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ON A COMMON FIXED POINT THEOREM OF WEAK** COMMUTING OPERATORS

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ABSTRACT

*In this present research article, we prove the existence of a common fixed point for three self mappings defined on a complete 2- metric space through weak **commutativity and Rotativity of maps. The result is an extension from metric space to 2-metric space settings.*

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Key words: *fixed point, 2- metric space, weak** commuting mapping, Rotativity of maps.*

INTRODUCTION

The notion of 2-metric space was introduced by *Gahler* [1] in 1963 as a generalization of area function for Euclidean triangles. Many fixed point theorems were established by various authors like *Brouwer*, *Banach*, *Schauder* etc. A point $x \in X$ is said to be a *fixed point* of a self-map $f : X \to X$ if $f(x) = x$, where X is a non- empty set. Theorems concerning fixed points of self-maps are known as fixed point theorems. Most of the fixed point theorems were proved for contraction mappings. It is well known that every contraction on a metric space is continuous. The converse is not necessarily true. The identity mapping on [0, 1] simply serves the counter example.

In this present work we consider Weak ** Commuting and Rotative self maps on a 2-metric space. Let T_1 and T_2 be two mappings from a metric space (X, d) into itself. T_1 and T_2 are said to commute if $T_1T_2x = T_2T_1x$, for all x in X. *Sessa* [5] introduced the concept of weak commutativity in metric spaces. In subsequent years the condition of weak commutativity was again made weaker. Weak* commutativity was introduced in metric space. In recent years weak** commutativity has been introduced and some theorems have been established. The existence of fixed point for weak**commutative self maps in 2-metric space are studied.

In this research article we present the concepts of weak** commutativity and Rotativity maps in 2-metric space.

1. PRELIMINARIES

In this section we define weak** commutativity, Idempotent maps and Rotative.

Definition-1.1: Two self maps A and S of a 2-metric space (X, d) are called *weak** commutative* if
$(1)\, A(x) \subset S(x)$ and
$(2)\, d\left(A^2S^2x, S^2A^2x, a\right) \le d\left(A^2Sx, SA^2x, a\right) \le d\left(AS^2x, S^2Ax, a\right) \le d\left(ASx, SAx, a\right) \le d\left(A^2x, S^2x, a\right)$
For all x ,a in X

Definition-1.2: A map $T : X \to X$ is called *idempotent*, if $T^2 = T$. We note that if the mappings are idempotent i.e. $A^2 = A$, $S^2 = S$ then our definition of weak** commutating reduces to weak commutating pair of mappings $\{S, A\}$.

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Definition-1.3: Let X be a 2-metric space and let T and I be mapping of X into itself.
The map T is called rotative with respect to I if $d\left(Tx, I^2x, a\right) \le d\left(Ix, T^2x, a\right)$ for all x in X and every a in X. Clearly if T and I are Idempotent maps, then definition is obvious.

2. COMMON FIXED POINT THEOREMS FOR A WEEK ** COMMUTING PAIR OF MAPPINGS

In this section, we have some results on common fixed points for Three self maps of a 2- complete metric space using the concept of week **commuting maps and Rotativity of maps.

Theorem 2.1: Let S, T and I be three Self mapping of complete 2-metric space (X, d) with d continuous such that for all x, y, a in X either

$$(1)\, d\left(S^2x, T^2y, a\right) \le \frac{d\left(I^2x, S^2x, a\right)d\left(I^2y, T^2y, a\right) + \beta d\left(I^2x, T^2y, a\right)d\left(I^2y, S^2x, a\right)}{d\left(I^2x, S^2x, a\right) + d\left(I^2y, T^2y, a\right)}$$

if $d\left(I^2x, S^2x, a\right) + d\left(I^2y, T^2y, a\right) \ne 0$

Where $1 \prec \alpha \prec 2$ and $\beta \ge 0$ or

$$(2)\, d\left(S^2x, T^2y, a\right) = 0 \text{ if } d\left(I^2x, S^2x, a\right) + d\left(I^2y, T^2y, a\right) = 0$$

Suppose that the range of I^2 contains the range of S^2 and T^2. If either

(A_1) I^2 is continuous, I is weak**commutating with S and T is rotative with respect to I or,
(A_2) I^2 is continuous, I is weak**commutating with T and S is rotative with respect to I or,
(A_3) S^2 is continuous, S is weak**commutating with I and T is rotative with respect to S or,
(A_4) T^2 is continuous, T is weak**commutating with I and S is rotative with respect to T.
Then S, T and I have a unique common fixed point z. further z is the unique common point of S and I and T and I.

Proof: Let x_0 be an arbitrary point in X.
Since the range of I^2 contains the range of S^2.

Let x_1 be a point in X Such that $S^2x_0 = I^2x_1$.

Since the range of I^2 contains the range of T^2

We can choose a point x_2 in X such that $T^2x_1 = I^2x_2$.

In general, having chosen the point x_{2n} such that

$$T^2x_{2n+1} = I^2x_{2n+2}$$
$$S^2x_{2n} = I^2x_{2n+1}$$

For $n = 0, 1, 2, 3........$

Put $d_{2n-1} = d\left(T^2x_{2n-1}, S^2x_{2n}, a\right)$ and $d_{2n} = d\left(S^2x_{2n}, T^2x_{2n+1}, a\right)$

For n= 1, 2, ...

Now we distinguish the three cases:

Case-I: Let $d_{2n-1} \ne 0$ and $d_{2n} \ne 0$ for n =1, 2 …then we have,

$$d_{2n-1} + d_{2n} = d\left(T^2x_{2n-1}, S^2x_{2n}, a\right) + d\left(S^2x_{2n}, T^2x_{2n+1}, a\right) \ne 0 \text{ for n=1, 2,..........}$$

Using inequality (1) we then have

$$d_{2n} = d\left(S^2x_{2n}, T^2x_{2n+1}, a\right)$$

$$\le \frac{\alpha d\left(T^2x_{2n-1}, S^2x_{2n}, a\right)d\left(S^2x_{2n}, T^2x_{2n+1}, a\right) + \beta d\left(T^2x_{2n-1}, T^2x_{2n+1}, a\right)d\left(S^2x_{2n}, S^2x_{2n}, a\right)}{d\left(T^2x_{2n-1}, S^2x_{2n}, a\right) + d\left(S^2x_{2n}, T^2x_{2n+1}, a\right)}$$

$$d_{2n} = \frac{\alpha d_{2n-1}.d_{2n}}{d_{2n-1} + d_{2n}}$$

Then $\dfrac{d_{2n}}{d_{2n}} \le \dfrac{\alpha d_{2n-1}}{d_{2n-1} + d_{2n}}$

$$\Rightarrow d_{2n} \prec \alpha d_{2n-1} - d_{2n-1}$$

$$= (\alpha - 1)d_{2n-1}$$

$$= cd_{2n-1}$$

$$\Rightarrow d_{2n} \le cd_{2n-1}$$

So, $d\left(S^2x_{2n}, T^2x_{2n+1}, a\right) = \left\{S^2x_0, T^2x_1, S^2x_2 ....T^2x_{2n-1}, S^2x_{2n}, T^2x_{2n+1} ....\right\}$ (3)

For n = 1, 2…where $c = (\alpha - 1)$

Similarly it can be proved that

$d\left(T^2x_{2n-1}, S^2x_{2n}, a\right) = d_{2n-1} \le cd_{2n-2} = cd\left(S^2x_{2n-1}, T^2x_{2n-1}, a\right)$ for n = 1, 2…

and since $0 \prec c \prec 1$. it follows that the sequence

$$\left\{S^2x_0, T^2x_1, S^2x_2 ....T^2x_{2n-1}, S^2x_{2n}, T^2x_{2n+1} ....\right\} \tag{4}$$

is a Cauchy sequence in the complete 2-metric space and so has a limit u in X.

Hence the sequence

$\left\{S^2x_{2n}\right\} = \left\{I^2x_{2n+1}\right\} and \left\{T^2x_{2n-1}\right\} = \left\{I^2x_{2n}\right\}$ Converge to the point u because they are subsequence of the sequence (4)

Suppose first of all that I^2 is continuous, then the sequence $\left\{I^4x_n\right\} and \left\{I^2S^2x_{2n}\right\}$

Converge to point I^2u.

if I weak**commutes with S, we have

$$d\left(S^2I^2x_{2n}, I^2u, a\right) \le d\left(S^2I^2x_{2n}, I^2u, I^2S^2x_{2n}\right) + d\left(S^2I^2x_{2n}, I^2S^2x_{2n}, a\right) + d\left(I^2S^2x_{2n}, I^2u, a\right)$$

$$\le d\left(S^2I^2x_{2n}, I^2u, I^2u\right) + d\left(S^2I^2x_{2n}, I^2u, a\right) + d\left(I^2u, I^2u, a\right)$$

Which implies on letting n tends to infinity that the sequence { $S^2I^2x_{2n}$} also converges to I^2u.

Now we claim that $T^2u = I^2u$. Supposed not, then we have $d\left(I^2u, T^2u, a\right) \succ 0$ and using inequality (1), we obtain

$$d\left(S^2I^2x_{2n}, T^2u, a\right) \le \frac{\alpha d\left(I^4x_{2n}, S^2I^2x_{2n}, a\right)d\left(I^2u, T^2u, a\right) + \beta d\left(I^4x_{2n}, T^2u, a\right)d\left(T^2u, S^2I^2x_{2n}, a\right)}{d\left(I^4x_{2n}, S^2I^2x_{2n}, a\right) + d\left(I^2u, T^2u, a\right)}$$

Letting $n \to \infty$ we deduce that $d\left(I^2u, T^2u, a\right) \le 0$, a contradiction,

Now suppose that $S^2u \ne T^2u$, then

$$d\left(S^2u, T^2u, a\right) \le (\alpha + \beta)\frac{d\left(I^2u, S^2u, a\right).d\left(I^2u, T^2u, a\right)}{d\left(I^2u, S^2u, a\right) + d\left(I^2u, T^2u, a\right)} = 0$$

A contradiction.

Thus $I^2u = S^2u = T^2u$.

A similar conclusion is obtained if I is weak**commute with T.

Let us suppose that S^2 is continuous instead of I^2. Then the sequence $\left\{S^4x_{2n}\right\}$ and $\left\{S^2I^2x_{2n}\right\}$ converge to a point S^2u.

Since S weak**commute with I. we have that the sequence $\{I^2 S^2 x_{2n}\}$ also converges to S^2u.

Since the range of I^2 contains the range of S^2, there exists a point u_1 such that $I^2u_1 = S^2u$. Then If $T^2u \neq S^2u = I^2u_1$, we have

$$d\left(S^4x_{2n},T^2u_1,a\right) \le \frac{\alpha d\left(I^2S^2x_{2n},S^2S^2x_{2n},a\right)d\left(I^2u,T^2u_1,a\right)+\beta d\left(I^2S^2x_{2n},T^2u_1,a\right)d\left(I^2u_1,S^2S^2x_{2n},a\right)}{d\left(I^2S^2x_{2n},S^2S^2x_{2n},a\right)+d\left(I^2u_1,T^2u_1,a\right)}$$

When $n \to \infty$ we have

$$d\left(S^2u,T^2u_1,a\right) \le \frac{\beta d\left(I^2u,T^2u_1,a\right)d\left(I^2u_1,S^2u,a\right)}{d\left(I^2u_1,T^2u_1,a\right)}$$

Which implies that $d\left(S^2u,T^2u_1,a\right) \le 0$, a contradiction.

Thus $S^2u = T^2u_1 = I^2u_1$.

Now suppose that

$S^2u_1 \neq T^2u_1 = I^2u_1$, then

We have

$$d\left(S^2u_1,T^2u_1,a\right) \le \frac{(\alpha+\beta)d\left(I^2u_1,S^2u_1,a\right)d\left(I^2u_1,T^2u_1,a\right)}{d\left(I^2u_1,S^2u_1,a\right)+d\left(I^2u_1,T^2u_1,a\right)} = 0,$$

A contradiction and so $S^2u_1 = T^2u_1 = I^2u_1$.

A similar conclusion is achieved if one assumes that T^2 is continuous and T is weak**commutating with I.

Case-II: Let $d_{2n-1} = 0$ for some n.

Then $I^2x_{2n} = T^2x_{2n-1} = S^2x_{2n}$.

We claim that $I^2x_{2n} = T^2x_{2n}$.

Since otherwise if $d\left(I^2x_{2n},T^2x_{2n},a\right) \succ 0$

Inequality (1) implies

$$\begin{aligned}0 \prec d\left(I^2x_{2n},T^2x_{2n},a\right) &= d\left(S^2x_{2n},T^2x_{2n},a\right)\\ &\le \frac{\alpha d\left(I^2x_{2n},S^2x_{2n},a\right)d\left(I^2x_{2n},T^2x_{2n},a\right)+\beta d\left(I^2x_{2n},T^2x_{2n},a\right)d\left(I^2x_{2n},S^2x_{2n},a\right)}{d\left(I^2x_{2n},S^2x_{2n},a\right)+d\left(I^2x_{2n},T^2x_{2n},a\right)} = 0\end{aligned}$$

A contradiction.

Thus $I^2x_{2n} = S^2x_{2n} = T^2x_{2n}$.

Case-III: Let $d_{2n} = 0$ for some n. then $I^2x_{2n+1} = S^2x_{2n} = T^2x_{2n+1}$.

And reasoning as in *case (II)*, $I^2x_{2n+1} = S^2x_{2n+1} = T^2x_{2n+1}$

Therefore in all cases it follows, there exists a point u such that $I^2u = S^2u = T^2u$.

If I week**commutes with S, we have

$$d\left(S^2Iu,IS^2u,a\right) \le d\left(SI^2u,I^2Su,a\right) \le d\left(SIu,ISu,a\right) \le d\left(S^2u,I^2u,a\right) = 0,$$

which implies that

$$S^2Iu = IS^2u, SI^2u = I^2Su, SIu = ISu \text{ and so } I^2Su = S^3u \quad (5).$$

Thus $d\left(I^2Su, S^2Su, a\right) + d\left(I^2u, T^2u, a\right) = 0$

And using Condition (II), we deduce that

$$I^2u = S^2Su = SI^2u = T^2u.$$

It follows $I^2u = z$ is a fixed point of S.

Further $d\left(I^2Iu, S^2Iu, a\right) + d\left(I^2u, T^2u, a\right) = 0$

And using (II), we deduce that $Iz = S^2Iu = IS^2u = T^2u = z$

Using inequality (I), on the assumption that

$$T^2z \neq z$$

We have $d\left(z, T^2z, a\right) = d\left(S^2z, T^2z, a\right)$

$$\leq \frac{(\alpha+\beta)\, d\left(I^2z, S^2z, a\right) d\left(I^2z, T^2z, a\right)}{d\left(I^2z, S^2z, a\right) + d\left(I^2z, T^2z, a\right)} = 0$$

A contradiction,

So, $T^2z = z$.

Now using the rotativity of T with respect to I (or with respect to S)

We have $d\left(Tz, z, a\right) = d\left(Tz, I^2z, a\right) \leq d\left(Iz, T^2z, a\right) = d\left(z, z, a\right) = 0$

And so z is a common fixed point of I , S and T .

Similarly it can be proved if we assumed that I week**commutes with T and rotativity of S with respect to I (or with respect to T).

Now suppose that z_1 is another common fixed point of I and S . then

$d\left(I^2z, S^2z_1, a\right) + d\left(I^2z, T^2z, a\right) = 0$ and condition (2) implies that

$$z_1 = S_{\ 1} = zS^2z_1 = T^2z = z.$$

We can similarly prove that z is the unique common fixed point of I and T .

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