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A STUDY OF INFINITESIMAL HOLOMORPHICALLY PROJECTIVE TRANSFORMATIONS IN KAHLERIAN SUBMANIFOLDS WITH BOCHNER CURVATURE TENSOR

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ABSTRACT

Main purpose of the author is to study of Infinitesimal Holomorphically Projective Transformations (IHPT) in Kahlerian Submanifolds with Bochner curvature tensor and discuss about its origin. We established and defined some important properties, results and theorems. The present paper is also devoted for the study of complex conformal curvature tensor and complex conformal connection with Bochner curvature tensor. At last the complex conformal connection does not vanishing curvature tensor then the Bochner curvature tensor of the manifolds also does not vanish.

Key Words: *Holomorphically Kahlerian submanifolds, Bochner curvature tensor, Complex conformal connection, Ricci tensor and Einstein space.*

1. INTRODUCTION

S. Tachibana and S. Ichihara [14] have studied of Infinitesimal Holomorphically Projective Transformations in Kahlerian Manifolds. S. Tachibana [13] has discussed on the Bochner Curvature Tensor .Totally real Submanifolds of a Kahlerian Manifolds have been studied by K. Yano [7] and K. Yano [8] has defined complex conformal connection with vanishing Bochner Curvature tensor and defined the Bochner curvature tensor of the manifold vanishes.

In this paper main purpose of the author is to study of IHPT in Kahlerian submanifolds with Bochner curvature tensor and established some important properties, results and theorems.

Firstly we defined Kahlerian manifolds and Bochner curvature tensor [13]. We discussed and defined Kahlerian submanifolds and discuss for the study of some properties of Kahlerian submanifolds. We establish some results of the theory of flat totally real submanifolds of a Kahlerian manifolds. We have studied of complex conformal curvature tensor and complex conformal connections with Bochner curvature tensor.

Definition 1.1. Kahlerian Manifolds: An $n = 2m$ dimensional Kahlerian space K^n is a Riemannian space which admits a tensor field $\varphi_\lambda^{\ \mu}$ satisfying

$$\varphi_\alpha^{\ \lambda}\varphi_\mu^{\ \alpha} = -\delta_\mu^{\ \lambda}, \quad \varphi_{\lambda\mu} = -\varphi_{\mu\lambda}, \quad \left(\varphi_{\lambda\mu} = g_{\mu\alpha}\varphi_\lambda^{\ \alpha}\right) \text{ and } \quad \nabla_\nu \varphi_\lambda^{\ \mu} = 0$$

Where ∇_ν means the operator of covariant differentiation.

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We define Riemannian curvature tensor $R^{\kappa}_{\lambda\mu\nu}$ is

$$R^{\kappa}_{\lambda\mu\nu} = \partial_\lambda \begin{Bmatrix} \kappa \\ \mu\nu \end{Bmatrix} - \partial_\mu \begin{Bmatrix} \kappa \\ \lambda\nu \end{Bmatrix} + \begin{Bmatrix} \kappa \\ \lambda\alpha \end{Bmatrix}\begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix} - \begin{Bmatrix} \kappa \\ \mu\alpha \end{Bmatrix}\begin{Bmatrix} \alpha \\ \lambda\nu \end{Bmatrix}$$

and $R_{\mu\nu} = R^{\alpha}_{\alpha\mu\nu}$, $R = g^{\lambda\mu} R_{\lambda\mu}$ are Ricci tensor and the scalar curvature respectively.

It is well known that these tensors satisfy the following identities:

$$R^{\kappa}_{\alpha\mu\nu}\varphi^{\alpha}_{\lambda} = -R^{\kappa}_{\lambda\alpha\nu}\varphi^{\alpha}_{\mu}, \quad R^{\kappa}_{\lambda\mu\alpha}\varphi^{\alpha}_{\nu} = R^{\alpha}_{\lambda\mu\nu}\varphi^{\kappa}_{\alpha}, \quad \varphi^{\alpha}_{\lambda} R_{\alpha\mu} = -R_{\lambda\alpha}\varphi^{\alpha}_{\mu}, \quad \varphi^{\alpha}_{\lambda} R^{\kappa}_{\alpha} = R^{\alpha}_{\lambda}\varphi^{\kappa}_{\alpha},$$

$$\nabla_\alpha R^{\alpha}_{\lambda\mu\nu} = \nabla_\lambda R_{\mu\nu} - \nabla_\mu R_{\lambda\nu} \quad \text{And} \quad \nabla_\lambda R = 2\nabla_\alpha R^{\alpha}_{\lambda}.$$

If we define a tensor $S_{\mu\nu}$ by $S_{\mu\nu} = \varphi^{\alpha}_{\mu} R_{\alpha\nu}$, then we have

$$S_{\mu\nu} = -S_{\nu\mu}, \ \varphi^{\alpha}_{\lambda} S_{\alpha\nu} = -S_{\lambda\alpha}\varphi^{\alpha}_{\nu}, \ S_{\mu\nu} = -(1/2)\varphi^{\alpha\beta} R_{\alpha\beta\mu\nu} \quad \text{and} \quad 2\nabla_\alpha S^{\alpha}_{\lambda} = \varphi^{\alpha}_{\lambda}\nabla_\alpha R$$

The differential form $S = (1/2) S_{\lambda\mu} dx^{\lambda} \wedge dx^{\mu}$ is closed.

It follows that:

$$\varphi^{\alpha}_{\lambda}\nabla_\alpha S_{\mu\nu} = -\nabla_\mu R_{\nu\lambda} + \nabla_\nu R_{\mu\lambda}$$

It is also known as 2-form *S* is harmonic, where *R* is a constant.

Definition 1.2. Bochner Curvature Tensor: A tensor $K^{\kappa}_{\lambda\mu\nu}$ is defined by

$$K^{\kappa}_{\lambda\mu\nu} = R^{\kappa}_{\lambda\mu\nu} + \frac{1}{n+4}\left(R_{\lambda\nu}\delta^{\kappa}_{\mu} - R_{\mu\nu}\delta^{\kappa}_{\lambda} + g_{\lambda\nu}R^{\kappa}_{\mu} - g_{\mu\nu}R^{\kappa}_{\lambda} + S_{\lambda\nu}\varphi^{\kappa}_{\mu} - S_{\mu\nu}\varphi^{\kappa}_{\lambda} + \varphi_{\lambda\nu}S^{\kappa}_{\mu} - \varphi_{\mu\nu}S^{\kappa}_{\lambda} + 2S_{\lambda\mu}\varphi^{\kappa}_{\nu} + 2\varphi_{\lambda\mu}S^{\kappa}_{\nu}\right)$$
$$-\frac{R}{(n+2)(n+4)}\left(g_{\lambda\mu}\delta^{\kappa}_{\mu} - g_{\mu\nu}\delta^{\kappa}_{\lambda} + \varphi_{\lambda\mu}\varphi^{\kappa}_{\mu} - \varphi_{\mu\nu}\varphi^{\kappa}_{\lambda} + 2\varphi_{\lambda\mu}\varphi^{\kappa}_{\nu}\right)$$

Which is constructed formally from $C^{\kappa}_{\lambda\mu\nu}$ by taking account of the form arisen balance between $W^{\kappa}_{\lambda\mu\nu}$ *and* $P^{\kappa}_{\lambda\mu\nu}$. Then we can prove that the tensor $K_{\lambda\mu\nu\varpi} = g_{\kappa\varpi} K^{\kappa}_{\lambda\mu\nu}$ has components of the tensor given by S. Bochner with respect to complex local co-ordinates. Hence it is known as Bochner curvature tensor.

Remark-1: If we put $L_{\lambda\mu} = R_{\lambda\mu} - \frac{R}{2(n+2)} g_{\lambda\mu}, \quad M_{\lambda\mu} = \varphi^{\alpha}_{\lambda} L_{\alpha\mu} = S_{\lambda\mu} - \frac{R}{2(n+2)}\varphi_{\lambda\mu}$

and $K^{\kappa}_{\lambda\mu\nu}$ has the following form :

$$K^{\kappa}_{\lambda\mu\nu} = R^{\kappa}_{\lambda\mu\nu} + \frac{1}{n+4}\left(L_{\lambda\mu}\delta^{\kappa}_{\mu} - L_{\mu\nu}\delta^{\kappa}_{\lambda} + g_{\lambda\nu}L^{\kappa}_{\mu} - g_{\mu\nu}L^{\kappa}_{\lambda} + M_{\lambda\nu}\varphi^{\kappa}_{\mu} - M_{\mu\nu}\varphi^{\kappa}_{\lambda} + \varphi_{\lambda\nu}M^{\kappa}_{\mu} - \varphi_{\mu\nu}M^{\kappa}_{\lambda} + 2M_{\lambda\mu}\varphi^{\kappa}_{\nu} + 2\varphi_{\lambda\mu}M^{\kappa}_{\nu}\right)$$

The following identities are obtained by the straight forward computations:

$$K^{\kappa}_{\lambda\mu\nu} = -K^{\kappa}_{\mu\lambda\nu}, \ K_{\lambda\mu\nu\varpi} = -K_{\lambda\mu\varpi\nu}, \ K^{\kappa}_{\lambda\mu\nu} + K^{\kappa}_{\mu\nu\lambda} + K^{\kappa}_{\nu\lambda\mu} = 0, \ K^{\alpha}_{\alpha\mu\nu} = 0, \ K^{\alpha}_{\lambda\mu\alpha} = 0,$$

$$K^{\alpha}_{\lambda\mu\nu}\varphi^{\kappa}_{\alpha} = K^{\kappa}_{\lambda\mu\alpha}\varphi^{\alpha}_{\nu}, K^{\kappa}_{\alpha\mu\nu}\varphi^{\alpha}_{\lambda} = -K^{\kappa}_{\lambda\alpha\nu}\varphi^{\alpha}_{\mu}, \ K^{\beta}_{\lambda\mu\alpha}\varphi^{\alpha}_{\beta} = 0 \ \text{and} \ K^{\beta}_{\alpha\mu\nu}\varphi^{\alpha}_{\beta} = 0$$

Next we introduce a tensor $K_{\lambda\mu\nu}$ is given by

$$K_{\lambda\mu\nu} = \nabla_\lambda R_{\mu\nu} - \nabla_\mu R_{\lambda\nu} + \frac{1}{2(n+2)}\left(g_{\lambda\nu}\delta^{\varepsilon}_{\mu} - g_{\mu\nu}\delta^{\varepsilon}_{\lambda} + \varphi_{\lambda\nu}\varphi^{\varepsilon}_{\mu} - \varphi_{\mu\nu}\varphi^{\varepsilon}_{\lambda} + 2\varphi_{\lambda\mu}\varphi^{\varepsilon}_{\nu}\right)\nabla_\varepsilon R$$

Then we can get the following identity

$$\nabla_\alpha K^{\alpha}_{\lambda\mu\nu} = \frac{n}{n+4} K_{\lambda\mu\nu}$$

Now consider a tensor $U^{\kappa}_{\lambda\mu\nu}$ is given by

$$U^{\kappa}_{\lambda\mu\nu} = R^{\kappa}_{\lambda\mu\nu} + \frac{R}{n(n+2)}\left(g_{\lambda\nu}\delta^{\kappa}_{\mu} - g_{\mu\nu}\delta^{\kappa}_{\lambda} + \varphi_{\lambda\nu}\varphi^{\kappa}_{\mu} - \varphi_{\mu\nu}\varphi^{\kappa}_{\lambda} + 2\varphi_{\lambda\mu}\varphi^{\kappa}_{\nu}\right)$$

We can obtain the following theorems:

Theorem 1: The Bochner curvature tensor coincides with $U^{\kappa}_{\lambda\mu\nu}$ of a Kahlerian space K^n if and only if K^n is an Einstein space.

Remark-2: The tensor of a Riemannian space is defined by

$$Z^{\kappa}_{\lambda\mu\nu} = R^{\kappa}_{\lambda\mu\nu} + \frac{R}{n(n-1)}\left(g_{\lambda\nu}\delta^{\kappa}_{\mu} - g_{\mu\nu}\delta^{\kappa}_{\lambda}\right)$$

It is called the concircular curvature tensor and it is invariant under any concircular correspondence. $U^{\kappa}_{\lambda\mu\nu}$ Corresponds to $Z^{\kappa}_{\lambda\mu\nu}$.

A Kahlerian space is called a space of constant holomorphic sectional curvature, if $U^{\kappa}_{\lambda\mu\nu}$ vanishes identically.

Theorem 2: The Bochner curvature tensor of a space of constant holomorphic sectional curvature vanishes identically. The following theorem is known.

Theorem 3: If a compact Kahlerian space K^{2m} with vanishing Bochner curvature tensor has positive definite Ricci form, then

$$b_{2l} = 1 \text{ and } b_{2l+1} = 0,\ 0 \leqq 2l, 2l+1 \leqq (m/2) + 2$$

Where b_i denotes the i[th] Betti number of K^{2m}.

2. KAHLERIAN SUBMANIFOLDS

Let $\bar{M}$ be a Kahlerian manifold of complex dimension *m* (of real dimension 2*m*) with almost complex structure *J* and with Kahlerian metric $\bar{g}$. Let *M* be a complex *n*-dimensional analytic sub-manifold of $\bar{M}$, that is the immersion $f : M \to \bar{M}$ is holomorphic $J.f_* = f_*.J$, where f_* is the differential of the immersion *f* and we denote by same *J* the induced complex structure on *M*. Then the Riemannian metric *g* induced on *M* is Hermitian. It is easy to see that the fundamental 2-form with this Hermitian metric *g* is the restriction of the fundamental 2-form of $\bar{M}$ and is closed. This shows that every complex analytic sub-manifold *M* of a Kahlerian manifold $\bar{M}$ is also a Kahlerian manifold with the induced metric. We call such a submanifold *M* of $\bar{M}$ a Kahlerian submanifold.

Lemma 2.1: Any Kahlerian submanifold *M* is a minimal submanifold.

Proof: For each $T_X(M)$, we can choose a basis $e_1, e_2, \ldots\ldots, e_n, Je_1, Je_2, \ldots\ldots, Je_n$. Then we have

$$\sum_{i=1}^{n} B(e_i, e_i) + B(Je_i, Je_i) = 0 \tag{2.1}$$

Which means that *M* is a minimal sub-manifold of Kahlerian manifolds.

3. TOTALLY REAL SUBMANIFOLDS OF A KAHLERIAN MANIFOLD

Let $M^{2m}, m \geq 2$ be a Kahlerian manifold of real dimension 2*m* covered by a system of co-ordinate neighborhoods {*U*; x^k}, where the sequel the indices (*i, j, k, h,… …*) run over the range [1, 2, 3, 4,......, 2*m*] and let $g_{\mu\lambda}, F^{\tau}_{\lambda}, \nabla_{\lambda}, K^{\lambda}_{\nu\mu\lambda}, K_{\mu\lambda}$ and γ the metric tensor, the complex structure tensor, the operator of covariant differentiation with respect to $g_{\mu\lambda}$, the curvature tensor, the Ricci tensor and the scalar curvature of M^{2m} respectively.

Let $M^n, n \geq 3$, be a Riemannian manifold of dimension *n* covered by a system of co-ordinate neighborhoods $\{V; y^h\}$ and in the sequel the indices $(\lambda, \alpha, \beta, \gamma, \ldots\ldots)$ run over the range $[1', 2', 3', 4', \ldots\ldots, n']$ and let $g_{\dot\beta\dot\alpha}, \nabla_{\dot\beta}, K^h_{\dot\gamma\dot\beta\dot\alpha}$ and γ the metric tensor, the operator of covariant differentiation with respect to g_{ji}, the curvature tensor, the Ricci tensor and the scalar curvature of M^n respectively.

We assume that M^n is isometrically immersed in M^{2m} and represent the immersion by $x^\gamma = x^\gamma(y^\gamma)$ and put $A^\gamma_{\dot\alpha} = \partial\dot\alpha x^\gamma$, then we have

(3.1) $$g_{\mu\lambda} A^{\mu\lambda}_{\dot\beta\dot\alpha} = g_{\dot\beta\dot\alpha}$$

Where

(3.2) $$A^{\mu\lambda}_{\dot\beta\dot\alpha} = A^\mu_{\dot\beta} A^\lambda_{\dot\alpha}$$

4. PROPERTIES OF SUBMANIFOLDS

Let us consider a Kahlerian manifold M^{2m} which admits a complex conformal connection ∇^*_α with connection coefficient $\Gamma^{*\gamma}_{\beta\alpha}$ and scalar function *q*. Then connected coefficient $\nabla^{*\gamma}_{\beta\alpha}$ is defined as

(4.1) $$\Gamma^{*\lambda}_{\beta\alpha} = \begin{Bmatrix} \lambda \\ \beta \quad \alpha \end{Bmatrix} + \delta^\lambda_\alpha q_\beta + \delta^\lambda_\beta q_\alpha - g_{\beta\alpha} q^\lambda + F^\lambda_\beta p_\alpha + F^\lambda_\alpha p_\beta - F_{\beta\alpha} p^\lambda$$

Wherein

(4.2) $$\partial_\alpha = \partial / \partial x^\alpha$$

(4.3) $$q_\alpha = \partial_\alpha q = \partial q / \partial x^\alpha$$

(4.4) $$p_\alpha = -q_\tau F^\tau_\alpha$$

(4.5) $$q^\lambda = q_\tau g^{\tau\lambda}$$

(4.6) $$p^\lambda = p_\tau g^{\tau\lambda}$$

The curvature tensor $K^\lambda_{\gamma\beta\alpha}$ of $\begin{Bmatrix} \lambda \\ \beta \quad \alpha \end{Bmatrix}$ and $K^{*\lambda}_{\gamma\beta\alpha}$ of $\Gamma^{*\lambda}_{\beta\alpha}$ is given by

(4.7) $$K^{*\lambda}_{\gamma\beta\alpha} = K^\lambda_{\gamma\beta\alpha} + \delta^\lambda_\beta q_{\gamma\alpha} - \delta^\lambda_\gamma q_{\beta\alpha} + q^\lambda_\beta g_{\gamma\alpha} - q^\lambda_\gamma g_{\beta\alpha} - F^\lambda_\gamma g_{\beta\alpha} + F^\lambda_\beta g_{\gamma\alpha} + p^\lambda_\beta F_{\gamma\alpha} - p^\lambda_\gamma F_{\beta\alpha}$$
$$-2F_{\gamma\beta}\left(q_\alpha p^\lambda - p_\alpha q^\lambda\right) + \left(\nabla_\gamma p_\beta - \nabla_\beta p_\gamma\right) F^\lambda_\alpha$$

Wherein

(4.8) $$p_{\beta\alpha} = \nabla_\beta p_\alpha - p_\beta q_\alpha - q_\beta p_\alpha + (1/2) q_\tau q^\tau F_{\beta\alpha}$$

(4.9) $$q_{\beta\alpha} = \nabla_\beta q_\alpha + p_\beta p_\alpha - q_\beta p_\alpha + (1/2) q^\tau q_\tau g_{\beta\alpha}$$

By virtue of the equations (4.8) and (4.9), we get

(4.10) $$q_{\beta\alpha} = p_{\beta\tau} F^\tau_\alpha$$

(4.11) $$q^\lambda_\gamma = q_{\gamma\tau} g^{\tau\lambda}$$

(4.12) $$p^\lambda_\gamma = p_{\gamma\tau} g^{\tau\lambda}$$

(4.13) $$p_{\beta\alpha} = -q_{\beta\tau} F^\tau_\alpha$$

We have

(4.14) $a_{\gamma\beta} = -\left(\nabla_{\gamma} p_{\beta} - \nabla_{\beta} p_{\gamma}\right)$

and

(4.15) $b_{\gamma\beta} = 2\left(q_{\beta} p_{\gamma} - p_{\beta} q_{\gamma}\right)$

The equation (4.14) can be written as

(4.16) $a_{\beta\alpha} = -2 p_{\beta\alpha} - \left\{\dfrac{2}{n+4}\right\} q_{\tau}^{\tau} F_{\beta\alpha}$

The equation (4.15) can be written as

(4.17) $b_{\beta\alpha} = -2 p_{\beta\alpha} + \left\{2/(n+4)\right\} q_{\tau}^{\tau} F_{\beta\alpha}$

From the equations (4.16) and (4.17), we obtain

(4.18) $a_{\beta\alpha} + b_{\beta\alpha} = -4 p_{\beta\alpha}$

Where in

(4.19) $p_{\beta\alpha} = -(1/4)\left(a_{\beta\alpha} + b_{\beta\alpha}\right)$

and

(4.20) $q_{\tau}^{\tau} = \nabla_{\tau} q^{\tau} + (n/2) q^{\tau} q_{\tau}$

As a consequence of the equation (4.7), we have

(4.21) $\overset{*}{K}_{\gamma\beta\alpha\lambda} = K_{\gamma\beta\alpha\lambda} + g_{\beta\lambda} q_{\gamma\alpha} - g_{\gamma\lambda} q_{\beta\alpha} + q_{\beta\lambda} g_{\gamma\alpha} - q_{\gamma\lambda} g_{\beta\alpha} - F_{\gamma\lambda} g_{\beta\alpha} + F_{\beta\lambda} g_{\gamma\alpha} + p_{\beta\lambda} F_{\gamma\alpha}$
$-P_{\gamma\lambda} F_{\beta\alpha} - F_{\gamma\beta} b_{\alpha\lambda} - a_{\gamma\beta} F_{\alpha\lambda}$

Wherein

(4.22) $K_{\gamma\beta\alpha\lambda} = K_{\gamma\beta\alpha}^{\tau} g_{\tau\lambda}$

(4.23) $\overset{*}{K}_{\gamma\beta\alpha\lambda} = K_{\gamma\beta\alpha}^{*\tau} g_{\tau\lambda}$

Contracting the equation (4.21) with $g^{\gamma\lambda}$ yields

(4.24) $\overset{*}{K}_{\beta\alpha} = K_{\beta\alpha} - 2(m+2) q_{\beta\alpha} - q_{\tau}^{\tau} g_{\beta\alpha}$

Wherein $\overset{*}{K}_{\beta\alpha}$ denotes the Ricci tensor with regard to $\overset{*}{\nabla}_{\alpha}$.

Contracting the equation (4.24) with $g^{\beta\alpha}$ yields

(4.25) $K^{*} = K - 4(m+1) q_{\tau}^{\tau}$.

Where K^{*} is the scalar curvature with regard to $\overset{*}{\nabla}_{\alpha}$.

5. COMPLEX CONFORMAL CURVATURE TENSOR AND COMPLEX CONFORMAL CONNECTION WITH BOCHNER CURVATURE TENSOR:

Consider that the Buchner curvature tensor is defined by [13]

(5.1) $A_{\gamma\beta\alpha}^{\lambda} = K_{\gamma\beta\alpha}^{\lambda} - \delta_{\beta}^{\lambda} L_{\gamma\alpha} + \delta_{\gamma}^{\lambda} L_{\beta\alpha} - L_{\beta}^{\lambda} g_{\gamma\alpha} + L_{\gamma}^{\lambda} g_{\beta\alpha} - F_{\beta}^{\lambda} M_{\gamma\alpha} + F_{\gamma}^{\lambda} M_{\beta\alpha}$
$+M_{\gamma}^{\lambda} F_{\beta\alpha} - M_{\beta}^{\lambda} F_{\gamma\alpha} - 2\left(M_{\gamma\beta} F_{\alpha}^{\lambda} + F_{\gamma\beta} M_{\alpha}^{\lambda}\right)$

Wherein

(5.2) $M_{\beta\alpha} = -L_{\beta\tau} F_{\alpha}^{\tau}$

(5.3) $L_{\beta\alpha} = \left\{1/8(m+1)(m+2)\right\} K g_{\beta\alpha} - \left\{1/2(m+2)\right\} K_{\beta\alpha}$

That is

(5.4) $H_{\beta\alpha} = -K_{\beta\tau}F_{\alpha}^{\tau}$

(5.5) $L_{\gamma}^{\lambda} = L_{\gamma\tau}g^{\tau\lambda}$

(5.6) $M_{\gamma}^{\lambda} = M_{\gamma\alpha}g^{\alpha\lambda}$

and

(5.7) $M_{\beta\alpha} = -\{1/2(m+2)\}H_{\beta\alpha} + \{1/8(m+1)(m+2)\}KF_{\beta\alpha}$

By virtue of the equation (5.1), we obtain

(5.8) $A_{\gamma\beta\alpha\lambda} = K_{\gamma\beta\alpha\lambda} - g_{\beta\lambda}L_{\gamma\alpha} + g_{\gamma\lambda}L_{\beta\alpha} - L_{\beta\lambda}g_{\gamma\alpha} + L_{\gamma\lambda}g_{\beta\alpha} - F_{\beta\lambda}M_{\gamma\alpha} + F_{\gamma\lambda}M_{\beta\alpha}$
$+M_{\gamma\lambda}F_{\beta\alpha} - M_{\beta\lambda}F_{\gamma\alpha} - 2\left(M_{\gamma\beta}F_{\alpha\lambda} + F_{\gamma\beta}M_{\alpha\lambda}\right).$

Wherein

(5.9) $A_{\gamma\beta\alpha\lambda} = A_{\gamma\beta\alpha}^{\tau}g_{\tau\lambda}$

Let M^n be a totally real submanifold of a Kahlerian manifold $M^{2m}(m>2)$ admits an induced fundamental tensor g_{rs} and let $\nabla_s, K_{uts}^{r}, K_{sr}$ and *R* the operator of covariant differentiation, the curvature tensor, the Ricci tensor and the scalar curvature of M^n.

In this regard, we have

(5.10) $\Gamma_{st}^{*r} = \left(\partial_t A_s^{\lambda} + \Gamma_{\beta\alpha}^{*\lambda}A_s^{\beta}A_t^{\alpha}\right)A_{\lambda}^{r},$

Wherein

(5.11) $A_t^{\lambda} = x^{\lambda}/d^t$

And

(5.12) $A_{\lambda}^{\alpha} = g^{rs}g_{\lambda r}A_s^{\alpha}$

Remark -5.1: It is to be noted that Γ_{st}^{*r} is the induced connection on M^n with the induced metric $\overset{*}{g}_{rs}$.

We denote operator ∇_s^*, the operator of covariant differentiation with regard to Γ_{st}^{*r} and K_{uts}^{*r}, K_{st}^{*} and K^* are the curvature tensor, the Ricci tensor and the scalar curvature of M^n with regard ∇_s^*. We put

(5.13) $\nabla_t^* A_s^{\lambda} = \partial_t A_s^{\lambda} + \Gamma_{\beta\alpha}^{*\lambda}A_t^{\beta}A_s^{\alpha} - \Gamma_{ts}^{*r}A_r^{\lambda}$

Where $\partial t = \partial/\partial d^t$

Wherein Γ_{st}^{*r} and $\Gamma_{\beta\alpha}^{*\lambda}$ are given by the equations (5.10) and (4.1).

Remark 5.2: It is to be noted that kind of covariant differentiation is called the vander Waerden-Bartolotti covariant differentiation with respect to the complex conformal connection.

Suppose $D_1^{\lambda}, \ldots\ldots, D_{2m-n}^{\lambda}$ are 2*m*–*n* unit orthogonal normal fields on M^n. Decomposing q^{λ} into its unique tangential and normal components along M^n, we obtain

(5.14) $q^{\lambda} = q^r A_r^{\lambda} + a^x D_x^{\lambda}$

The summation in the index *x* will run over the range $x = 1, 2, 3, \ldots\ldots, 2m-n$ .Wherein

(5.15a) $q^r = g^{rs}q_s$

(5.15b) $q_s = \nabla_s q = \partial_s q$

The second fundamental tensor H_{ts}^{*x} of ∇_t^* relative to the normal D_x^λ is given by

(5.16) $$H_{tsx}^* = g_{\beta\lambda}\left(\nabla_t^* A_s^\beta\right) D_x^\lambda.$$

The Gauss curvature equation and Gauss equation of M^n with regard to the complex conformal connection is defined as

(5.17) $$K_{\gamma\beta\alpha\lambda}^* A_u^\lambda A_t^\beta A_s^\alpha A_r^\lambda = K_{utsr}^* - B_{utsr}^*$$

(5.18) $$K_{\gamma\beta\alpha\lambda} A_u^\lambda A_t^\beta A_s^\alpha A_r^\lambda = K_{utsr} - B_{utsr}$$

Wherein

(5.19) $$B_{utsr}^* = H_{ur}^{*x} H_{tsx}^* - H_{tr}^{*x} H_{usx}^*$$

Weyl's conformal curvature tensor D_{utsr} of M^n is defined as

(5.20) $$D_{utsr} \stackrel{\text{def}}{=} K_{utsr} + R\left\{1/(n-1)(n-2)\right\}\left(g_{ur}g_{ts} - g_{us}g_{tr}\right)$$
$$-\left\{1/(n-1)\right\}\left(g_{ur}K_{ts} + g_{ts}K_{ur} - g_{us}K_{tr} - g_{tr}K_{us}\right), \quad n > 3.$$

Theorem 5.1: Let M^{2m} ($m > 2$) be a Kahlerian manifold admitting the complex conformal connection (4.1). If the Ricci tensor with respect to the complex conformal connection vanishes, the Bochner curvature tensor is identically equal to the curvature tensor of the complex conformal connection.

Proof: If $K_{\beta\alpha}^* = 0$ and $K^* = 0$, then by virtue of the equations (4.24) and (4.25), we obtain

(5.21) $$K_{\beta\alpha} = 2(m+2)q_{\beta\alpha} + q_\tau^\tau g_{\beta\alpha}$$

and

(5.22) $$K = 4(m+1)q_\tau^\tau$$

Inserting the equations (5.21) and (5.22) into the equations (5.2), (5.3), (5.4), (5.5) and (5.6),
We obtain

(5.23) $$L_{\beta\alpha} = -q_{\beta\alpha}$$

(5.24) $$M_{\beta\alpha} = -p_{\beta\alpha}$$

Inserting the equations (5.23) and (5.24) into the equation (5.8), we obtain

(5.25) $$A_{\gamma\beta\alpha\lambda} = K_{\gamma\beta\alpha\lambda} + g_{\beta\lambda}q_{\gamma\alpha} - g_{\gamma\lambda}q_{\beta\alpha} + q_{\beta\lambda}g_{\gamma\alpha} - q_{\gamma\lambda}g_{\beta\alpha} + F_{\beta\lambda}p_{\gamma\alpha} - F_{\gamma\lambda}p_{\beta\alpha}$$
$$+ p_{\beta\lambda}F_{\gamma\alpha} - p_{\gamma\lambda}F_{\beta\alpha} + 2\left(p_{\gamma\beta}F_{\alpha\lambda} + F_{\gamma\beta}p_{\alpha\lambda}\right)$$

By virtue of the equations (4.16) and (4.17), we obtain

(5.26) $$\alpha_{\gamma\beta}F_{\alpha\lambda} + F_{\gamma\beta}\beta_{\alpha\lambda} = -2\left(p_{\gamma\beta}F_{\alpha\lambda} + F_{\gamma\beta}p_{\alpha\lambda}\right)$$

Using the equations (5.26) and (4.16), then the equation (5.25) reduced in the form

(5.27) $$A_{\gamma\beta\alpha\lambda} = K_{\gamma\beta\alpha\lambda}^*$$

In this regard, the following theorem is

Theorem 5.2: In a Kahlerian manifold M^{2n} a scalar function q is such that the complex Conformal connection (4.1) is of zero curvature, the Bochner curvature tensor of the manifold Vanishes.

Proof: From theorem 5.1, we have

(5.28) $$A_{\gamma\beta\alpha\lambda} = K_{\gamma\beta\alpha\lambda}^*$$

If

(5.29) $$K^*_{\gamma\beta\alpha\lambda} = 0$$

and

(5.30) $$K^*_{\beta\alpha} = 0$$

Inserting the equations (5.29) and (5.30) into the equation (5.28), we obtain

(5.31) $$A_{\gamma\beta\alpha\lambda} = 0$$

This shows that the Bochner curvature tensor in a Kahlerian manifold becomes zero.

6. CONFORMALLY FLAT TOTALLY REAL SUBMANIFOLDS OF A KAHLERIAN MANIFOLDS WITH BOCHNER CURVATURE TENSOR:

In this section, we have studied the properties of the submanifolds of a Kahlerian manifolds with Bochner Curvature Tensor. If a Kahlerian manifold M^{2m} ($m > 2$) admits a conformal change of Hermitian metric, the totally real sub-manifold M^n admits a conformal change of a Riemannian metric.

Suppose M^{2m} admits a conformal change of a Hermitian metric that is

(6.1) $$g^*_{\beta\alpha} = e^{2q} g_{\beta\alpha}$$

(6.2) $$F^{*\gamma}_{\alpha} = F^{\gamma}_{\alpha}$$

(6.3) $$F^*_{\beta\alpha} = e^{2q} F_{\beta\alpha}$$

Wherein q is a scalar function. The scalar function q in a Kahlerian manifold is such that the complex conformal connection of (4.1) is zero curvature then the Bochner curvature tensor in a Kahlerian manifold will be zero i.e.

$$K^*_{\gamma\beta\alpha\lambda} = 0$$

Multiplying both sides of the equation (6.1) by $A^{\beta}_{s}\, A^{\alpha}_{r}$, we obtain

(6.4) $$g^*_{sr} = e^{2q} g_{sr}$$

and

(6.5) $$F^*_{\beta\alpha} A^{\beta}_{s}\, A^{\alpha}_{r} = 0$$

Wherein the induced metric g^*_{sr} is given by

(6.6) $$g^*_{sr} = g^*_{\beta\alpha} A^{\beta}_{s}\, A^{\alpha}_{r}$$

(6.7) $$F_{\beta\alpha} A^{\beta}_{s}\, A^{\alpha}_{r} = 0$$

The tensor field H_{utsr} of type (0, 4) is defined as

(6.8) $$H_{utsr} = B_{utsr} - \{1/(n-1)\}\left(g_{ur}B_{ts} + g_{ts}B_{ur} - g_{us}B_{tr} - g_{tr}B_{us}\right) + \{1/(n-1)(n-2)\}B\left(g_{ur}g_{ts} - g_{us}g_{tr}\right)$$

Lemma 6.1: $M^n(n>3)$ be a totally real submanifold of a Kahlerian manifold $M^{2m}(m>2)$, then the following condition

(6.9) $$H^*_{utsr} = H_{utsr}$$ holds good.

Proof: The equation (5.17) can be written as

(6.10) $$K^*_{utsr} = B^*_{utsr} + K^*_{\gamma\beta\alpha\lambda} A^{\lambda}_{u} A^{\beta}_{t} A^{\alpha}_{s} A^{\lambda}_{r}$$

Inserting the equation (4.21) into the equation (6.10), we get

(6.11) $$K^*_{utsr} = B^*_{utsr} + K_{\gamma\beta\alpha\lambda} A^{\gamma}_{u} A^{\beta}_{t} A^{\alpha}_{s} A^{\lambda}_{r} + g_{tr}P_{us} - g_{ur}P_{ts} + P_{tr}g_{us} - g_{ts}P_{ur}.$$

Wherein

(6.12) $$P_{ts} = Q_{\beta\alpha} A_t^{\beta} A_s^{\alpha}$$

By virtue of the equations (5.17), (5.18) and (6.11), we get

(6.13) $$\overset{*}{K}_{utsr} = \overset{*}{B}_{utsr} + K_{utsr} - B_{utsr} + g_{tr}P_{us} - g_{ur}P_{ts} + P_{tr}g_{us} - g_{ts}P_{ur}$$

Contracting the equation (6.13) with regard to the indices *u* and *r* yields

(6.14) $$\overset{*}{K}_{ts} = \overset{*}{B}_{ts} + K_{ts} - B_{ts} - (n-2)P_{ts} - Pg_{ts}$$

Wherein $P = g^{ts}P_{ts}$

and $\overset{*}{B}_{ts}$ is given by

$$\overset{*}{B}_{ts} = g^{*ts}\overset{*}{B}_{utsr} = H^{*r}_{rx}H^{*x}_{ts} - H^{*ux}_{t}\overset{*}{H}_{usx}$$

Contracting the equation (6.14) with g^{*ts}, we obtain

(6.15) $$e^{2q}K^{*} = e^{2q}B^{*} + K - B - 2(n-1)P$$

We define D_{utsr} similar to that of the equation (5.20) by

(6.16) $$\overset{*}{D}_{utsr} = \overset{*}{K}_{utsr} + \{1/(n-1)(n-2)\}K^{*}\left(\overset{*}{g}_{ur}\overset{*}{g}_{ts} - \overset{*}{g}_{us}\overset{*}{g}_{tr}\right)$$
$$-\{1/(n-1)\}\left(\overset{*}{g}_{ur}\overset{*}{K}_{ts} + \overset{*}{g}_{ts}\overset{*}{K}_{ur} - \overset{*}{g}_{us}\overset{*}{K}_{tr} - \overset{*}{g}_{tr}\overset{*}{K}_{us}\right)$$

Inserting the equations (6.13), (6.14) and (6.15) into the equation (6.16) and using the equations (6.1), (6.2) and (6.3), we obtain

$$\overset{*}{D}_{utsr} = D_{utsr} + \overset{*}{H}_{utsr} - H_{utsr}$$

Remark 6.1: It is to be noted that if we take

$$\overset{*}{D}_{utsr} = D_{utsr}$$

Then we get the relation (6.9).

The projective curvature tensor of *n* dimensional Riemannian space M^n is given by

$$W^{\kappa}_{\lambda\mu\nu} = R^{\kappa}_{\lambda\mu\nu} + \frac{1}{n-1}\left(R_{\lambda\nu}\delta^{\kappa}_{\mu} - R_{\mu\nu}\delta^{\kappa}_{\lambda}\right)$$

Which is invariant under any projective correspondence, where $R^{\kappa}_{\lambda\mu\nu}, R_{\mu\nu}$ are the Riemannian curvature tensor, the Ricci tensor.

The conformal curvature tensor of M^n is given by

$$C^{\kappa}_{\lambda\mu\nu} = R^{\kappa}_{\lambda\mu\nu} + \frac{1}{n-2}\left(R_{\lambda\nu}\delta^{\kappa}_{\mu} - R_{\mu\nu}\delta^{\kappa}_{\lambda} + g_{\lambda\nu}R^{\kappa}_{\mu} - g_{\mu\nu}R^{\kappa}_{\lambda}\right) - \frac{R}{(n-1)(n-2)}\left(g_{\lambda\nu}\delta^{\kappa}_{\mu} - g_{\mu\nu}\delta^{\nu}_{\lambda}\right)$$

Where $g_{\lambda\nu}$ is the Riemannian metric of M^n and $R^{\kappa}_{\lambda} = g^{\kappa\alpha}R_{\lambda\alpha}$, $R = g^{\lambda\mu}R_{\lambda\mu}$.

Let K^n be an (n=2*m*) dimensional Kahlerian space with the structure tensor $g_{\lambda\mu}$ and $\varphi^{\kappa}_{\lambda}$. It is known that the tensor

$$P^{\kappa}_{\lambda\mu\nu} = R^{\kappa}_{\lambda\mu\nu} + \frac{1}{n+2}\left(R_{\lambda\nu}\delta^{\kappa}_{\mu} - R_{\mu\nu}\delta^{\kappa}_{\lambda} + S_{\lambda\nu}\varphi^{\kappa}_{\mu} - S_{\mu\nu}\varphi^{\kappa}_{\lambda} + 2S_{\lambda\mu}\varphi^{\kappa}_{\nu}\right)$$

is called the Holomorphically projective curvature tensor of K^n, is invariant under any Holomorphically projective correspondence. $P^{\kappa}_{\lambda\mu\nu}$ May be considered as the tensor corresponding to $W^{\kappa}_{\lambda\mu\nu}$. Under this situation it is natural to ask what tensor of K^n does correspond to $C^{\kappa}_{\lambda\mu\nu}$.

On the other hand, S. Buchner has introduced a tensor in K^n given by

$$K_{jh^*lk^*} = R_{jh^*lk^*} - \frac{1}{m+2}\left(R_{h^*l}g_{jk^*} + R_{jh^*}g_{lk^*} + g_{h^*l}R_{jk^*} + g_{jh^*}R_{lk^*}\right) + \frac{R}{2(m+1)(m+2)}\left(g_{h^*l}g_{jk^*} + g_{jh^*}g_{lk^*}\right)$$

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