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WEAK AND STRONG CONVERGENCE OF SEQUENCE OF ITERATES OF GENERALIZED CONTRACTION

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ABSTRACT

The aim of this paper is to study some properties of generalized contraction mappings and obtain some result for the weak & strong convergence of sequence of iterates for mappings of this type.

Keywords: *Nonexpansive mapping, quasi-nonexpansive mappings, asymptotically regular mapping, generalized contraction, reasonable wanderer.*

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INTRODUCTION & PRELIMINARIES

The aim of this paper is to study some properties of generalized contraction mappings and obtain some result for the weak and strong convergence of sequence of iterates for mappings of this type. The paper is divided into four sections. In section 1, we have shown that, if T is a generalized contraction mapping of closed, bounded and convex subset of a uniformly convex Banach space into itself with non-empty fixed points set, then the mapping T_λ defined by $T_\lambda = \lambda I + (1-\lambda)T$, for any λ such that $0 < \lambda < 1$ is asymptotically regular. In section 2, we prove for Hilbert space the mapping T_λ as defined above is a reasonable wanderer. Finally in section 3 and 4 we have obtain some results for weak and strong convergence of sequence of iterates of such kind of mappings.

Definition 1.1: Let C be a closed, bounded and convex subset of a Banach space X. A mapping $T: C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\| \quad \text{for all } x, y \in X$$

Definition 1.2: A mapping $T: C \to C$ is said to be quasi-nonexpansive if

$$\|Tx - Ty\| \le a\|x - y\| + b\|x - Tx\| + c\|y - Ty\|$$

for all $x, y \in C$, $a \ge 0, b \ge 0, c \ge 0$ and $a + b + c \le 1$.

The following example shows that there are quasi-nonexpansive mappings which are not nonexpansive.

Example 1.1: Let $X = [0,1]$ and let $Tx = \frac{1}{3}x$ for $0 \le x < 1$ and $T(1) = \frac{1}{6}$, then T is quasi-nonexpansive , but it is not nonexpansive.

Definition 1.3: A mapping $T: C \to C$ is said to be a generalized contraction if for any $x, y \in C$ we have

$$\|Tx - Ty\| \le a\|x - y\| + b[\|x - Tx\| + \|y - Ty\|] + c[\|x - Ty\| + \|y - Tx\|]$$

Where $a, b, c \in [0,1]$ and $a + 2b + 2c = 1$.

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The following example shows that there are generalized contraction mappings which are not quasi-nonexpansive mappings.

Example 1.2: Let

$$M_1 = \left\{\frac{m}{n} : m = 0,1,3,9,\cdots, n = 1,4,7,\cdots,3k+1,\cdots\right\}$$
$$M_2 = \left\{\frac{m}{n} : m = 1,3,9,27,\cdots, n = 2,5,\cdots,3k+2,\cdots\right\}$$

and let $M = M_1 \cup M_2$ with the usual metric. Define $T: M \to M$ as follows

$$T(x) = \begin{cases} \frac{2}{3}x & for\ \ x\ in\ M_1 \\ \frac{1}{3}x & for\ x\ in\ M_2 \end{cases}$$

Then T is generalized contraction. Indeed, if both x and y are in M_1 or in M_2, then $\|Tx - Ty\| \le \frac{2}{3}\|x - y\|$. If x in M_1 and y in M_2, then for $x = 1$ & $y = \frac{1}{2}$ we have

$$\|Tx - Ty\| = \frac{2}{3} - \frac{1}{6} = \frac{1}{2}$$

Also

$$a\|x - y\| + b[\|x - Tx\| + \|y - Ty\|] + c[\|x - Ty\| + \|y - Tx\|]$$
$$= a.\frac{1}{2} + b\left(\frac{1}{3} + \frac{1}{3}\right) + c\left(\frac{5}{6} + \frac{1}{6}\right) = \frac{1}{2} \text{ for } \ a = \frac{1}{2}\ , b = \frac{1}{4}\ , c = \frac{1}{12}$$

Thus

$$\|Tx - Ty\| \le a\|x - y\| + b[\|x - Tx\| + \|y - Ty\|] + c[\|x - Ty\| + \|y - Tx\|]$$

However, T is not quasi-nonexpansive. Indeed, taking $a = b = c = \frac{1}{3}$ we have

$$\frac{1}{3}\{\|x - y\| + \|x - Tx\| + \|y - Ty\|\} = \frac{1}{3}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{3}\right) = \frac{7}{18}$$

It follows that $\|Tx - Ty\| \nleq \frac{1}{3}\{\|x - y\| + \|x - Tx\| + \|y - Ty\|\}$

Definition 1.4: A Banach space X is said to be uniformly convex if given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\|x - y\| \ge \varepsilon$ for $\|x\| \le 1$ and $\|y\| \le 1$
implies $\left\|\frac{1}{2}(x + y)\right\| \le 1 - \delta(\varepsilon)$.

Definition 1.5: Let X be a Banach space and C be a closed, convex subset of X. A mapping $T: C \to C$ is called asymptotically regular at x if and only if $\|T^n x - T^{n+1}x\| \to 0$ as $n \to \infty$.

RESULTS

Theorem 1.1: *Let D be a nonempty, bounded and convex subset of a uniformly convex Banach space X. Let $T: D \to D$ be a generalized contraction mapping. Let us suppose that $F = \{x \in D : Tx = x\}$ is non-empty. Then the mapping T_λ defined by $T_\lambda = \lambda T + (1 - \lambda)I$ for any λ such that $0 < \lambda < 1$ is asymptotically regular with the same fixed point as T.*

Proof: It is clear that $F(T) = F(T_\lambda)$, where $F(T)$ and $F(T_\lambda)$ are the fixed point sets of T and T_λ respectively. Indeed, x in $F(T)$ implies $Tx = x$. Thus $T_\lambda x = \lambda x + (1 - \lambda)x = x$, hence x in $F(T_\lambda)$, i.e. $F(T)$ is contained in $F(T_\lambda)$. Conversely y in $F(T_\lambda)$ implies $T_\lambda y = y = \lambda Ty + (1 - \lambda)y$, which implies $\lambda Ty = \lambda y$, or $Ty = y$. Thus $F(T_\lambda)$ contained in $F(T)$. Hence $F(T) = F(T_\lambda)$.

Let x_0 in D, $x_{n+1} = T_\lambda(x_n)$, $n = 0,1,2,\cdots$ .Since $T_\lambda x - x = \lambda(x - Tx)$ for x in D, it is enough to show that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. Now let x in D and y in $F(T)$, hence in $F(T_\lambda)$, then

$$\begin{aligned} \|Tx - y\| &= \|Tx - Ty\| \\ &\le a\|x - y\| + b[\|x - Tx\| + \|y - Ty\|] + c[\|x - Ty\| + \|y - Tx\|] \\ &\le a\|x - y\| + b\|x - Tx\| + c[\|x - y\| + \|y - Tx\|] \\ &\le \frac{a+b+c}{1-b-c}\|x - y\| \le \|x - y\| \qquad \text{as } a + 2b + 2c = 1. \end{aligned}$$

Thus we obtain

(1) $$\|Tx - y\| \le \|x - y\|$$

Also

(2) $$\begin{aligned} \|y - T_\lambda x\| &= \|y - (1 - \lambda)x - \lambda Tx\| = \|(1 - \lambda)(y - x) + \lambda(y - Tx)\| \\ &\le (1 - \lambda)\|x - y\| + \lambda\|y - Tx\| \le (1 - \lambda)\|x - y\| + \lambda\|x - y\| \\ &= \|y - x\|. \end{aligned}$$

So the sequence $\{\|y - x_n\|\}$ is bounded by $= \|y - x_0\|$. If $y = x_n$ for some n, then from (2), $\{x_n\}$ converges to y and the proof is complete. So we may assume that $y \neq x_n$ for all $n = 0,1,2,\cdots$.Suppose that $\lambda \leq \frac{1}{2}$. Now

(3) $$\begin{aligned}\|y - x_{n+1}\| &= \|\lambda(y - x_n + y - Tx_n) + (1 - 2\lambda)(y - x_n)\| \\ &\leq \lambda\|y - x_n + y - Tx_n\| + (1 - 2\lambda)\|y - x_n\| \\ &= (\|y - x_n\|)\lambda \left\|\frac{y - x_n + y - Tx_n}{\|y - x_n\|}\right\| + (1 - 2\lambda)\|y - x_n\|\end{aligned}$$

Let

(4) $$a = (y - x_n)/\|y - x_n\| \quad \text{and} \quad b = (y - Tx_n)/\|y - x_n\|$$

then

$$\|a + b\| = \|y - x_n + y - Tx_n/\|y - x_n\|\|$$

and $\|a\| \leq 1$, $\|b\| \leq 1$. Thus from (3) we get

(5) $$\begin{aligned}\|y - x_{n+1}\| &= 2\lambda \left\|\frac{1}{2}(a + b)\right\| \|y - x_n\| + (1 - 2\lambda)\|y - x_n\| \\ &= \left\{2\lambda \left\|\frac{1}{2}(a + b)\right\| + (1 - 2\lambda)\right\} \|y - x_n\|\end{aligned}$$

Since X is uniformly convex

$$\delta(\varepsilon) = inf\left\{1 - \frac{1}{2}(x + y): \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\right\}$$

is positive for ε in (0,2]. Also $\delta(0) = 0$. From (4) we have

$$\left\|\frac{1}{2}(a + b)\right\| \leq 1 - \delta(\|x_n - Tx_n\|/\|y - x_n\|)$$

Since δ is monotonically nondecreasing on [0,2]

(6) $$\left\|\frac{1}{2}(a + b)\right\| \leq 1 - \delta(\|x_n - Tx_n\|/M)$$

Thus from (5) and (6) we have

(7) $$\begin{aligned}\|y - x_{n+1}\| &\leq 2\lambda\|y - x_n\|\big(1 - \delta(\|x_n - Tx_n\|/M)\big) + (1 - 2\lambda)\|y - x_n\| \\ &= \{2\lambda - 2\lambda\delta(\|x_n - Tx_n\|/M)(1 - 2\lambda)\}\|y - x_n\| \\ &= \big(1 - 2\lambda\delta(\|x_n - Tx_n\|/M)\big)\|y - x_n\|\end{aligned}$$

From (7) and by induction we obtain

(8) $$\|y - x_{n+1}\| \leq \prod_{j=0}^{n}\Big(1 - 2\lambda\delta\big(\|x_j - Tx_j\|/M\big)\Big) M$$

Suppose $\{x_n - Tx_n\}$ does not converge to zero. Then there exists a subsequence $\{x_{k(n)}\}$ of $\{x_n\}$ such that $\{x_{k(n)} - Tx_{k(n)}\}$ converges to some constant α in $(0, \infty)$. Since δ is monotonically nondecreasing and $\Big(1 - 2\lambda\delta\big(\|x_j - Tx_j\|/M\big)\Big)$ belongs to [0, 1] for each j, we have from (8) for sufficiently large n

$$\left\|y - x_{k(n+1)}\right\| \leq 1 - 2\delta\lambda(\alpha/2M)^n M$$

So $\{x_{k(n)}\}$ converges to y, but then from (1) we get the convergence of $\{Tx_{k(n)}\}$ to y . Therefore $\{x_{k(n)} - Tx_{k(n)}\}$ converges to zero, a contradiction to the choice of. If $\lambda \geq \frac{1}{2}$ then $1 - \lambda \leq \frac{1}{2}$, we can apply the same argument as above by replacing (4) as

$$\begin{aligned}\|y - x_{n+1}\| &= \|(1 - \lambda)(y - x_n + y - Tx_n) + (2\lambda - 1)(y - Tx_n)\| \\ &\leq (1 - \lambda)\|y - x_n + y - Tx_n\| + (2\lambda - 1)\|y - Tx_n\| \\ &= (1 - \lambda)\|y - x_n\| \left\|\frac{1}{2}(a + b)\right\| + (2\lambda - 1)\|y - Tx_n\|\end{aligned}$$

By induction the roles of λ and $1 - \lambda$ we can obtain as earlier a contradiction. Thus T_λ is asymptotically regular.

Remark 1.1: A theorem similar to our Theorem 1.1 for nonexpansive mappings was proved by Schaefer [15].

Definition 2.1: Let H be a Hilbert space and C be a closed, convex subset of H. A mapping $T: C \to C$ is said to be *reasonable wanderer* in C if starting at any point x_0 in C, its successive steps $x_n = T^n x_0$ $(n = 1,2,3, \dots \dots)$ are such that the sum of square of their lengths is finite, i.e. $\sum_{n=0}^{\infty}\|x_{n+1} - x_n\| < \infty$.

Theorem 2.1: *Let H be a Hilbert space and C be a closed, convex subset of H. A mapping $T: C \to C$ be a generalized contraction mapping. Suppose F, the fixed point set of T in C is nonempty. Let $T_\lambda = \lambda I + (1 - \lambda)T$ for any given λ with $0 < \lambda < 1$, then T_λ is a reasonable wonderer from C into C with the same fixed point as T.*

Proof: For any $x \in C$, set $x_n = T_\lambda^n x$ and let y be a fixed point of T and, hence of T_λ. Then

(9) $$x_{n+1} - y = \lambda x_n + (1-\lambda)Tx_n - \lambda y - (1-\lambda)y = \lambda(x_n - y) + (1-\lambda)(Tx_n - y)$$

Now

(10) $$\begin{aligned}\|Tx_n - y\| &= \|Tx_n - Ty\| \\ &\le a\|x_n - y\| + b[\|x_n - Tx_n\| + \|y - Ty\|] + c[\|x_n - Ty\| + \|y - Tx_n\|] \\ &\le a\|x_n - y\| + b\|x_n - Tx_n\| + c[\|x_n - y\| + \|y - Tx_n\|] \\ &\le \frac{a+b+c}{1-b-c}\|x_n - y\| \le \|x_n - y\| \qquad \text{as } a + 2b + 2c = 1.\end{aligned}$$

Thus we obtain

(11) $$\|Tx_n - y\| \le \|x_n - y\|$$

Also for any a, we have

(12) $$a(x_n - Tx_n) = a(x_n - y + y - Tx_n) = a(x_n - y) - a(Tx_n - y)$$

Using (9) we get

(13) $$\begin{aligned}\| x_{n+1} - y\|^2 &= \lambda^2\|x_n - y\|^2 + (1-\lambda)^2\|Tx_n - y\|^2 + 2\lambda(1-\lambda)(Tx_n - y, x_n - y) \\ &\le \lambda^2\|x_n - y\|^2 + (1-\lambda)^2\|x_n - y\|^2 + 2\lambda(1-\lambda)(Tx_n - y, x_n - y) \\ &= \{\lambda^2 + (1-\lambda)^2\}\|x_n - y\|^2 + 2\lambda(1-\lambda)(Tx_n - y, x_n - y)\end{aligned}$$

Using (13) we get

(14) $$\begin{aligned}a^2\|x_n - Tx_n\|^2 &= a^2\|x_n - y\|^2 + a^2\|Tx_n - y\|^2 - 2a^2(Tx_n - y, x_n - y) \\ &\le 2a^2\|x_n - y\|^2 - 2a^2(Tx_n - y, x_n - y)\end{aligned}$$

Adding (13) and (14) we obtain

(15) $$\| x_{n+1} - y\|^2 + a^2\|x_n - Tx_n\|^2 \le \{2a^2 + \lambda^2 + (1-\lambda)^2\}\|x_n - y\|^2 + 2\{\lambda(1-\lambda) - a^2\}(Tx_n - y, x_n - y)$$

If we assume that a is such that $a^2 \le \lambda(1-\lambda)$, then from (15) and using Cauchy-Schwarz inequality we get

(16) $$\begin{aligned}&\| x_{n+1} - y\|^2 + a^2\|x_n - Tx_n\|^2 \\ &\le \{2a^2 + \lambda^2 + (1-\lambda)^2\}\|x_n - y\|^2 + \{2\lambda(1-\lambda) - 2a^2\}\|x_n - y\|^2 = \|x_n - y\|^2\end{aligned}$$

Letting $a^2 = \lambda(1-\lambda) > 0$ and summing up (16) from $n = 0$ to $n = N$ we get

$$\begin{aligned}\lambda(1-\lambda)\sum_{n=0}^{N}\|x_n - Tx_n\|^2 &\le \sum_{n=0}^{N}\{\|x_n - y\|^2 - \| x_{n+1} - y\|^2\} \\ &\le \|x_0 - y\|^2 - \| x_{N+1} - y\|^2 \le \|x_0 - y\|^2\end{aligned}$$

Hence $\sum_{n=0}^{\infty}\|x_n - Tx_n\|^2 < \infty$. Since $x_{n+1} - x_n = (1-\lambda)(x_n - Tx_n)$, we obtain

$$\begin{aligned}\lambda(1-\lambda)\sum_{n=0}^{\infty}\|x_n - Tx_n\|^2 &= \lambda(1-\lambda)\sum_{n=0}^{\infty}\frac{1}{(1-\lambda)^2}\|x_{n+1} - x_n\|^2 \\ &= \frac{\lambda}{(1-\lambda)}\sum_{n=0}^{\infty}\|x_{n+1} - x_n\|^2 \le \|x_0 - y\|^2\end{aligned}$$

Hence

$$\sum_{n=0}^{\infty}\|x_{n+1} - x_n\|^2 \le \frac{1-\lambda}{\lambda}\|x_0 - y\|^2 < \infty$$

i.e. T_λ is reasonable wanderer.

Remark 2.1: A similar theorem for nonexpansive mapping was proved by Browder and Petryshyn [2]. Unfortunately the theorem is not stated correctly there. The mapping. T_λ should be defined by $T_\lambda = \lambda I + (1-\lambda)T$ instead of $T_\lambda = I + (1-\lambda)T$.

Lemma 3.1: *Let H be a Hilbert space and C be a nonempty, convex subset of H. Let $T: C \to C$ be a generalized contraction mapping. Suppose F, the fixed point set of T in C is nonempty, then F is convex.*

Proof: We may assume that F consists of more than one point; otherwise the result is proved. Let $x, y \in F$. It is enough to show that $z = \lambda x + (1-\lambda)y,\ 0 < \lambda < 1$ belongs to F. Since T is generalized contraction, from (1) we have

$$\|Tz - x\| \le \|z - x\| \quad \text{and} \quad \|Tz - y\| \le \|z - y\|$$

Now $z - x = \lambda x + (1-\lambda)y - x = -(1-\lambda)(x - y)$. Hence $x - z = (1-\lambda)(x - y)$, and $z - y = \lambda(x - y)$. Thus we obtain

$$\begin{aligned}\|x - y\| &\le \|x - Tz\| + \|Tz - y\| \le \|x - z\| + \|z - y\| \\ &= (1-\lambda)\|x - y\| + \lambda\|x - y\| = \|x - y\|\end{aligned}$$

Hence

$$\|x - Tz\| + \|Tz - y\| = \|x - Tz + Tz - y\|$$

If $x - Tz = 0$, then $\|Tz - y\| = \|x - y\| \leq \lambda\|x - y\|$, whence $1 \leq \lambda$, which is not true. Similarly $Tz - y = 0$ implies that $1 \leq 1 - \lambda$, whence $\lambda \leq 0$, which is not true. Since H is strictly convex, therefore there exists $\alpha > 0$ such that $Tz - x = \alpha(y - Tz)$, whence $Tz = (1 - \beta)x + \beta y$, where $\beta = \alpha/(1 + \alpha)$. We have $Tz - x = \beta(y - x)$ and so

$$\beta\|y - x\| = \|Tz - x\| \leq \|z - x\| = (1 - \lambda)\|x - y\|$$

which gives $\beta \leq 1 - \lambda$. Using $Tz - y = (1 - \beta)(x - y)$, a similar argument gives $\beta \geq 1 - \lambda$. Thus $\beta = 1 - \lambda$ and so $Tz = \lambda x + (1 - \lambda)y = z$, i.e. z belongs to F.

Lemma 3.2: ([5], Proposition 2.5, pp.53). *Let X be a Banach space and g a convex continuous real-valued function on X. Then g is weakly lower semi continuous.*

Lemma 3.3: ([5] Proposition 1.4, pp.32). *Let X be a topological space and C be a compact subset of it. Let $g: X \rightarrow R$ be a lower semi continuous function in X. Then there exists x_0 in C such that $g(x_0) = \inf_{x \in C} g(x)$.*

Definition 3.1: Let X be a Banach space. A mapping $T: X \rightarrow X$ is said to be *demiclosed* if for any sequence $\{x_n\}$ such that $x_n \rightharpoonup x$ (i.e. x_n converges weakly to x) and $Tx_n \rightarrow y$ then $y = Tx$.

Theorem 3.1: *Let H be a Hilbert space and T be a generalized contraction asymptotically regular mapping of H into itself. Suppose T is continuous and $I - T$ is demiclosed. Let F, the fixed point set of T in H be nonempty. Then for each x_0 in H, the sequence of iterates $\{T^n x_0\}$ converges weakly to a point of F.*

Proof: Since F is nonempty we see that a ball B about some fixed point and containing x_0 is mapped into itself by T; consequently B contains the sequence of iterates $\{T^n x_0\}$. So we may restrict ourselves to mappings of a ball into itself. It follows from Lemma 3.1 that F is convex. The continuity of T implies that F is closed. Thus F being closed, bounded and convex is weakly compact.

Define in F the following mapping $g: F \rightarrow R^+$ (R^+ = nonnegative real numbers)

(17) $$g(y) = \inf_n\|T^n x_0 - y\| = \lim_{n\rightarrow\infty}\|T^n x_0 - y\|$$

(In (17) lim = inf, because the sequence $\{\|T^n x_0 - y\|\}$ is nonincreasing, see (18)). The mapping g so defined is continuous. Indeed,

$$g(z) = lim\|T^n x_0 - z\| \leq lim\|T^n x_0 - y\| + \|y - z\| = g(y) + \|y - z\|$$

From this inequality it follows that $|g(y) - g(z)| \leq \|y - z\|$. On the other hand, g is a convex function. In fact

$$\begin{aligned} g(\lambda y + (1 - \lambda)z) &= lim\|T^n x_0 - (\lambda y + (1 - \lambda)z)\| \\ &= lim\|\lambda T^n x_0 - \lambda y + (1 - \lambda)(T^n x_0 - z)\| \\ &\leq \lambda lim\|T^n x_0 - y\| + (1 - \lambda) lim\|T^n x_0 - z\| \\ &= \lambda g(y) + (1 - \lambda)g(z) \end{aligned}$$

Thus using Lemma 3.2 we see that g is weakly lower semi continuous. Now applying Lemma 3.3 we conclude that there exists a point u in F such that

$$g(u) = \alpha = \inf_{y \in F} g(y)$$

Now we claim that u is unique. Suppose there exists another point v of F such that $g(v) = \alpha$ .Since g is convex, for $0 \leq \lambda \leq 1$ we have

$$g(\lambda u + (1 - \lambda)v) \leq \lambda g(u) + (1 - \lambda)g(v) = \lambda\alpha + (1 - \lambda)\alpha = \alpha$$

Thus

$$\begin{aligned} \alpha \geq g(\lambda u + (1 - \lambda)v) &= inf\|T^n x_0 - \lambda u + (1 - \lambda)v\| \\ &= inf\|\lambda(T^n x_0 - u) + (1 - \lambda)(T^n x_0 - v)\| \\ &\geq \lambda\, inf\|T^n x_0 - u\| + (1 - \lambda)\, inf\|T^n x_0 - v\| \\ &= g(u) + (1 - \lambda)g(v) = \lambda\alpha + (1 - \lambda)\alpha = \alpha \end{aligned}$$

Hence $g(\lambda u + (1 - \lambda)v) = \alpha$. Since $u \in F$ and T is generalized contraction, it follows that

(18) $$\begin{aligned} \|T^n x_0 - u\| &= \|T^n x_0 - Tu\| \\ &\leq a\|T^{n-1}x_0 - u\| + b[\|T^{n-1}x_0 - T^n x_0\| + \|u - Tu\|] + c[\|T^{n-1}x_0 - Tu\| + \|u - T^n x_0\|] \\ &\leq a\|T^{n-1}x_0 - u\| + b\|T^{n-1}x_0 - T^n x_0\| + c[\|T^{n-1}x_0 - u\| + \|u - T^n x_0\|] \\ &\leq \tfrac{a+b+c}{1-b-c}\|T^{n-1}x_0 - u\| \leq \|T^{n-1}x_0 - u\| \quad \text{as } a + 2b + 2c = 1. \end{aligned}$$

Thus we obtain

(19) $$\|T^n x_0 - y\| \leq \|T^{n-1}x_0 - u\|$$

Similarly for $v \in F$ it follows that

(20) $$\|T^n x_0 - v\| \le \|T^{n-1} x_0 - v\|$$

So, the sequence $\{\|T^n x_0 - u\|\}$ and $\{\|T^n x_0 - v\|\}$ are nonincreasing. Therefore

$$\|x_n - u\| = \|T^n x_0 - u\| \to \alpha \text{ and } \|x_n - v\| = \|T^n x_0 - v\| \to \alpha$$

Thus from the uniform convexity of H, we conclude that $\|(x_n - u) - (x_n - v)\| \to 0$, i.e. $u = v$.

Finally it remains to show that the sequence $\{T^n x_0\}$ converges weakly to u. Suppose not , then by the reflexivity of H and the boundedness of the sequence $\{T^n x_0\}$, there exists a convergent subsequence of $\{T^{n(j)} x_0\}$ whose limit say z is different from u. Since T is asymptotically regular, it follows that the sequence $\{(I-T)(T^{n(j)} x_0)\}$ tends to zero as $n \to \infty$. Since by hypothesis $I - T$ is demiclosed, $(I-T)z = 0$, i.e. z is a fixed point of T. We claim that $z = u$. Indeed, we have

$$\begin{aligned} \|T^{n(j)} x_0 - u\|^2 &= \|T^{n(j)} x_0 - z + z - u\|^2 \\ &= \|T^{n(j)} x_0 - z\|^2 + \|z - u\|^2 + 2Re(T^{n(j)} x_0 - z, z - u) \end{aligned}$$

Taking limits we obtain

$$g(u) = g(z) + \|z - u\|$$

which is possible only if $z = u$. This completes the proof of the theorem.

Theorem 3.2: *Let X be a reflexive Banach space and T an asymptotically regular generalized contraction mapping from X into itself. Suppose T is continuous and $I - T$ is demiclosed. Let F(T), the fixed point set of T in X be nonempty. Then, for each x_0 in X, every subsequence of $\{T^n x_0\}$ contains a further subsequence which converges weakly to a fixed point of T. In particular, if F(T) consists of precisely one point then the whole sequence $\{T^n x_0\}$ converges to this point.*

Proof: Let y be in $F(T)$. Since T is generalized contraction, it follows that $\|T^n x_0 - y\| \le \|x_0 - y\|$. So the sequence $\{T^n x_0\}$ is bounded. Thus it follows from the reflexivity of X that every subsequence of $\{T^{n(j)} x_0\}$ contains a further subsequence, which we again denoted by $\{T^{n(j)} x_0\}$ such that $T^{n(j)} x_0 \to y$. Now we show that y is a fixed point of T. Indeed, since $T^{n(j)} x_0 \to y$, it follows that $(I-T)T^{n(j)} x_0 \to (I-T)y$. On the other hand since T is asymptotically regular it follows that

$$(I-T)T^{n(j)} x_0 = T^{n(j)} x_0 - T^{n(j)+1} x_0 \to 0 \quad \text{as } n \to \infty$$

Thus $(I-T)y = 0$, i.e. y is a fixed point of T. If $F(T)$ contains only one point y then the whole sequence must converge to y.

Remark 3.1: A theorem similar to our Theorem 3.1 for nonexpansive mapping was proved by Opial [10] and a theorem similar to our Theorem 3.2 was obtained by Browder and Petryshyn [1].

In the sequel we will prove some strong convergence theorems for sequence of iterates for the generalized contraction mapping. We will assume that T is continuous in the present section.

Theorem 4.1: *Let X be a Banach space and T a generalized contractive asymptotically regular mapping of X into itself. Suppose that F(T), the fixed point set of T in X is nonempty. Let us assume that T satisfies the following condition:*

(A) $(I - T)$ maps bounded closed sets into closed sets.

Then for any point x_0 in X, the sequence $\{T^n x_0\}$ converges strongly to some point in F(T).

Proof: Let y be a fixed point of T, since T is generalized contraction, it follows that

$$\|T^{n+1} x_0 - y\| \le \|T^n x_0 - y\|, \qquad n = 1,2,3, \ldots.$$

So the sequence $\{T^n x_0\}$ is bounded. Let D be the strong closure of $\{T^n x_0\}$. By condition (A) it follows that $(I-T)(D)$ is closed. This together with the fact that T is asymptotically regular implies that zero belongs to $(I-T)(D)$. So there exists z in D such that $(I-T)z = 0$. But this implies that either $z = T^n x_0$ for some n, or there exists a sequence $\{T^{n(j)} x_0\}$ converging to z. Since z is a fixed point of T, we conclude that in either case the whole sequence $\{T^n x_0\}$ converges to z.

Corollary 4.1: *Let X be a Banach space and T a generalized contractive asymptotically regular mapping of X into itself. Suppose that F(T), the fixed point set of T in X is nonempty. Let us assume that T satisfies the following condition:*

(A) $(I - T)$ maps bounded closed sets into closed sets.

Then for any point x_0 in X, the sequence $\{x_n\}$ defined by
$$x_{n+1} = \lambda x_n + (1-\lambda)Tx_n, \qquad 0 < \lambda < 1$$
converges strongly to a fixed point of T.

Proof: Let λ be such that $0 < \lambda < 1$. Let $T_\lambda = \lambda I + (1-\lambda)T$. It follows from Theorem 1.1 that T_λ is asymptotically regular. T satisfies condition (A) if and only if T_λ is also does. Indeed, we just observe that $I - T_\lambda = (1-\lambda)(I-T)$. Let us observe that T_λ is not generalized contraction, however, for any y in $F(T)$ it follows from (2) that $\|T_\lambda x - y\| \le \|x - y\|$. From this we conclude that the sequence $\{T_\lambda^n x_0\}$ is bounded, hence the corollary follows from Theorem 4.1.

Definition 4.1: A continuous mapping T from a Banach space X into itself is said to be *demicompact* if every bounded sequence $\{x_n\}$ such that $\{(I-T)(x_n)\}$ converges strongly, contains a strongly convergent subsequence $\{x_{n(j)}\}$.

Remark 4.1: It follows from Proposition II.4 ([5], pp.47) that a demicompact mapping T of a Banach space X into itself satisfies condition (A). Thus we have the following corollary.

Corollary 4.2: *Let X be a uniformly convex Banach space. Let T be a generalized contractive demicompact mapping of X into itself. Then, for each point x_0 in X, the sequence $\{x_n\}$ defined by*
$$x_{n+1} = \lambda x_n + (1-\lambda)Tx_n, \qquad 0 < \lambda < 1$$
converges strongly to a fixed point of T.

Remark 4.2: Theorem 4.1, Corollary 4.1 and Corollary 4.2 for nonexpansive mappings were proved by Browder and Petryshyn [1].

As our final result we prove the following result:

Theorem 4.2: *Let X be strictly convex Banach space and D be compact convex subset of X. Let $T: D \to D$ be continuous generalized contraction. Then the fixed point set $F(T)$ of T is nonempty and compactmoreover for any x_0 in D and any λ such that $0 < \lambda < 1$, $\{T_\lambda^n x_0\}$ converges to a fixed point of T, where*
$$T_\lambda x = (1-\lambda)x + \lambda Tx, \qquad x \in X$$

Proof: By the continuity of T and the Schauder-Tychonoff theorem, it follows that $F(T)$, the fixed point set of T is compact and nonempty. Let $n \ge 0, x_n = T_\lambda^n x_0$. Since D is compact, $\{x_n\}$ has a convergent subsequence $\{x_{k(n)}\}$ which converges to some point $x \in D$. We need to show that x is a fixed point of T. From (1) it follows that $\{\|x_n - y\|\}$, where y is a fixed point of T is monotonically non-increasing. So by the continuity of norm and T_λ we have

(21) $$\|x-y\| = \lim_{n\to\infty}\|x_{k(n+1)} - y\| \le \lim_{n\to\infty}\|x_{k(n)+1} - y\| = \lim_{n\to\infty}\|T(x_{k(n)}) - y\| = \|T_\lambda x - y\|$$

By (21) and (2) we obtain

(22) $$\|T_\lambda x - y\| = \|x-y\|$$

Moreover,

(23) $$\|T_\lambda x - y\| = \|(1-\lambda)x + \lambda Tx - y\| = \|(1-\lambda)(x-y) + \lambda(Tx-y)\| \le (1-\lambda)\|x-y\| + \lambda\|Tx-y\| = \|x-y\|$$

Combining (22) and (23) we conclude that all inequalities in (23) are equalities. So

(24) $$\|(1-\lambda)(x-y) + \lambda(Tx-y)\| = (1-\lambda)\|x-y\| + \lambda\|Tx-y\|$$

And

(25) $$\|Tx-y\| = \|x-y\|$$

By (24) and strict convexity of X, either $x = y$ *or* $Tx - y = t(x-y)$ for some $t > 0$. From (25) it follows that $t = 1$. Thus $Tx - y = x - y$ or $x = Tx$. Hence x is a fixed point of T. It follows from (23) that the sequence $\{\|x_n - \|\}$ is monotonically non-increasing, hence $\{x_n\}$ converges to x.

Remark 4.3: The above result was proved by Edelstein [6] for nonexpansive mappings.

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