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SOME COMMON FIXED POINT THEOREMS WITH PPF DEPENDENCE IN BANACH SPACES

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ABSTRACT

In this paper, we establish the results concerning the existence of common fixed points with PPF dependence for the pairs of operators in Banach spaces satisfying a new inequality initiated by Constantin [6]. The novalty of the present work lies in the fact that the domain and the range spaces of the operators are not same and the results are obtained via constructive method. Our results extend and unify the results of Bernfeld et al. [2], Dhage [7], Sintunavarat and kumam [14] and many others.

Keywords: *PPF dependent fixed point, PPF dependent coincidence point, Razumikhin class.*

2010 MSC: *34K10, 47H09, 47H10, 54H25.*

INTRODUCTION

The theory of fixed points has a broad set of applications in various field of mathematics. In 1922, polish mathematician Stephen Banach published his famous contraction mapping principle. Since, then this principle has been extended and generalized in several ways either by using the contractive condition or imposing some additional conditions on an ambient spaces. In particular, this principle is used to demonstrate the existence and uniqueness of a solution of differential equations, integral equations, functional equations, partial differential equations and others.

On the other hand, the study of fixed points and Banach contraction principle, to the case of non self mappings, is one of the most interesting topic in this field. In this sequel Bernfeld *et al.* [2] introduced the concept of past present future (in short PPF) dependent fixed point or the fixed point with PPF dependence which is one type of fixed points for mapping that have different domains and ranges. They also proved some PPF dependent fixed point theorem in the Razumikhin class for Banach type contraction mappings. Some basic fixed point theorems along this line such as those established in [1] and [4] are integral equations which may depend upon the past history, present data and future consideration. The properties of a special Razumikhin class of functions are employed in the development of fixed point theory with PPF dependence in abstract spaces. After word, a number of people appeared in which PPF dependent fixed point theorems have been discussed (see [1, 8, 9, 10]) and references there in).

In main purpose of this paper is to generalize and extend the result of Bernfeld *et al.* [2], Dhage [7], sintunavarat and kumam [14] for a pair of operators satisfying a new type of inequality initiated by constantin [6] in Banach spaces and obtain some interesting commom fixed point theorems with PPF dependence.

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PRELIMINARIES

In this section, we recall some basic concepts and definitions. Throughout this paper, let E denotes a Banach spaces with the norm$\| \cdot \|_E$, I denotes a closed interval [a, b] in R and E_o = c(I,E) denotes the set of all continuous E- valued functions on I equips with the supremum norms$\| \cdot \|_{Eo}$ defined by

$$\| \varphi \|_{Eo} = \sup_{t \in I} \| \varphi(t) \|_E$$

A point $\varphi \epsilon E_O$ is said to be a PPF dependent fixed point or a fixed point with PPF dependence of an operator T: Eo → E if $T\varphi = \varphi(c)$ for some $C \epsilon I$. For a fixed element, the Razumikhin class or minimal class of functions in Eo is defined by

$$R_c = \{ \varphi \epsilon E_O : \| \varphi \|_{Eo} = \| \varphi(t) \|_E \}$$

It is easy to see that, if the function $\widetilde{\emptyset} \epsilon E_O$ is a constant function then $\widetilde{\emptyset} \epsilon R_c$.

The class R_c is algebraically closed with respect to difference if $\varphi - \xi \epsilon R_c$ when ever $\varphi, \xi \epsilon R_c$. Similarly, R_c is topologically closed if it is closed with respect to the topology on E_o generated by the norm $\| \cdot \|_{Eo.}$

The Razumikhin class play an important role in proving the existence of PPF dependent fixed points with different domain and range of operators in abstract spaces.

Definition 2.1: [2]. An operator T: E_o → E is called Banach type contraction if there is a real number $0 < \alpha < 1$ such that

$$\| T\varphi - T\xi \|_E \leq \alpha \| \varphi - \xi \|_{Eo} \qquad \text{for all } \varphi, \xi \epsilon E_o.$$

The following PPF dependent fixed point theorem is proved in Bernfeld *et al.* [2].

Theorem 2.1: Suppose that T: E_o→ E is a Banach type contraction. Then the following statements holds:
(a) If R_c is algebraically closed with respect to difference, then for a given $\varphi_o \epsilon R_c$ and $c \in I$, every sequence $\{\varphi_n\}$ of iterates of T defined by

$$T\varphi_n = \varphi_{n+1}(c)$$
$$\|\varphi_n - \varphi_{n+1}\|_{Eo} = \|\varphi_n(c) - \varphi_{n+1}(c)\|_E$$

For n = 0, 1, 2……… Converges to a PPF dependent fixed point of T.

(b) If R_c is topologically closed, then T has a unique fixed point in R_c.

Definition 2.2: [7]. An operator T: E_o → E is called strong kannan type contraction if

$$\| T\varphi - T\xi \|_E \leq \alpha [\| \varphi(c) - T\varphi \|_E + \| \xi(c) - T\xi \|_E]$$

for all $\varphi, \xi \epsilon E_o$ and some $c \in I$, where $0 < \alpha < \frac{1}{2}$.

The following PPF dependent fixed point theorem proved in Dhage [7]

Theorem 2.2: Suppose that T: E_o → E is a strong kannan type contraction. Then the following statement holds
(a) If R_c is algebraically closed with respect to difference, then for a given $\varphi_o \in R_c$ and $c \in I$, every sequence $\{\varphi_n\}$ of iterates of T defined by (2.1) converges to a PPF dependent fixed point of T.
(b) If R_c is topologically closed, then T has a unique PPF dependent fixed point in R_c.

Remark 2.1:
(i) The statement (a) in the above theorem theorem 2.2., it is assumed that the Razumikhin class R_c of functions in E_o is algebraically closed with respect to the difference, that is $\varphi - \xi \epsilon R_c$, whenever $\varphi, \xi \epsilon R_c$. Otherwise the construction of the sequence $\{\varphi_n\}$ made there is not possible because of the fact that

$$\| \varphi - \xi \|_{Eo} = \|\varphi(c) - \xi(c)\|_E = \|(\varphi - \xi)(c)\|_E.$$

(ii) If the Razumikhin class R_c is not topologically closed then the limit of the sequence $\{\varphi_n\}$ may be outside of R_c. Therefore, PPF dependent fixed point of T may not be unique.

Definition 2.3: [14]. An operator T: E_o → E is said to satisfy a condition of rational type contraction if there exist real numbers $\alpha, \beta \epsilon [0,1)$ with $\alpha + \beta < 1$ and c ∈ I such that

$$\| T\varphi - T\xi \|_E \leq \alpha \| \varphi - \xi \|_{Eo} + \beta \frac{\| \varphi(c) - T\varphi \|_E + \| \xi(c) - T\xi \|_E}{1 + \| T\varphi - T\xi \|_E}$$

For all $\varphi, \xi \epsilon E_o$.

The following PPF dependent fixed point theorem is proved in sintunavarat and kumam [14].

Theorem 2.3: Let T: $E_o \to E$ be a rational type contraction. If R_c is topologically closed and algebraically closed with respect to closed with respect to difference, then T has a unique PPF dependent fixed point in R_c.

Moreover, for a fixed $\varphi_o \epsilon Rc$, if a sequence $\{\varphi_n\}$ of iterates of T defined by (2.1) then φ_n converges to a PPF dependent fixed point of T.

MAIN RESULTS

Following constantin [6], we recall the following definition.

Definition 3.1: Consider the set $\mathcal{L}$ of all real continuous functions g: $[0,\infty)^5 \to [0,\infty)$ satisfying the following properties:
(i) g is non decreasing in 4^{th} and 5^{th} variables;
(ii) There is an $\lambda_1 > 0$ an $\lambda_2 > 0$ such that $= \lambda_1 . \lambda_2 < 1$ and if u, v $\in [0,\infty)$ satisfying

$u \leq g$ (v, v, u, u+v, 0)
Or $u \leq g$ (v, u, v, u+v, 0)
then $u \leq \lambda_1 v$

and if u, v $\in [0,\infty)$ satisfying

$u \leq g$ (v, v, u, 0, u+v)
Or $u \leq g$ (v, u, v, 0, u+v)
then $u \leq \lambda_2 \ v$;

(iii) If u $\in [0,\infty)$ is such that $u \leq g$ (u, 0, 0, u, u) or $u \leq g$ (0, u, 0, u, u)
Or $u \leq g$ (0, 0, u, u, u), then u = 0.

Definition 3.2: Let S, T: $E_o \to E$ be two operators. A point $\varphi^* \epsilon E_o$ is called a PPF dependent common fixed point of S and T if $s\varphi^* = \varphi^* (c) = T \varphi^*$ for some c $\in I$.

Now we are able to prove our main result of this section.

Theorem 3.1: Let S, T: $E_o \to E$ be two operators. If there exists a g $\in \mathcal{L}$ such that for all $\varphi, \xi \epsilon E_o$ for some c $\in I$.

$$\| S\varphi - T \xi \|_E \leq g (\| \varphi - \xi\|_{Eo} , \| \varphi(c) - S\varphi \|_E, \| \xi(c) - T \xi \|_E, \| \varphi(c) - T \xi \|_E , \| \xi(c) - S\varphi \|_E) \tag{3.1}$$

Then the following statements hold.

(a) If R_c is algebraically closed with respect to difference, then for a given $\varphi_o \in E_o$ $and\ c \in I$, every sequence $\{\varphi_n\}$ of iterates of T defined by

$$S\varphi_{2n} = \varphi_{2n+1}(c), \quad T\varphi_{2n+1} = \varphi_{2n+2}(c);$$

and $\|\varphi_n - \varphi_{n+1}\|_{Eo} = \|\varphi_n(c) - \varphi_{n+1}(c)\|_E$ for n = 0, 1, 2,…

Converges to a PPF dependent common fixed point of S and T.

(b) If R_c is topologically closed, then S and T has a unique PPF dependent fixed point in R_c.

Proof:
(a) Let $\varphi_o \epsilon E_o$ be arbitrary and define a sequence $\{\varphi_n\}$ in E_o as follows. By hypothesis, $S\varphi_o \epsilon E$. Suppose that $S\varphi_o = x_1$, choose $\varphi_1 \epsilon E_o$ such that $x_1 = \varphi_1 (c)$ and $\|\varphi_1 - \varphi_o \|_{Eo} = \|\varphi_1(c) - \varphi_o(c)\|_E$. Again by hypothesis, $T\varphi_1 \epsilon E$. Suppose that $T\varphi_1 = \varphi_2$, choose $\varphi_2 \epsilon E_o$ such that $x_2 = \varphi_2 (c)$ and $\|\varphi_2 - \varphi_1 \|_{Eo} = \|\varphi_2(c) - \varphi_1(c)\|_E$. Proceeding in this way , by induction, we obtain

$$S\varphi_{2n} = \varphi_{2n+1}(c) ; \quad T\varphi_{2n+1} = \varphi_{2n+2}(c)$$

and $\|\varphi_n - \varphi_{n+1}\|_{Eo} = \|\varphi_n(c) - \varphi_{n+1}(c)\|_E$ for n = 0, 1, 2, …..

Now for n=0 and using (3.1) we have the following,

$$\begin{aligned}
\| \varphi_1 - \varphi_2 \|_{Eo} &= \| \varphi_1(c) - \varphi_2(c)\|_E = \| S\varphi_o - T\varphi_1\|_E \\
&\leq g (\| \varphi_0 - \varphi_1 \|_{Eo} , \| \varphi_0(c) - S\varphi_0\|_E , \| \varphi_1(c) - T\varphi_1\|_E , \| \varphi_0(c) - T\varphi_1\|_E , \\
&\quad \| \varphi_1(c) - S\varphi_0\|_E) \\
&\leq g (\| \varphi_0 - \varphi_1 \|_{Eo} , \| \varphi_0(c) - \varphi_1(c)\|_E , \| \varphi_1(c) - \varphi_2(c)\|_E , \| \varphi_0(c) - \varphi_2(c)\|_E, \\
&\quad \| \varphi_1(c) - \varphi_1(c)\|_E) \\
&\leq g (\| \varphi_0 - \varphi_1 \|_{Eo} , \| \varphi_0 - \varphi_1 \|_{Eo}, \| \varphi_1 - \varphi_2 \|_{Eo} , \| \varphi_0 - \varphi_1 \|_{Eo} + \| \varphi_1 - \varphi_2 \|_{Eo} , 0)
\end{aligned}$$

Which implies in view of definition (3.1), that
$\| \varphi_1 - \varphi_2 \|_{Eo} \leq \lambda_1 \| \varphi_0 - \varphi_1 \|_{Eo}$
Again

$$\begin{aligned}
\| \varphi_2 - \varphi_3 \|_{Eo} &= \| \varphi_2(c) - \varphi_3(c)\|_E = \| S\varphi_2 - T\varphi_1\|_E \\
&\leq g (\| \varphi_2 - \varphi_1 \|_{Eo} , \| \varphi_2(c) - S\varphi_2\|_E , \| \varphi_1(c) - T\varphi_1\|_E , \| \varphi_2(c) - T\varphi_1\|_E, \\
&\quad \| \varphi_1(c) - S\varphi_2\|_E)
\end{aligned}$$

$$\leq g\,(\| \varphi_2 - \varphi_1 \|_{E_o}, \| \varphi_2(c) - \varphi_3(c)\|_E, \| \varphi_1(c) - \varphi_2(c)\|_E, \| \varphi_2(c) - \varphi_2(c)\|_E, \| \varphi_1(c) - \varphi_3(c)\|_E)$$
$$\leq g\,(\| \varphi_2 - \varphi_1 \|_{E_o}, \| \varphi_2(c) - \varphi_3(c)\|_E, \| \varphi_1(c) - \varphi_2(c)\|_E, 0, \| \varphi_1(c) - \varphi_2(c)\|_E + \| \varphi_2(c) - \varphi_3(c)\|_E)$$
$$\leq g\,(\| \varphi_2 - \varphi_1 \|_{E_o}, \| \varphi_2 - \varphi_3 \|_{E_o}, \| \varphi_1 - \varphi_2 \|_{E_o}, 0, \| \varphi_1 - \varphi_2 \|_{E_o} + \| \varphi_2 - \varphi_3 \|_{E_o})$$

Which implies in view of definition (3.1), that

$$\| \varphi_2 - \varphi_3 \|_{E_o} \leq \lambda_2 \| \varphi_1 - \varphi_2 \|_{E_o} \leq \lambda_1 \lambda_2 \| \varphi_0 - \varphi_1 \|_{E_o} \leq \lambda \| \varphi_0 - \varphi_1 \|_{E_o}$$

Proceeding in this way, by induction, we get

$$\|\varphi_n - \varphi_{n+1}\|_{E_o} \leq \lambda \| \varphi_{n-1} - \varphi_n \|_{E_o} \leq \ldots\ldots\ldots.. \leq \lambda^n \| \varphi_0 - \varphi_1 \|_{E_o} \text{ for all n = 0, 1, 2…}$$

Now, we shall show that $\{\varphi_n\}$ is a cauchy sequence. If m > n, then by triangular inequality, we have

$$\|\varphi_n - \varphi_m\|_{E_o} \leq \|\varphi_n - \varphi_{n+1}\|_{E_o} + \ldots\ldots\ldots + \| \varphi_{m-1} - \varphi_m \|_{E_o}$$
$$\leq [\lambda^n + \lambda^{n+1} + \ldots\ldots\ldots\ldots + \lambda^{m-1}] \| \varphi_0 - \varphi_1 \|_{E_o}$$
$$\leq \frac{\lambda^n}{\lambda+1} \| \varphi_0 - \varphi_1 \|_{E_o}$$

Hence $\lim\limits_{m>n\to\infty} \|\varphi_n - \varphi_m\|_{E_o} = 0$

As a result, the sequence $\{\varphi_n\}$ is a cauchy sequence. Since E_o is complete, $\{\varphi_n\}$ and every subsequence of it converges to a limit point φ^* in E_o, that is $\lim\limits_{n\to\infty}\varphi_n = \varphi^*$ and that $\lim\limits_{n\to\infty}\varphi_{2n+1} = \varphi^* \lim\limits_{n\to\infty}\varphi_{2n+2}$.

We prove that φ^* is a PPF dependent fixed point of S and T. By inequality (3.1), we have

$$\| S\varphi^* - \varphi^*(c)\|_E \leq \| S\varphi^* - \varphi_{2n+2}(c)\|_E + \| \varphi_{2n+2}(c) - \varphi^*(c) \|_E$$

$$\leq \| S\varphi^* - T\varphi_{2n+1}(c)\|_E + \|\varphi_{2n+2} - \varphi^*\|_{E_o}$$

$$\leq g\,(\|\varphi^* - \varphi_{2n+1} \|_{E_o}, \| \varphi^*(c) - S\varphi^*\|_E, \| \varphi_{2n+1}(c) - T\varphi_{2n+1}(c)\|_E, \| \varphi^*(c) - T\varphi_{2n+1}\|_E, \| \varphi_{2n+1}(c) - S\varphi^*\|_E) + \|\varphi_{2n+2} - \varphi^*\|_{E_o}$$

$$\leq g\,(\|\varphi^* - \varphi_{2n+1} \|_{E_o}, \| \varphi^*(c) - S\varphi^*\|_E, \| \varphi_{2n+1}(c) - \varphi_{2n+2}(c)\|_E, \| \varphi^*(c) - \varphi_{2n+2}\|_E, \| \varphi_{2n+1}(c) - S\varphi^*\|_E) + \|\varphi_{2n+2} - \varphi^*\|_{E_o}$$

Taking the linit superior as n → ∞ in the above inequality, we obtain

$$\| S\varphi^* - \varphi^*(c)\|_E \leq g\,(0, \| \varphi^*(c) - S\varphi^*\|_E, 0, 0, \| \varphi^*(c) - S\varphi^*\|_E)$$

Which gives a contradiction, in view a definition 3.1.

$$\| S\varphi^* - \varphi^*(c)\|_E = 0$$

Hence it follows that $S\varphi^* = \varphi^*(c)$. Similarly it is proved that $T\varphi^* = \varphi^*(c)$.

(b) To prove uniqueness of PPF dependent fixed point in R_c, let φ^* and ξ^* be two fixed points, then

$$\| \varphi^* - \xi^* \|_{E_o} = \| \varphi^*(c) - \xi^*(c)\|_E$$
$$= \| S\varphi^* - T\xi^*\|_E$$
$$\leq g\,(\|\varphi^* - \xi^*\|_{E_o}, \| \varphi^*(c) - S\varphi^*\|_E, \|\xi^*(c) - T\xi^*\|_E, \| \varphi^*(c) - T\xi^*\|_E, \|\xi^*(c) - S\varphi^*\|_E)$$
$$\leq g\,(\|\varphi^* - \xi^*\|_{E_o}, 0, 0, \| \varphi^*(c) - \xi^*(c)\|_E, \|\xi^*(c) - \varphi^*(c)\|_E)$$
$$\leq g\,(\|\varphi^* - \xi^*\|_{E_o}, 0, 0, \|\varphi^* - \xi^*\|_{E_o}, \|\varphi^* - \xi^*\|_{E_o})$$

Which implies, by definition 3.1, that

$$\|\varphi^* - \xi^*\|_{E_o} = 0 \quad \text{i.e. } \varphi^* = \xi^*$$

This completes the proof of the theorem 3.1.

The following corollary is an analogue of Banach's contraction principle and extends and unify Theorem 2.1 for pair of operators.

Corollary 3.1: Let S, T: $E_o \rightarrow E$ be two operators. If there exists a real number $0 \leq \lambda < 1$ such that

$$\| S\varphi - T\xi \|_E \leq \lambda \| \varphi - \xi \|_{E_o},$$

for all $\varphi, \xi \in E_o$ then the following statements holds.

(a) If R_c is algebraically closed with respect to difference then for a given $\varphi_o \in E_o$ $and\ c \in I$, every sequence $\{\varphi_n\}$ of iterates defined as in (3.2) converges to a PPF dependent common fixed point of S and T.

(b) If R_c is topologically closed then S and T have a unique PPF dependent fixed point in R_c.

Proof: The assertion follows using Theorem 3.1 with g(u, v, w, x, y) = λ u for some $\lambda \in [0,1)$ and all u,v,w,x,y $\in [0, \infty)$.

The following corollary is considered as a PPF dependent version of Kannan's result in [12] and extends Theorem 2.2 for pair of operators.

Corollary 3.2: Suppose that S, T: $E_o \rightarrow E$ be two operators. If there exists a real number $0 \leq \lambda < \frac{1}{2}$ such that

$$\| S\varphi - T\xi \|_E \leq a [\ \| \varphi(c) - S\varphi \|_E + \| \xi(c) - T\xi \|_E\ ]$$

for all $\varphi, \xi \in E_o$ and for some $c \in I$. Then the following statements holds.

(a) If R_c is algebraically closed with respect to difference then for a given $\varphi_o \in E_o$ $and\ c \in I$, every sequence $\{\varphi_n\}$ of iterates defined as in (3.2) converges to a PPF dependent common fixed point of S and T.

(b) If R_c is topologically closed then S and T have a unique PPF dependent fixed point in R_c .

Proof: The assertion follows using Theorem 3.1 with

$$g(u, v, w, x, y) = a(v + w)$$

for some $a \in [0, \frac{1}{2})$ and all u, v, w, x, y $\in [0, \infty)$.

Indeed $g \in \mathcal{L}$ is continuous and condition (i) is obvious.

First we have

$$g(v, v, u, u+v, 0) = a(v + u)$$

So if $u \leq g(v, v, u, u+v, 0)$ then $u \leq a(v + u)$, which implies that $u \leq \frac{a}{1-a}v$ with $\frac{a}{1-a} < 1$. Similarly , if $u \leq g(v, u, v, 0, u+v)$ then $u \leq a(v + u)$, gives implies that $u \leq \frac{a}{1-a}v$ with $\frac{a}{1-a} < 1$. There fore S and T satisfies condition (ii).

Next, if $u \leq g(u, 0, 0, u, u) = a(0 + 0)$, then u = 0. Thus T and S satisfies condition (iii).

The following corollary is considered as a PPF dependent version of Bianchini's result in [3].

Corollary 3.3: Suppose that S, T: $E_o \rightarrow E$ be two operators. If there exists a real number $0 \leq h < 1$ such that

$\| S\varphi - T\xi \|_E \leq h\, max\{ \| \varphi(c) - S\varphi \|_E , \| \xi(c) - T\xi \|_E \}$

for all $\varphi, \xi \in E_o$ and for some $c \in I$. Then the following statements holds.

(a) If R_c is algebraically closed with respect to difference, then for a given $\varphi_o \in E_o$ $and\ c \in I$, every sequence $\{\varphi_n\}$ of iterates defined as in (3.2) converges to a PPF dependent common fixed point of S and T.

(b) If R_c is topologically closed then S and T have a unique PPF dependent fixed point in R_c .

Proof: The assertion follows using Theorem 3.1 with

$$g(u, v, w, x, y) = h \max \{v + w\}$$

for some $h \in [0, 1)$ and all u, v, w, x, y $\in [0, \infty)$. Indeed, $g \in \mathcal{L}$ is continuous and condition (i) is obvious.

First we have

$$g(v, v, u, u+v, 0) = h \max \{v + u\}$$

Then if $u \leq g(v, v, u, u+v, 0) = h \max \{v + u\}$ implies that $u \leq hv$ or $u \leq hu$.

Therefore $u \leq hv$ as $h \in [0, 1)$. Similarly if $u \leq g(v, v, u, u+v, 0) = h \max \{v, u\}$ implies that $u \leq hv$, $h \in [0, 1)$. Thus S and T satisfies condition (ii).

Moreover, if $u \leq g(u, 0, 0, u, u) = h \max \{0 + 0\}$, gives u = 0. Therefore S and T satisfies condition (iii).

The following corollary is considered as a PPF dependent analogue of Reich's result in [13].

Corollary 3.4: Let S, T: $E_o \to E$ be two operators satisfying
$\| S\varphi - T\xi \|_E \leq a \ \| \varphi - \xi \|_{E_o} + b \| \varphi(c) - S\varphi \|_E + c \| \xi(c) - T\xi \|_E$
for all $\varphi, \xi \in E_o$ and for some a, b, $c \geq 0$ with a+b+c< 1. Then the following statements holds.

(a) If R_c is algebraically closed with respect to difference then for a given $\varphi_o \in E_o$ *and* $c \in I$, every sequence $\{\varphi_n\}$ of iterates defined as in (3.2) converges to a PPF dependent common fixed point of S and T.
(b) If R_c is topologically closed then S and T have a unique PPF dependent fixed point in R_c.

Proof: The assertion follows from Theorem3.1. with g (u, v, w, x, y) = au+bv+cw for some a, b, c ≥0 with a+b+c <1 and all u, v, w, x, y ∈[0,∞) .

Since g∈ $\mathcal{L}$ is continuous, condition (i) is obvious.

Now, we have g (v, v, u, u+v, 0) = av +bv+ cu.

So, if u ≤ g (v, v,u,u+v,0)) = av+bv+cu gives

$$u \leq \frac{a+b}{1-c} v \text{ with } \frac{a+b}{1-c} < 1 .$$

Similarly, if u ≤ g (v, u, v, 0, u+v) = av +bu+ cv which implies that

$$u \leq \frac{a+b}{1-b} v \quad \text{with } \frac{a+c}{1-b} < 1 \text{ that is } \quad u \leq \lambda v \quad \text{where}$$
$$\lambda = \max \left\{ \frac{a+b}{1-c}, \frac{a+c}{1-b} \right\} < 1.$$

Therefore condition (ii) is satisfied.

Moreover, if u ≤ g (u, 0, 0, u, u) = au, then u = 0 since a<1.

Therefore S and T satisfies condition (iii).

The following corollary is considered as a PPF dependent version of Chattarjee's result in [5].

Corollary 3.5: Let S, T: $E_o \to E$ be two operators. If there exists a real number $0 \leq h < \frac{1}{2}$ such that;

$$\| S\varphi - T\xi \|_E \leq h \max \{ \| \varphi(c) - T\xi \|_E , \| \xi(c) - S\varphi \|_E \}$$

for all $\varphi, \xi \in E_o$ and for some $c \in I$. Then the following statements holds.

(a) If R_c is algebraically closed with respect to difference then for a given $\varphi_o \in E_o$ *and* $c \in I$, every sequence $\{\varphi_n\}$ of iterates defined as in (3.2) converges to a PPF dependent common fixed point of S and T.
(b) If R_c is topologically closed then S and T have a unique PPF dependent fixed point in R_c.

Proof: The assertion follows from Theorem3.1. with g (u, v, w, x, y) = h max {x, y} for some h ∈ [0, $\frac{1}{2}$) and all u, v, w, x, y ∈[0,∞). Indeed, g ∈ $\mathcal{L}$ is continuous. First we have g (v, v, u, u+v, 0) = h max {u+v,0} so, if u ≤ g (v, v, u, u+v, 0) = h max {u+v,0}, implies that $u \leq \frac{h}{1-h} v$ with $\frac{h}{1-h} < 1$.

Similarly, if u ≤ g (v, u, v, 0, u+v) = h max {0,u+v}, gives $u \leq \frac{h}{1-h} v$ with $\frac{h}{1-h} < 1$.

Therefore S and T satisfies condition (ii).

Moreover, if u ≤ g (u, 0, 0, u, u) = h max {u,u}, implies that u = 0 as h ∈ [0, $\frac{1}{2}$) .

Therefore S and T satisfies condition (iii).

Corollary 3.7: [*, Theorem 3.3]. Let S, T: $E_o \to E$ be two operators. If there exists a real number $0 \leq h < 1$ such that

$$\| S\varphi - T\xi \|_E \leq h \max \Big\{ \| \varphi - \xi \|_{E_o} , \| \varphi(c) - S\varphi \|_E , \| \xi(c) - T\xi \|_E , \frac{1}{2} \left[\| \varphi(c) - T\xi \|_E + \| \xi(c) - S\varphi \|_E \right] \Big\}$$

for all $\varphi, \xi \in E_o$ and for some $c \in I$. Then the following statements hold.

(a) If R_c is algebraically closed with respect to difference then for a given $\varphi_o \in E_o$ *and* $c \in I$, every sequence $\{\varphi_n\}$ of iterates defined as in (3.2) converges to a PPF dependent common fixed point of S and T.
(b) If R_c is topologically closed then S and T have a unique PPF dependent fixed point in R_c.

Proof: The assertion follows from Theorem 3.1. with g (u, v, w, x, y) = h max {u,v,w, $\frac{1}{2}$ (x+y)}. Since $g \in \mathcal{L}$ is continuous, condition (i) is obvious. Now we have g (v, v, u, u+v, 0) = h max {v, v, u, $\frac{1}{2}$ (u+v)}. So if u $\leq$ g (v, v, u, u+v, 0) = h max {v, v, u, $\frac{1}{2}$ (u+v)} implies that u $\leq hv$ or
u $\leq \frac{h}{2}$ (u+v) . Therefore u $\leq kv$ with k = max { h, $\frac{h}{2-h}$ } <1.

Similarly, if u $\leq$ g (v, u, v, 0, u+v) = h max {v, u, v, $\frac{1}{2}$ (u+v)} gives u $\leq kv$ with k = max { h, $\frac{h}{2-h}$ } <1. Therefore S and T satisfies condition (ii).

Moreover, if u $\leq$ g (u, 0, 0, u, u) = h max {u, 0, 0, $\frac{1}{2}$ (u+v)}. Then u = 0 with $0 \leq h < 1$

Therefore condition (iii) is satisfied.

The following corollary is considered as a PPF dependent version of Hardy and Roger's result in [11].

Corollary 3.8: Let S, T: $E_o \to E$ be two operators satisfying
$\| S\varphi - T\xi \|_E \leq a_1 \| \varphi - \xi \|_{E_o} + a_2 \| \varphi(c) - S\varphi \|_E + a_3 \| \xi(c) - T\xi \|_E + a_4 \| \varphi(c) - T\xi \|_E + a_5 \| \xi(c) - S\varphi \|_E$
for all $\varphi, \xi \in E_o$, $c \in I$ ans fore some $a_1, a_2, a_3, a_4, a_5 \geq 0$ with

$$\max\{ a_1 + a_2 + a_3 + 2a_4 , a_1 + a_2 + a_3 + 2a_5\} < 1.$$

Then the following statements hold.

(a) If R_c is algebraically closed with respect to difference then for a given $\varphi_o \in E_o$ *and* $c \in I$, every sequence $\{\varphi_n\}$ of iterates defined as in (3.2) converges to a PPF dependent common fixed point of S and T.
(b) If R_c is topologically closed then S and T have a unique PPF dependent fixed point in R_c.

Proof: The assertion follows from Theorem 3.1. with g (u, v, w, x, y) = $a_1 u + a_2 v + a_3 w + a_4 x + a_5 y$. Since $g \in \mathcal{L}$ is continuous, condition (i) is obvious. Now we have

$$g(v,v,u,u+v,0) = a_1 v + a_2 v + a_3 u + a_4(u+v).$$

so, if

$$u \leq g(v,v,u,u+v,0) = a_1 v + a_2 v + a_3 u + a_4(u+v) \text{ implies that } u \leq hv \text{ with } h = \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} < 1.$$

Similarly if u $\leq$ g (v, u, v, 0, u+v) = $a_1 v + a_2 u + a_3 v + a_5(u+v)$. Then

$$u \leq kv \text{ with hk} = \frac{a_1 + a_3 + a_5}{1 - a_2 - a_5} < 1 \text{ i.e. } u \leq \lambda v \text{ where } \lambda = \max\{ h, k\} < 1.$$

Therefore S and T satisfies condition (ii).

Moreover, if u $\leq$ g (u, 0, 0, u, u,) = $a_1 u + a_4 u + a_5 u$. Then u = 0 as $a_1 + a_4 + a_5 < 1$. Thus condition (iii) satisfied.

The following corollary extends and unify theorem 2.3 of Sintunavarat and Kumam [14] for pairs of operators.

Corollary 3.9: Let S, T: $E_o \to E$ be two operators. If there exists $\alpha, \beta \epsilon [0,1)$ with $\alpha + \beta < 1$ such that

$$\| S\varphi - T\xi \|_E \leq \alpha \| \varphi - \xi \|_{E_o} + \beta \frac{\| \varphi(c) - S\varphi \|_E \,.\, \| \xi(c) - T\xi \|_E}{1 + \| \varphi - \xi \|_{E_o}}$$

for all $\varphi, \xi \in E_o$ and for some $c \in I$. Then the following statements hold.

(a) If R_c is algebraically closed with respect to difference then for a given $\varphi_o \in E_o$ *and* $c \in I$, every sequence $\{\varphi_n\}$ of iterates defined as in (3.2) converges to a PPF dependent common fixed point of S and T.
(b) If R_c is topologically closed then S and T have a unique PPF dependent fixed point in R_c .

Proof: The assertion follows from Theorem 3.1. with

$$g(u, v, w, x, y) = \alpha u + \beta \frac{u.w}{1+u}.$$

Since $g \in \mathcal{L}$ is continuous, condition (i) is obvious. First we have

$$g(v,v,u,u+v,0) = \alpha v + \beta \frac{v.u}{1+v}$$

So if

$$u \leq g(v,v,u,u+v,0) = \alpha v + \beta \frac{v.u}{1+v}.$$

Then

$$u \leq \alpha v + \frac{\beta uv}{1+v}$$

implies

$$u \leq \alpha v + \frac{\beta u(1+v)}{1+v}$$

i.e. $u \leq \alpha v + \beta u.$ Then $u \leq \frac{\alpha}{1-\beta} v$

with $\frac{\alpha}{1-\beta} < 1$. Similarly $u \leq g(v, u, v, 0, u+v) = \alpha v + \beta \frac{uv}{1+v}$

Then $u \leq \frac{\alpha}{1-\beta} v$ with $\frac{\alpha}{1-\beta} < 1$. Therefore S and T satisfies condition (ii).

Moreover, if $u \leq g(u,0,0,u,u,) = \alpha u + \beta \frac{0.0}{1+u} = \alpha u$. Then u = 0 as $\alpha < 1$. Therefore S and T satisfies condition (iii).

On taking S = T in theorem 3.1 we obtain the following corollary as a special case of theorem 3.1.

Corollary 3.10. Suppose that $T: E_o \to E$ be an operator. If there is a $g \in \mathcal{L}$ such that for all $\varphi, \xi \in E_o$ and for some $c \in I$

$$\| T\varphi - T\xi \|_E \leq g(\| \varphi - \xi\|_{E_o}, \| \varphi(c) - T\varphi \|_E, \| \xi(c) - T\xi \|_E, \| \varphi(c) - T\xi \|_E, \| \xi(c) - T\varphi \|_E).$$

Then the following statements hold.

(a) If R_c is closed with respect to difference then for a given $\varphi_o \in E_o$, every sequence $\{\varphi_n\}$ of iterates defined by

$$T\varphi_n = \varphi_{n+1}(c)$$
$$\|\varphi_n - \varphi_{n+1}\|_{E_o} = \|\varphi_n(c) - \varphi_{n+1}(c)\|_E$$

for n = 0, 1, 2,….converges to a PPF dependent common fixed point of T.

(b) If R_c is algebraically and topologically closed then for a given $\varphi_o \in E_o$, every sequence $\{\varphi_n\}$ defined as in (3.3) converges to a unique PPF dependent fixed point in R_c .

Remark 3.1:

(a) We note that the operators in theorem 3.1 and corollary 3.10. aer not required to satisfy any continuity condition on the domains of their definitions.
(b) Corollary 3.10. includes theorem 2.1, theorem 2.2 and theorem 2.3 as a special case in view of (a) and definition 3.1.

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