$-closed set in $X$. Therefore $B$ is $(\psi^*g^*)^*$-open in $X$.

VI. CONCLUSION

In this paper we have introduced $(\psi^*g^*)^*$-closed sets and $(\psi^*g^*)^*$-open sets and studied some properties. This class of sets can be used to discuss the notion of Continuity, Compactness and connectedness and also can be extended to other topological spaces like Fuzzy & Bitopological Spaces.

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