Applications of $b^\sharp$-Open set

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ABSTRACT

Using the concept of $b^\sharp$-open sets we introduce and study topological properties of $b^\sharp$-limit points, $b^\sharp$-derived sets, $b^\sharp$-closure, $b^\sharp$-border, $b^\sharp$-Frontier and $Db^\sharp$- exterior and discuss their relations with one another.

Keywords: $b^\sharp$-limit points, $b^\sharp$-derived sets, $b^\sharp$-closure, $b^\sharp$-border, $b^\sharp$-Frontier and $Db^\sharp$- exterior.

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1. INTRODUCTION

In the year 1996, Andrijivic introduced [1] and studied b-open sets. Following this Usha Paraeswari et.al [2] introduced the concept of $b^\sharp$-open sets. In this paper we introduce the notions of $b^\sharp$-limit points, $b^\sharp$-derived sets, $b^\sharp$-closure, $b^\sharp$-border, $b^\sharp$-Frontier and $Db^\sharp$- exterior by using the concept of $b^\sharp$-open set.

2. PRELIMINARIES

Throughout this paper X denotes a topological space on which no separation axiom is assumed. For any subset A of X, $cl(A)$ denotes the closure of A and $int(A)$ denotes the interior of A in the topological space X. Further $X \setminus A$ denotes the complement of A in X. The following definitions and results are very useful in the subsequent sections.

Definition 2.1 [2]: A subset A of a space X is called $b^\sharp$-open if $A = cl(int(A)) \cup int(cl(A))$ and their complement is called $b^\sharp$-closed. That is A is $b^\sharp$-closed if $A = cl(int(A)) \cap int(cl(A))$.

Definition 2.2[3]: The $b^\sharp$-interior of A, denoted by $b^\sharp-int(A)$, is defined to be the union of all $b^\sharp$-open sets contained in A. That is $b^\sharp-int(A) = \bigcup \{B: B \subseteq A \text{ and } B \text{ is } b^\sharp\text{-open}\}$.

The next Lemma gives the properties of $b^\sharp$-interior.

Lemma 2.3[3]:

(i) $b^\sharp-int(\emptyset) = \emptyset$.

(ii) $b^\sharp-int(X) = X$.

(iii) $b^\sharp-int(A) \subseteq A$.

(iv) $b^\sharp$-interior of a set A is not always $b^\sharp$-open.

(v) If A is $b^\sharp$-open then $b^\sharp-int(A) = A$.

Lemma 2.4[3]: Let X be a space. Then for any two sub sets A and B of X we have

(i) If $A \subseteq B$ then $b^\sharp-int(A) \subseteq b^\sharp-int(B)$.

(ii) $b^\sharp-int(b^\sharp-int(A)) = b^\sharp-int(A)$.

(iii) $b^\sharp-int(A \setminus B) \subseteq b^\sharp-int(A) \cap b^\sharp-int(B)$.

(iv) $b^\sharp-int(A \cup B) \supseteq b^\sharp-int(A) \cup b^\sharp-int(B)$.

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Definition 2.5[3]: The $b^s$-closure of $A$, denoted by $b^s$-cl$(A)$, is defined to be the intersection of all $b^s$-closed sets containing $A$. That is $b^s$-cl$(A)$=$\bigcap \{B: A \subseteq B$ and $B$ is $b^s$-closed$\}$.

Lemma 2.6[3]: Let $X$ be a space. Then for any sub set $A$ of $X$ we have
(i) $X \setminus b^s$-int$(A)$ = $b^s$-cl$(X \setminus A)$.
(ii) $X \setminus b^s$-cl$(A)$ = $b^s$-int$(X \setminus A)$.

Remarks 2.7[3]:
(i) $b^s$-cl$(\emptyset)$ = $\emptyset$.
(ii) $b^s$-cl$(X)$ = $X$.
(iii) $A$ is $b^s$-closed if and only if $b^s$-cl$(A)$ = $A$.
(iv) $b^s$-closure of a set $A$ is not always $b^s$-closed.
(v) If $A$ is $b^s$-closed then $b^s$-cl$(A)$ = $A$.

Lemma 2.8[3]: Let $X$ be a space. Then for any two sub sets $A$ and $B$ of $X$ we have
(i) If $A \subseteq B$ then $b^s$-cl$(A)$ $\subseteq$ $b^s$-cl$(B)$.
(ii) $b^s$-cl$(b^s$-cl$(A))$ = $b^s$-cl$(A)$.
(iii) $b^s$-cl$(A \cup B)$ $\supseteq$ $b^s$-cl$(A) \cup b^s$-cl$(B)$.
(iv) $b^s$-cl$(A \cap B)$ $\subseteq$ $b^s$-cl$(A) \cap b^s$-cl$(B)$.

3. $b^s$- limit points

Definition 3.1: Let $A$ be a subset of a topological space $(X, \tau)$ and $x$ be a point of $X$. A point $x \in X$ is said to be a $b^s$-limit point of $A$ if every $b^s$-neighborhood of $x$ intersects $A$ in some point other than $x$ itself. That is $U \cap (A \setminus \{x\}) \neq \emptyset$ for all $U \in b^s$-$O(X, \tau)$.

The set of all $b^s$-limit points of $A$ is called the $b^s$-derived set of $A$ and is denoted by $D_{b^s}$$(A)$.

Remark 3.2: A subset $A$ of $X$, a point $x \in X$ is not a $b^s$-limit point of $A$ if and only if there exists a $b^s$-open set $G$ in $X$ such that $x \in G$ and $G \cap (A \setminus \{x\}) = \emptyset$ that is $x \in G$ and $G \cap A = \emptyset$ or $G \cap A = \{x\}$ that is $x \in G$ and $G \cap A \subseteq \{x\}$.

Theorem 3.3: Let $\tau_1$ and $\tau_2$ be topologies on $X$ such that $\tau_1$ $\subseteq$ $\tau_2$. For any subset $A$ of $X$, every $b^s$-limit point of $A$ with respect to $\tau_2$ is a $b^s$-limit point of $A$ with respect to $\tau_1$.

Proof: Let $x$ be a $b^s$-limit point of $A$ with respect to $\tau_2$. Then $U \cap (A \setminus \{x\}) \neq \emptyset$ for every $U \in \tau_2$ such that $x \in U$. But $\tau_1$ $\subseteq$ $\tau_2$, we have $U \cap (A \setminus \{x\}) \neq \emptyset$ for every $U \in \tau_1$ such that $x \in U$. Hence $x$ is a $b^s$-limit point of $A$ with respect to $\tau_1$.

Theorem 3.4: For any sub sets $A$ and $B$ of $(X, \tau)$ the following holds.
(i) If $A \subseteq B$ then $D_{b^s}$$(A)$ $\subseteq$ $D_{b^s}$$(B)$.
(ii) $D_{b^s}$$(A \cup B)$ $\subseteq$ $D_{b^s}$$(A) \cup D_{b^s}$$(B)$.
(iii) $D_{b^s}$$(A \cap B)$ $\subseteq$ $D_{b^s}$$(A) \cap D_{b^s}$$(B)$.
(iv) $D_{b^s}$$(D_{b^s}$$(A))/A$ $\subseteq$ $D_{b^s}$$(A)$.
(v) $D_{b^s}$$(A \cup D_{b^s}$$(A))$ $\subseteq$ $A \cup D_{b^s}$$(A)$.

Proof: Let $x \in D_{b^s}$$(A)$ and let $U \in \tau$ with $x \in U$. Then $U \cap (A \setminus \{x\}) \neq \emptyset$. Since $A \subseteq B$, we have $U \cap (B \setminus \{x\}) \neq \emptyset$. This implies that $x \in D_{b^s}$$(B)$. This proves (i).

Now to prove (ii). Since $A \cup B \subseteq A \cup B$ and $B \subseteq A \cup B$. Using (i), $D_{b^s}$$(A \cup B)$ $\subseteq$ $D_{b^s}$$(A \cup B)$ and $D_{b^s}$$(B)$ $\subseteq$ $D_{b^s}$$(A \cup B)$ that is $D_{b^s}$$(A \cup B)$ $\subseteq$ $D_{b^s}$$(A \cup B)$. This proves (ii).

Next we have to prove (iii). Since $A \cap B \subseteq A \cap B$ and $B \subseteq A \cap B$. Using (i), $D_{b^s}$$(A \cap B)$ $\subseteq$ $D_{b^s}$$(A \cap B)$ and $D_{b^s}$$(B)$ $\subseteq$ $D_{b^s}$$(A \cap B)$ that is $D_{b^s}$$(A \cap B) \subseteq D_{b^s}$$(B)$. Thus we get $D_{b^s}$$(A \cap B)$ $\subseteq$ $D_{b^s}$$(A) \cap D_{b^s}$$(B)$. Hence (iii). Next to prove (iv). Let $x \in D_{b^s}$$(D_{b^s}$$(A))/A$ and let $U \in \tau$ with $x \in U$. Then $U \cap (D_{b^s}$$(A \setminus \{x\}) \neq \emptyset$. Let $y \in U \cap (D_{b^s}$$(A \setminus \{x\})$. Then $y \in U$ and $y \in D_{b^s}$$(A)$ and $U \cap (A \setminus \{y\}) \neq \emptyset$. If we take $z \in U \cap (A \setminus \{y\})$, then $x \neq z$ because $x \notin A$. Hence $U \cap (A \setminus \{x\}) \neq \emptyset$. Therefore $x \in D_{b^s}$$(A)$. Hence (iv).
Next to prove (v). Let \( x \in D b^\# (A \cup D b^\# (A)) \). If \( x \in A \), the result is obvious. Assume that \( x \notin A \). Then \( U \cap (A \cup D b^\# (A)) \neq \emptyset \) for all \( U \in \tau^{b^\#} \) with \( x \in U \). Hence \( U \cap (A \cup D b^\# (A)) \neq \emptyset \). The first case implies \( x \in D b^\# (A) \). Then the second case implies \( x \in D b^\# (D b^\# (A)) \). Since \( x \notin A \), by (iv) \( x \in D b^\# (D b^\# (A)) / A \subseteq D b^\# (A) \). This proves (v).

The reverse inclusion of (i) and the converse of (ii), (iii) and (iv) are not true as shown by the following examples.

**Example 3.5:** Let \( X = \{a, b, c, d\} \). Consider the topology \( \tau = \{\emptyset, X, \{a, b, c\}, \{a\}, \{b, c\}\} \). The \( b^\# \)-open sets are \( \emptyset, X, \{d, b, c\}, \{a, d\} \) and the \( b^\# \)-closed sets are \( \emptyset, X, \{a\}, \{b, c\} \). Let \( A = \{a, d\} \) and \( B = \{b, c\} \). Then \( D b^\# (A) = \{a, b, c\} \) and \( D b^\# (B) = \{b, c\} \). So \( D b^\# (B) \subseteq D b^\# (A) \) but \( B \not\subseteq A \).

Also \( D b^\# (A \cup B) = \{b, d\} \). Again let \( A_1 = \{a, b\} \) and \( B_1 = \{a, c\} \). Then \( D b^\# (A_1) = \{c, d\} \) and \( D b^\# (B_1) = \{b, d\} \). Therefore \( D b^\# (A) \cap D b^\# (B) \subseteq D b^\# (A \cap B) \).

**Theorem 3.6:** Let \( A \) be a sub set of \((X, \tau)\) and \( x \in X \). Then the following are equivalent.

(i) If for all \( U \in \tau^{b^\#} \), \( x \in U \) then \( A \cap U \neq \emptyset \).

(ii) \( x \in b^\# \text{-cl}(A) \).

**Proof:** Suppose (i) holds. If \( x \notin b^\# \text{-cl}(A) \), then there exists a \( b^\# \)-closed set \( F \) such that \( A \subseteq F \) and \( x \notin F \). Hence \( X/F \) is a \( b^\# \)-open set containing \( x \) and \( A \cap (X/F) = \emptyset \). This is a contradiction to our assumption. This proves (i) \( \Rightarrow \) (ii). The proof of (ii) \( \Rightarrow \) (i) is from the Definition 3.1.

**Corollary 3.7:** For any sub set \( A \) of \( X \) we have \( D b^\# (A) \subseteq b^\# \text{-cl}(A) \).

**Proof:** Let \( x \in D b^\# (A) \). By Definition 3.1, there exists \( x \in U \) such that \( U \cap (A/{x}) \neq \emptyset \). So by Theorem 3.6, \( x \in b^\# \text{-cl}(A) \).

**Theorem 3.8:** For any sub set \( A \) of \( X \), \( b^\# \text{-cl}(A) = A \cup D b^\# (A) \).

**Proof:** Let \( x \in b^\# \text{-cl}(A) \). Assume that \( x \notin A \) and let \( U \in \tau^{b^\#} \) with \( x \in U \). Then \( U \cap (A/{x}) \neq \emptyset \) and so \( x \in D b^\# (A) \). Hence \( b^\# \text{-cl}(A) \subseteq A \cup D b^\# (A) \). Conversely since \( A \subseteq b^\# \text{-cl}(A) \) and \( D b^\# (A) \subseteq b^\# \text{-cl}(A) \). This proves the theorem.

**Definition 3.9[3]:** A space \( X \) is said to be \( b^\# \)-closed preserving if every \( b^\# \)-closure of a subset is \( b^\# \)-closed.

**Theorem 3.10:** Let \( A \) and \( B \) be a sub sets of \((X, \tau)\). If \( A \) is \( b^\# \)-closed preserving then \( b^\# \text{-cl}(A \cup B) \subseteq A \cup b^\# \text{-cl}(B) \).

**Proof:** If \( A \) is \( b^\# \)-closed preserving then \( b^\# \text{-cl}(A) = A \) and so \( b^\# \text{-cl}(A \cup B) \subseteq b^\# \text{-cl}(A) \cup b^\# \text{-cl}(B) = A \cup b^\# \text{-cl}(B) \).

**Theorem 3.11:** For every sub set \( A \) of \( X \) we have \( A \) is \( b^\# \)-closed then \( D b^\# (A) \subseteq A \).

**Proof:** Assume that \( A \) is \( b^\# \)-closed. Let \( x \in X/A \). Then \( X/A \) is \( b^\# \)-open, \( (X/A) \cap (A/{x}) = \emptyset \). Therefore \( x \) is not a \( b^\# \)-limit point of \( A \). Hence \( D b^\# (A) \subseteq A \).

**Corollary 3.12:** The converse of the above theorem is true if \( A \) is \( b^\# \)-closed preserving.

**Theorem 3.13:** Let \( A \) be a sub set of \((X, \tau)\). If a point \( x \in A \) is a \( b^\# \)-limit point of \( A \setminus B \) then \( x \) is also a \( b^\# \)-limit point of \( A \).

**Proof:** If \( x \) is a \( b^\# \)-limit point of \( A \) then by Definition 3.1, there exists a \( b^\# \)-open set \( U \) such that \( x \in U \) and \( U \cap [(A/{x})/{x}] \neq \emptyset \). That is \( x \) is a \( b^\# \)-limit point of \( A \).

4. \( b^\# \)-interior, \( b^\# \)-border and \( b^\# \)-Frontier

**Definition 4.1:** Let \( A \) be a sub set of a topological space \((X, \tau)\). A point \( x \in X \) is called a \( b^\# \)-interior point of \( A \) if there exists a \( b^\# \)-open set \( U \) such that \( x \in U \subseteq A \). The set of all \( b^\# \)-interior points of \( A \) is called \( b^\# \)-interior of \( A \) and is denoted by \( b^\# \text{-int}(A) \).
**Definition 4.2:** For any sub set A of X, the set $b^#\text{-}\text{b}(A) = A/ b^#\text{-}\text{int}(A)$ is called the $b^#\text{-}\text{border}$ of A and the set $b^#\text{-}\text{Fr}(A) = b^#\text{-}\text{cl}(A)/ b^#\text{-}\text{int}(A)$ is called the $b^#\text{-}\text{Frontier}$ of A.

**Remark 4.3:** If A is a $b^#\text{-}\text{closed}$ preserving sub set of X then $b^#\text{-}\text{b}(A) = b^#\text{-}\text{Fr}(A)$.

**Proposition 4.4:** For a sub set A of X the following statements holds.

(i) $b^#\text{-}\text{int}(A) \cap b^#\text{-}\text{b}(A) = \phi$.

(ii) $b^#\text{-}\text{int}(b^#\text{-}\text{b}(A)) = \phi$.

(iii) $b^#\text{-}\text{b}(b^#\text{-}\text{b}(A)) = b^#\text{-}\text{b}(A)$.

(iv) $b^#\text{-}\text{b}(A) = A \cap b^#\text{-}\text{cl}(X/A)$.

**Proof:** By Definition of 4.2, (i) holds. Now to prove (ii).

If $x \in b^#\text{-}\text{int}(b^#\text{-}\text{b}(A))$ then $x \in b^#\text{-}\text{b}(A) \subseteq A$ and $x \in b^#\text{-}\text{int}(A)$. Thus $x \in b^#\text{-}\text{int}(A) \cap b^#\text{-}\text{b}(A) = \phi$ which is a contradiction. Hence $b^#\text{-}\text{int}(b^#\text{-}\text{b}(A)) = \phi$. This proves (ii).

Now to prove (iii). By Definition 4.2, $b^#\text{-}\text{b}(b^#\text{-}\text{b}(A)) = b^#\text{-}\text{b}(A)/ b^#\text{-}\text{int}(b^#\text{-}\text{b}(A))$.

Using (ii), $b^#\text{-}\text{b}(b^#\text{-}\text{b}(A)) = b^#\text{-}\text{b}(A)$. This proves (iii). Now to prove (iv). Using Definition 4.2, $b^#\text{-}\text{b}(A) = A/ b^#\text{-}\text{int}(A) = A/ (X/b^#\text{-}\text{cl}(X/A)) = A \cap b^#\text{-}\text{cl}(X/A)$. This proves (iv).

**Theorem 4.5:** For a sub set A of $(X, \tau)$, the following conditions holds.

(i) $b^#\text{-}\text{int}(A) \cap b^#\text{-}\text{Fr}(A) = \phi$.

(ii) $b^#\text{-}\text{b}(A) \subseteq b^#\text{-}\text{Fr}(A)$.

(iii) $b^#\text{-}\text{Fr}(A) = b^#\text{-}\text{cl}(A)/ b^#\text{-}\text{int}(A) = (A/ b^#\text{-}\text{int}(A)) \cup (Db^#\text{-}(A)/ b^#\text{-}\text{int}(A))$.

(iv) $b^#\text{-}\text{Fr}(A) = b^#\text{-}\text{cl}(A) \cap b^#\text{-}\text{cl}(X/A)$.

(v) $b^#\text{-}\text{Fr}(A) = b^#\text{-}\text{Fr}(X/A)$.

(vi) $b^#\text{-}\text{Fr}(b^#\text{-}\text{int}(A)) \subseteq b^#\text{-}\text{Fr}(A)$.

(vii) $b^#\text{-}\text{int}(A) = A/ b^#\text{-}\text{Fr}(A)$.

**Proof:** Using Definition 4.2, $b^#\text{-}\text{int}(A) \cap b^#\text{-}\text{Fr}(A) = b^#\text{-}\text{int}(A) \cap (b^#\text{-}\text{cl}(A)/ b^#\text{-}\text{int}(A)) = \phi$. This proves (i). Now to prove (ii).

Since $A \subseteq b^#\text{-}\text{cl}(A)$ we have $b^#\text{-}\text{b}(A) = A/ b^#\text{-}\text{int}(A) \subseteq b^#\text{-}\text{cl}(A)/ b^#\text{-}\text{int}(A) = b^#\text{-}\text{Fr}(A)$. This proves (ii). Now to prove (iii).

By Definition 4.2, $b^#\text{-}\text{Fr}(A) = b^#\text{-}\text{cl}(A)/ b^#\text{-}\text{int}(A) = (A \cup Db^#\text{-}(A)/ b^#\text{-}\text{int}(A)) = (A/ b^#\text{-}\text{int}(A)) \cup (Db^#\text{-}(A)/ b^#\text{-}\text{int}(A)) = b^#\text{-}\text{b}(A) \cup (Db^#\text{-}(A)/ b^#\text{-}\text{int}(A))$. Hence (iii) is proved. Now to prove (iv).

Using Lemma 2.6, we have $b^#\text{-}\text{cl}(A) \cap b^#\text{-}\text{cl}(X/A) = b^#\text{-}\text{cl}(A) \cap (X/b^#\text{-}\text{cl}(A)) = b^#\text{-}\text{cl}(A) = b^#\text{-}\text{Fr}(A)$. This proves (iv). Using (iv), $b^#\text{-}\text{Fr}(X/A) = b^#\text{-}\text{cl}(X/A) \cap b^#\text{-}\text{cl}(X/A) = b^#\text{-}\text{Fr}(A)$. Hence (v) is proved.

Using Lemma 2.4, $b^#\text{-}\text{Fr}(b^#\text{-}\text{int}(A)) = b^#\text{-}\text{cl}(b^#\text{-}\text{int}(A))/ b^#\text{-}\text{int}(b^#\text{-}\text{int}(A)) \subseteq b^#\text{-}\text{cl}(A)/ b^#\text{-}\text{int}(A) = b^#\text{-}\text{Fr}(A)$. This proves (vi). Now $A/ b^#\text{-}\text{Fr}(A) = A/ (b^#\text{-}\text{cl}(A)/ b^#\text{-}\text{int}(A)) = A \cap ((X/b^#\text{-}\text{cl}(A)) \cup b^#\text{-}\text{int}(A)) = \phi \cup (A \cup b^#\text{-}\text{int}(A)) = b^#\text{-}\text{int}(A)$. This completes the proof.

The converse of (ii) and (vi) of Theorem 4.5 is not true in general as seen in the following Example.

**Example 4.6:** Consider the same topological space in Example 3.5. Let $A = \{c\}$. Then $b^#\text{-}\text{Fr}(A) = \{b, c\}$, $b^#\text{-}\text{b}(A) = \{c\}$, $b^#\text{-}\text{int}(A) = \phi$ and $b^#\text{-}\text{Fr}(b^#\text{-}\text{int}(A)) = \phi$. Thus $b^#\text{-}\text{Fr}(A) \subseteq b^#\text{-}\text{b}(A)$ and $b^#\text{-}\text{Fr}(A) \subseteq b^#\text{-}\text{Fr}(b^#\text{-}\text{int}(A))$.

5. $b^#\text{-}\text{exterior}$

**Definition 5.1:** For a sub set A of $(X, \tau)$, the $b^#\text{-}\text{interior}$ of $X/A$ is called the $b^#\text{-}\text{exterior}$ of A and is denoted by $b^#\text{-}\text{ext}(A)$, that is $b^#\text{-}\text{ext}(A) = b^#\text{-}\text{int}(X/A)$.
Theorem 5.2: For sub sets $A$ and $B$ of $X$ the following assertions are valid.

(i) $b^s$-ext$(A) = X/ b^s$-cl$(A)$.

(ii) $b^s$-ext$(b^s$-ext$(A))= b^s$-int$(b^s$-cl$(A)) \supseteq b^s$-int$(A)$.

(iii) $A \subseteq B$ implies $b^s$-ext$(A) \subseteq b^s$-ext$(B)$.

(iv) $b^s$-ext$(A \cup B) \subseteq b^s$-ext$(A) \cap b^s$-ext$(B)$.

(v) $b^s$-ext$(A \cap B) \supseteq b^s$-ext$(A) \cup b^s$-ext$(B)$.

(vi) $b^s$-ext$(X) = \phi$, $b^s$-ext$(\phi) = X$.

(vii) $X = b^s$-int$(A) \cup b^s$-ext$(A) \cup b^s$-Fr$(A)$.

Proof: By Definition 5.1 and Lemma 2.6, $b^s$-ext$(A) = b^s$-int$(X/A) = X/b^s$-cl$(A)$. This proves (i). Now to prove (ii). Using Lemma 2.6, we get $b^s$-ext$(b^s$-ext$(A)) = b^s$-ext$(b^s$-int$(X/A)) = b^s$-ext$(b^s$-int$(X/b^s$-int$(X/A))) = b^s$-int$(b^s$-cl$(A)) \supseteq b^s$-int$(A)$. This proves (ii).

Now to prove (iii). Assume that $A \subseteq B$. Then $b^s$-ext$(B) = b^s$-int$(X/B) \subseteq b^s$-int$(X/A) = b^s$-ext$(A)$. Hence (iii) is proved.

Now to prove (iv), $b^s$-ext$(A \cup B) = b^s$-int$(X/(A \cup B)) = b^s$-int$(X/A \cap X/B) \subseteq b^s$-int$(X/A) \cap b^s$-int$(X/B) = b^s$-ext$(A) \cap b^s$-ext$(B)$. Hence (iv) is proved.

Now to prove (v), $b^s$-ext$(A \cap B) = b^s$-int$(X/A \cap X/B) = b^s$-int$(X/A) \cup b^s$-int$(X/B) = b^s$-ext$(A) \cup b^s$-ext$(B)$. Thus (v) is proved. Using Definition 5.1, (vi) is proved.

Now to prove (vii), $b^s$-int$(A) \cup b^s$-ext$(A) \cup b^s$-Fr$(A) = b^s$-int$(A) \cup (b^s$-cl$(A)/ b^s$-int$(A)) \cup b^s$-ext$(A) = (b^s$-int$(A) \cup b^s$-cl$(A))/\cup b^s$-ext$(A) = b^s$-cl$(A) \cup b^s$-cl$(X/A) = X$ since by Definition 4.2 and Lemma 2.6.

Examples can be easily constructed for the reverse inclusion of Theorem 5.2(iii) and (iv).

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