

α^* - REGULAR & α^* - NORMAL SPACE

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ABSTRACT

The purpose of this paper is to introduce new space namely α^ -regular, α^* -normal, using α^* - open sets and investigate their properties. We also study the relationships among themselves.*

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I. INTRODUCTION

Separation axioms are useful in classifying topological spaces. Maheswari and Prasad introduces the notation of s-regular and s- normal spaces using semi-open sets, Dorsett introduces the concept of semi – regular and semi – normal spaced and investigate their propertied.

In this paper, we define α^* - regular, α^* - normal, using α^* - open sets and investigate their basic properties. We further study the relationships among themselves

II. PRILIMINARIES

Throughout this paper (X, τ) will always denote a topological space on which no separation axioms are assumed, unless explicitly stated. If A is a subset of the space (X, τ) , $Cl(A)$ and $Int(A)$ respectively denote the closure and the interior of A in X .

Definition 2.1[7]: A subset A of a topological space (X, τ) is called

- (i) generalized closed (briefly g - closed) if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X .
- (ii) generalized open (briefly g - open) if $X \setminus A$ is g - closed in X .

Definition 2.2: Let A be a subset of X . Then

- (i) generalized closure [5] of A is defined as the intersection of all g - closed sets containing A and is denoted by $Cl^*(A)$.
- (ii) generalized interior of A is defined as the union of all g - open subsets of A and is denoted by $Int^*(A)$.

Definition 2.3: A subset A of a topological space (X, τ) is called

- (i) α^* -open [8] $A \subseteq Int^*(Cl(Int^*(A)))$.
- (ii) α^* -closed [8] if $Cl^*(Int(Cl^*(A))) \subseteq A$.

The class of all α^* -open (resp. α^* -closed) sets is denoted by $\alpha^*O(X, \tau)$ (resp. $\alpha^*C(X, \tau)$).

The α^* -interior of A is defined as the union of all α^* -open sets of X contained in A . It is denoted by $\alpha^*Int(A)$. The α^* -closure of A is defined as the intersection of all α^* -closed sets in X containing A . It is denoted by $\alpha^*Cl(A)$.

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Definition 2.4: A topological space X is said to be regular if for every pair consisting of a point x and a closed set B not containing x , there are disjoint open sets U and V in X containing x and B respectively. [5]

Definition 2.5: A topological space X is said to be normal if for every pair of disjoint closed sets A and B in X , there are disjoint open sets U and V in X containing A and B respectively. [5]

Definition 2.6: A function $f: X \rightarrow Y$ is said to be

- (i) closed [5] if $f(V)$ is closed in Y for every closed set V in X .
- (ii) α^* -continuous [6] if $f^{-1}(V)$ is α^* -open in X for every open set V in Y .
- (iii) α^* -open [7] if $f(V)$ is α^* -open in Y for every open set V in X .
- (iv) pre- α^* -open [7] if the image of every α^* -open set of X is α^* -open in Y .
- (v) contra-pre- α^* -open [7] if $f(V)$ is α^* -closed in Y for every α^* -open set V in X .
- (vi) pre- α^* -closed [7] if $f(V)$ is α^* -closed in Y for every α^* -closed set V in X .

III. REGULAR SPACES ASSOCIATE WITH α^* -OPEN SETS

In this section we introduce the concepts of α^* -regular and α^* -regular spaces. Also we investigate their basic properties and study their relationship with already existing concepts.

Definition 3.1: A Space X is said to be α^* -regular if for every pair consisting of a point x and a α^* -closed set B not containing x , there are disjoint α^* -open sets U & V in X containing x & B respectively.

Theorem 3.2: In a topological space X , the following are equivalent.

- 1) X is α^* -regular.
- 2) For $x \in X$ and every α^* -open set U containing x , there exist a α^* -open set V containing x such that $\alpha^*Cl(V) \subseteq U$.
- 3) For every set A & α^* -open set B such that $A \cap B \neq \emptyset$, there exists a α^* -open set U such that $A \cap U \neq \emptyset$ and $\alpha^*Cl(U) \subseteq B$.
- 4) For every non - empty set A and α^* -closed sets B such that $A \cap B = \emptyset$, there exist disjoint α^* -open set U and V such that $A \cap U \neq \emptyset$ and $B \subseteq V$.

Proof:

(i) \Rightarrow (ii): Let U be a α^* -open set containing x . Then $B = X \setminus U$ is a α^* -closed not containing x . Since X is α^* -regular, there exist disjoint α^* -open sets V and W containing x and B respectively. If $y \in B$, W is a α^* -open set containing y that does not intersect V . Therefore $\alpha^*Cl \subseteq U$.

(ii) \Rightarrow (iii): Let $A \cap B \neq \emptyset$ and B is α^* -open. Let $x \in A \cap B$. Then by assumption, there exists a α^* -open set U containing x such that $\alpha^*Cl \subseteq B$. Since $x \in A$, $A \cap U \neq \emptyset$. This proves (iii).

(iii) \Rightarrow (iv): Suppose $A \cap B = \emptyset$, where A is non - empty and B is α^* -closed. Then $X \setminus B$ is α^* -open set and $(X \setminus B) \cap A \neq \emptyset$. By (iii) there exist a α^* -open set U such that $A \cap U \neq \emptyset$, and $\alpha^*Cl(U) \subseteq X \setminus B$. Put $V = X \setminus \alpha^*Cl(U)$. Hence V is a α^* -open set containing B such that $U \cap V = U \cap (X \setminus \alpha^*Cl(U)) \subseteq X \setminus U = \emptyset$. This proves (iv).

(iv) \Rightarrow (i): Let B be α^* -closed and $x \notin B$. Take $A = \{x\}$. Then $A \cap B = \emptyset$. By (iv), there exist disjoint α^* -open sets U and V such that $U \cap A \neq \emptyset$ and $B \subseteq V$. Since $U \cap V = \emptyset$, $x \in U$. This proves that X is α^* -regular.

Theorem 3.3: Let X be a α^* -regular space.

- (i) Every α^* -open set in X is a union of α^* -closed sets.
- (ii) Every α^* -closed set in X is an intersection of α^* -open sets.

Proof:

- (i) Suppose X is α^* -regular. Let G be a α^* -open set and $x \in G$. Then $F = X \setminus G$ is α^* -closed and $x \notin F$. Since X is α^* -regular, there exist disjoint α^* -open sets U_x and V in X such that $x \in U_x$ and $F \subseteq V$. Since $U_x \cap F \subseteq U_x \cap V = \emptyset$, we have $U_x \subseteq X \setminus F = G$. Take $V_x = \alpha^*Cl(U_x)$. Then V_x is α^* -closed and $V_x \cap V = \emptyset$. Now $F \subseteq V$ implies that $V_x \cap F \subseteq V_x \cap V = \emptyset$. It follows that $x \in V_x \subseteq X \setminus F = G$. This proves that $G = \bigcup \{V_x : x \in G\}$. Thus G is a union of α^* -closed sets.

- (ii) Follows from (i) and set theoretic properties.

Theorem 3.4: If f is a α^* -irresolute and pre- α^* -closed injection of a topological space X into a α^* -regular space Y , then X is α^* -regular.

Proof: Let $x \in X$ and A be a α^* -closed set in X not containing x . Since f is pre- α^* -closed, $f(A)$ is a α^* -closed set in Y not containing $f(x)$. Since Y is α^* -regular, there exist disjoint α^* -open sets V_1 and V_2 in Y such that $f(x) \in V_1$ and $f(A) \subseteq V_2$. Since f is α^* -irresolute, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint α^* -open sets in X containing x and A respectively. Hence X is α^* -regular.

Theorem 3.5: If f is a α^* -continuous and closed injection of a topological space X into a regular space Y and if every α^* -closed set in X is closed, then X is α^* -regular.

Proof: Let $x \in X$ and A be a α^* -closed set in X not containing x . Then by assumption, A is closed in X . Since f is closed, $f(A)$ is a closed set in Y not containing $f(x)$. Since Y is regular, there exist disjoint open sets V_1 and V_2 in Y such that $f(x) \in V_1$ and $f(A) \subseteq V_2$. Since f is α^* -continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint α^* -open sets in X containing x and A respectively. Hence X is α^* -regular.

Theorem 3.6: If $f: X \rightarrow Y$ is a α^* -irresolute bijection which is pre- α^* -open and X is α^* -regular. Then Y is also α^* -regular.

Proof: Let $f: X \rightarrow Y$ be a α^* -irresolute bijection which is α^* -open and X be α^* -regular. Let $y \in Y$ and B be a α^* -closed set in Y not containing y . Since f is α^* -irresolute, $f^{-1}(B)$ is a α^* -closed set in X not containing $f^{-1}(y)$. Since X is α^* -regular, there exist disjoint α^* -open sets U_1 and U_2 containing $f^{-1}(y)$ and $f^{-1}(B) \subseteq U_2$ respectively. Since f is pre- α^* -open, $f(U_1)$ and $f(U_2)$ are disjoint α^* -open sets in Y containing y and B respectively. Hence Y is α^* -regular.

Theorem 3.7: If f is a continuous α^* -open bijection of a regular space X into a space Y and if every α^* -closed set in Y is closed, then Y is α^* -regular.

Proof: Let $y \in Y$ and B be a α^* -closed set in Y not containing y . Then by assumption, B is closed in Y . Since f is a continuous bijection, $f^{-1}(B)$ is a closed set in X not containing the point $f^{-1}(y)$. Since X is regular, there exist disjoint open sets U_1 and U_2 in X such that $f^{-1}(y) \in U_1$ and $f^{-1}(B) \subseteq U_2$. Since f is α^* -open, $f(U_1)$ and $f(U_2)$ are disjoint α^* -open sets in Y containing y and B respectively. Hence Y is α^* -regular.

IV. NORMAL SPACES ASSOCIATED WITH α^* -OPEN SETS

In this section we introduce a normal spaces namely α^* -normal spaces and investigate their basic properties.

Definition 4.1: A space X is said to be α^* -normal if for every pair of disjoint α^* -closed sets A and B in X , there are disjoint α^* -open sets U and V in X containing A and B respectively.

Theorem 4.2: In a topological space X , the following are equivalent:

- (i) X is α^* -normal.
- (ii) For every α^* -closed set A in X and every α^* -open set U containing A , there exists a α^* -open set V containing A such that $\alpha^*Cl(V) \subseteq U$.
- (iii) For each pair of disjoint α^* -closed sets A and B in X , there exists a α^* -open set U containing A such that $\alpha^*Cl(U) \cap B = \emptyset$.
- (iv) For each pair of disjoint α^* -closed sets A and B in X , there exist α^* -open sets U and V containing A and B respectively such that $\alpha^*Cl(U) \cap \alpha^*Cl(V) = \emptyset$.

Proof:

(i) \Rightarrow (ii): Let U be a α^* -open set containing the α^* -closed set A . Then $B = X \setminus U$ is a α^* -closed set disjoint from A . Since X is α^* -normal, there exist disjoint α^* -open sets V and W containing A and B respectively. Then $\alpha^*Cl(V)$ is disjoint from B , since if $y \in B$, the set W is a α^* -open set containing y disjoint from V . Hence $\alpha^*Cl(V) \subseteq U$.

(ii) \Rightarrow (iii): Let A and B be disjoint α^* -closed sets in X . Then $X \setminus B$ is a α^* -open set containing A . By (ii), there exists a α^* -open set U containing A such that $\alpha^*Cl(U) \subseteq X \setminus B$. Hence $\alpha^*Cl(U) \cap B = \emptyset$. This proves (iii).

(iii) \Rightarrow (iv): Let A and B be disjoint α^* -closed sets in X . Then, by (iii), there exists a α^* -open set U containing A such that $\alpha^*Cl(U) \cap B = \emptyset$. Since $\alpha^*Cl(U)$ is α^* -closed, B and $\alpha^*Cl(U)$ are disjoint α^* -closed sets in X . Again by (iii), there exists a α^* -open set V containing B such that $\alpha^*Cl(U) \cap \alpha^*Cl(V) = \emptyset$. This proves (iv).

(iv) \Rightarrow (i): Let A and B be the disjoint α^* -closed sets in X . By (iv), there exist α^* -open sets U and V containing A and B respectively such that $\alpha^*Cl(U) \cap \alpha^*Cl(V) = \emptyset$. Since $U \cup V \subseteq \alpha^*Cl(U) \cap \alpha^*Cl(V)$, U and V are disjoint α^* -open sets containing A and B respectively. Thus X is α^* -normal.

Theorem 4.3: For a space X , then the following are equivalent:

- (i) X is α^* -normal.
- (ii) For any two α^* -open sets U and V whose union is X , there exist α^* -closed subsets A of U and B of V whose union is also X .

Proof:

(i) \Rightarrow (ii): Let U and V be two α^* -open sets in a α^* -normal space X such that $X = U \cup V$. Then $X \setminus U$, $X \setminus V$ are disjoint α^* -closed sets. Since X is α^* -normal, there exist disjoint α^* -open sets G_1 and G_2 such that $X \setminus U \subseteq G_1$ and $X \setminus V \subseteq G_2$. Let $A = X \setminus G_1$ and $B = X \setminus G_2$. Then A and B are α^* -closed subsets of U and V respectively such that $A \cup B = X$. This proves (ii).

(ii) \Rightarrow (i): Let A and B be disjoint α^* -closed sets in X . Then $X \setminus A$ and $X \setminus B$ are α^* -open sets whose union is X . By (ii), there exists α^* -closed sets F_1 and F_2 such that $F_1 \subseteq X \setminus A$, $F_2 \subseteq X \setminus B$ and $F_1 \cup F_2 = X$. Then $X \setminus F_1$ and $X \setminus F_2$ are disjoint α^* -open sets containing A and B respectively. Therefore X is α^* -normal.

Theorem 4.4: If f is an injective and α^* -irresolute and pre- α^* -closed mapping of a topological space X into a α^* -normal space Y , then X is α^* -normal.

Proof: Let f be an injective and α^* -irresolute and pre- α^* -closed mapping of a topological space X into a α^* -normal space Y . Let A and B be disjoint α^* -closed sets in X . Since f is a pre- α^* -closed function, $f(A)$ and $f(B)$ are disjoint α^* -closed sets in Y . Since Y is α^* -normal, there exist disjoint α^* -open sets V_1 and V_2 in Y containing $f(A)$ and $f(B)$ respectively. Since f is α^* -irresolute, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint α^* -open sets in X containing A and B respectively. Hence X is α^* -normal.

Theorem 4.5: If f is an injective and α^* -continuous and closed mapping of a topological space X into a normal space Y and if every α^* -closed set in X is closed, then X is α^* -normal.

Proof: Let A and B be disjoint α^* -closed sets in X . By assumption, A and B are closed in X . Then $f(A)$ and $f(B)$ are disjoint closed sets in Y . Since Y is normal, there exist disjoint open sets V_1 and V_2 in Y such that $f(A) \subseteq V_1$ and $f(B) \subseteq V_2$. Then $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint α^* -open sets in X containing A and B respectively. Hence X is α^* -normal.

Theorem 4.6: If $f: X \rightarrow Y$ is a α^* -irresolute injection which is pre- α^* -open and X is α^* -normal, then Y is also α^* -normal.

Proof: Let $f: X \rightarrow Y$ be a α^* -irresolute surjection which is α^* -open and X be α^* -normal. Let A and B be disjoint α^* -closed sets in Y . Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint α^* -closed sets in X . Since X is α^* -normal, there exist disjoint α^* -open sets U_1 and U_2 containing $f^{-1}(A)$ and $f^{-1}(B)$ respectively. Since f is pre- α^* -open, $f(U_1)$ and $f(U_2)$ are disjoint α^* -open sets in Y containing A and B respectively. Hence Y is α^* -normal.

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