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STEREOGRAPHIC *l* – AXIAL REFLECTED LOG - LOGISTIC DISTRIBUTION

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ABSTRACT

In this paper, we introduce a new symmetric circular distribution by using a technique of modified inverse stereographic projection on a reflected log-logistic distribution and its characteristic function.Population characteristics are also evaluated. And it is further extended to Stereographic l – axial Reflected Log – logistic distribution.

Keywords: *Circular data, Stereographic projection, Symmetric distribution, Trigonometric moments.*

1. INTRODUCTION

The Log-logistic distribution is proposed by Shoukri *et al.* (1988) and has been used in hydrology for modeling stream flow rates and precipitation. Extreme values like maximum rainfall in oneday and river discharge per month or per year often follow the Log-logistic distribution.The Log-logistic distribution has been used as a simple model of the distribution of wealth or income in Economics where it is known as the Fisk distribution. The log-logistic has been used by Gago-Benítez *et al.* in 2013 as a model for the period of time beginning when some data leaves a software user application in a computer and the response is received by the same application after travelling through and being processed by other computers, applications and network segments, most or all of them without hard real time guarantees (for example, when an application is displaying data coming from a remote sensor connected to the Internet).

The Reflected Log-logistic distribution has received considerable attention as an appropriate model in various fields. The Reflected Log-logistic distribution can be introduced by reflecting Log – logistic distribution symmetrically to $\left(-\infty,0\right]$. Glancing the literature, it appears that little attention was paid to construct circular models induced by inverse stereographic projection. Minh and Farnum (2003) proposed a new method of generating probability distributions by applying Stereographic Projection, which maps every point on the unit circle onto the point on the real line. Basing on this, Toshihiro Abe *et al.* (2010) constructed symmetric unimodal distributions by inverse stereographic projection. On the lines of Minh and Farnum and Toshihiro Abe *et al.* (2003; 2010), here an attempt is made to derive circular model by inducing inverse stereographic projection on Reflected Log – logistic distribution.

This paper is organized into five sections. In section 2 methodology of modified inverse stereographic projection is explained. Section 3 is devoted to present the proposed model i.e., Stereographic Reflected Log-logistic distribution. In section 4 its characteristic function and population characteristics are also evaluated and Program listings developed in MATLAB for graphs and computations of population characteristics are presented. Stereographic *l* – axial Reflected Log – logistic distribution is derived in section 5.

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2. METHODOLOGY OF MODIFIED INVERSE STEREOGRAPHIC PROJECTION [Phani et al., 2012]

Probability distributions (both circular and linear) can be generated by applying Stereographic Projection, which yields one to one correspondence between the points on the unit circle and those on the real line. Inverse Stereographic Projection is defined by a one to one mapping given by $T(\theta) = x = u + v\tan\left(\frac{\theta}{2}\right)$, where $x \in (-\infty, \infty)$, $\theta \in [-\pi, \pi)$, $u \in \mathbb{R}$, *and* $v > 0$. Suppose *x* is randomly chosen on the interval $(-\infty, \infty)$. Let $F(x)$ and $f(x)$ denote the cumulative distribution and the probability density functions of the random variable *X* respectively. Then $T^{-1}(x) = \theta = 2\tan^{-1}\left\{\frac{(x-u)}{v}\right\}$ is a random point on the unit circle. Let $G(\theta)$ and $g(\theta)$ denote the cumulative distribution and the probability density functions of this random point θ respectively. Then $G(\theta)$ and $g(\theta)$ can be written in terms of $F(x)$ and $f(x)$ using the following Theorem.

Theorem: For $v > 0$,

i) $$G(\theta) = F\left(u + v\tan\left(\frac{\theta}{2}\right)\right) = F(x(\theta)) \tag{2.1}$$

ii) $$g(\theta) = v\left(\frac{1+\tan^2\left(\frac{\theta}{2}\right)}{2}\right) f\left(u + v\tan\left(\frac{\theta}{2}\right)\right) \tag{2.2}$$

3. STEREOGRAPHIC REFLECTED LOG – LOGISTIC DISTRIBUTION

Definition: A continuous random variable X is said to have a **Reflected Log-logistic model** with scale parameter $\alpha > 0$ and shape parameter $\beta > 0$, if its probability density function and distribution function are respectively given by

$$f(x) = \frac{\alpha\left(\frac{|x|}{\beta}\right)^{\alpha-1}}{2\beta\left[1+\left(\frac{|x|}{\beta}\right)^{\alpha}\right]^2}\ ,\ \alpha, \beta > 0,\ -\infty < x < \infty \tag{3.1}$$

$$F(x) = \frac{1}{2}\left[\frac{2+\left(\left|\frac{x}{\beta}\right|\right)^{-\alpha}}{1+\left(\left|\frac{x}{\beta}\right|\right)^{-\alpha}}\right], \alpha, \beta > 0,\ -\infty < x < \infty \tag{3.2}$$

Then applying inverse Stereographic projection defined by a one-one mapping $x = v\tan\left(\frac{\theta}{2}\right), -\pi < \theta < \pi,\ v > 0$, we get a two parametric Symmetric circular distribution on a unit circle ,whose pdf and cdf are respectively given by

$$g(\theta) = \frac{\alpha\sigma\sec^2\left(\frac{\theta}{2}\right)\left(\sigma\left|\tan\left(\frac{\theta}{2}\right)\right|\right)^{\alpha-1}}{4\left(1+\left(\sigma\left|\tan\left(\frac{\theta}{2}\right)\right|\right)^{\alpha}\right)^2}\ \text{, where } \alpha, > 0, \sigma = \frac{v}{\beta} > 0, -\pi < \theta < \pi \tag{3.3}$$

and

$$G(\theta)=\begin{cases} 1-\dfrac{1}{2}\left[\dfrac{2+\left(\sigma\left|\tan\left(\dfrac{\theta}{2}\right)\right|\right)^{-\alpha}}{1+\left(\sigma\left|\tan\left(\dfrac{\theta}{2}\right)\right|\right)^{-\alpha}}\right], \ \alpha,\sigma>0, -\pi<\theta<0 \\ \dfrac{1}{2}\left[\dfrac{2+\left(\sigma\left|\tan\left(\dfrac{\theta}{2}\right)\right|\right)^{-\alpha}}{1+\left(\sigma\left|\tan\left(\dfrac{\theta}{2}\right)\right|\right)^{-\alpha}}\right], \ \alpha,\sigma>0, \ 0<\theta<\pi \end{cases} \tag{3.4}$$

Definition: A random variable X_S on unit circle is said to have Stereographic Reflected Log-logistic Distribution with shape parameter $\alpha > 0$, scale parameter $\sigma > 0$ and concentration parameter $v > 0$, denoted by SRLLG(α,σ), then the probability density and cumulative distribution functions are given by

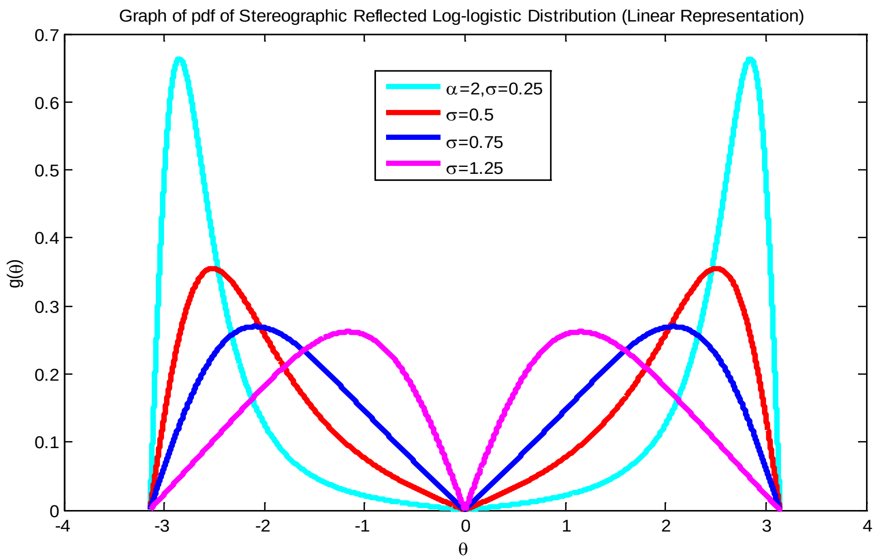
$$g(\theta)=\frac{\alpha\sigma\sec^2\left(\dfrac{\theta}{2}\right)\left(\sigma\left|\tan\left(\dfrac{\theta}{2}\right)\right|\right)^{\alpha-1}}{4\left(1+\left(\sigma\left|\tan\left(\dfrac{\theta}{2}\right)\right|\right)^{\alpha}\right)^2}, \text{where } \alpha,\sigma>0, -\pi<\theta<\pi \tag{3.5}$$

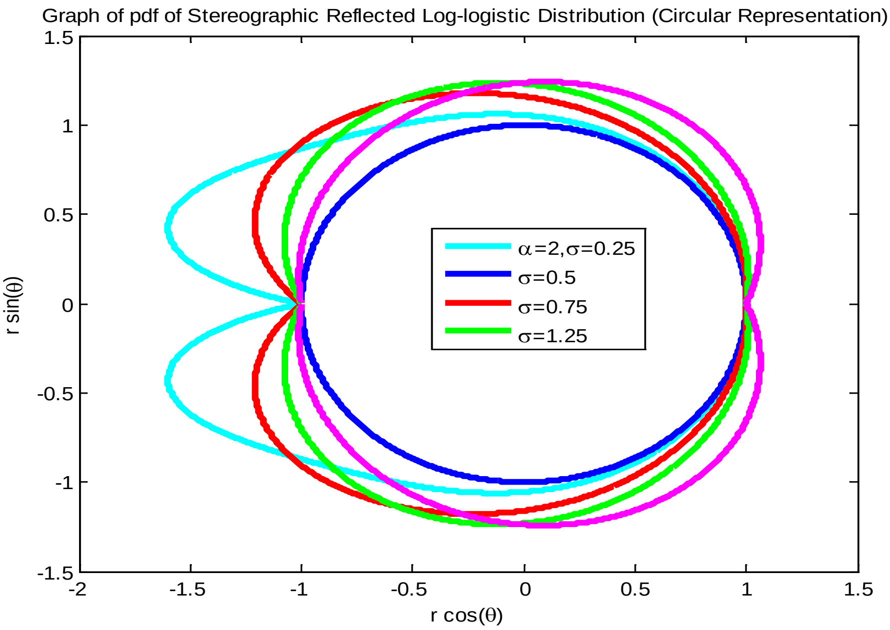
and

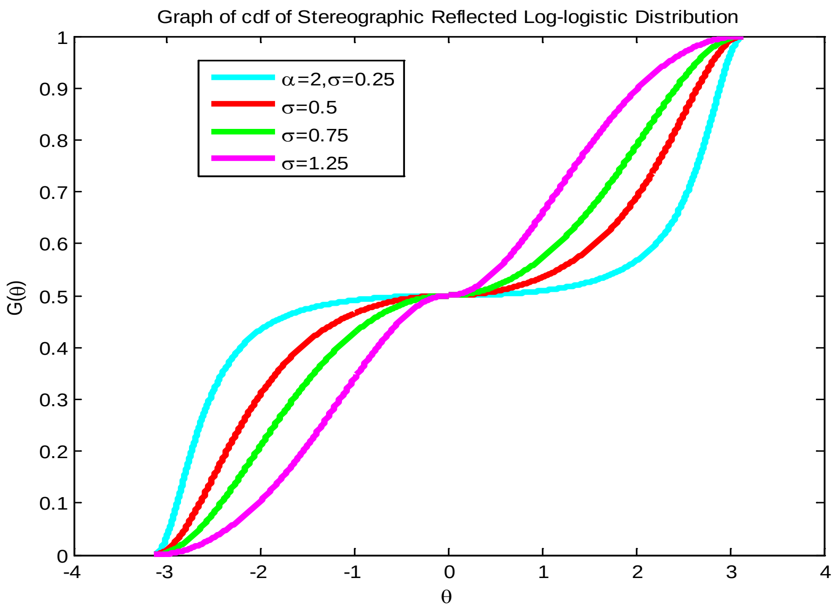
$$G(\theta)=\begin{cases} 1-\dfrac{1}{2}\left[\dfrac{2+\left(\sigma\left|\tan\left(\dfrac{\theta}{2}\right)\right|\right)^{-\alpha}}{1+\left(\sigma\left|\tan\left(\dfrac{\theta}{2}\right)\right|\right)^{-\alpha}}\right], \ \alpha,\sigma>0, -\pi<\theta<0 \\ \dfrac{1}{2}\left[\dfrac{2+\left(\sigma\left|\tan\left(\dfrac{\theta}{2}\right)\right|\right)^{-\alpha}}{1+\left(\sigma\left|\tan\left(\dfrac{\theta}{2}\right)\right|\right)^{-\alpha}}\right], \ \alpha,\sigma>0, \ 0<\theta<\pi \end{cases} \tag{3.6}$$

Hence the proposed new model SRLLG(α,σ) is a circular model named as "**STEREOGRAPHIC REFLECTED LOG - LOGISTIC MODEL".** It can be seen that the Stereographic Reflected Log - logistic Model is symmetric model.

Graphs of probability density function and cumulative distribution function of Stereographic Reflected Log-logistic Distribution for various values of α and σ are presented here.


Graph of pdf of Stereographic Reflected Log-logistic Distribution (Linear Representation)
α=2,σ=0.25
σ=0.5
σ=0.75
σ=1.25
g(θ)
θ


Graph of pdf of Stereographic Reflected Log-logistic Distribution (Circular Representation)
α=2,σ=0.25
σ=0.5
σ=0.75
σ=1.25
r sin(θ)
r cos(θ)


Graph of cdf of Stereographic Reflected Log-logistic Distribution
α=2,σ=0.25
σ=0.5
σ=0.75
σ=1.25
G(θ)
θ

4. THE CHARACTERISTIC FUNCTION OF STEREOGRAPHIC REFLECTED LOG-LOGISTIC MODEL

$$\Phi_{X_S}(p) = \int_{-\pi}^{\pi} e^{ip\theta} g(\theta) d\theta$$

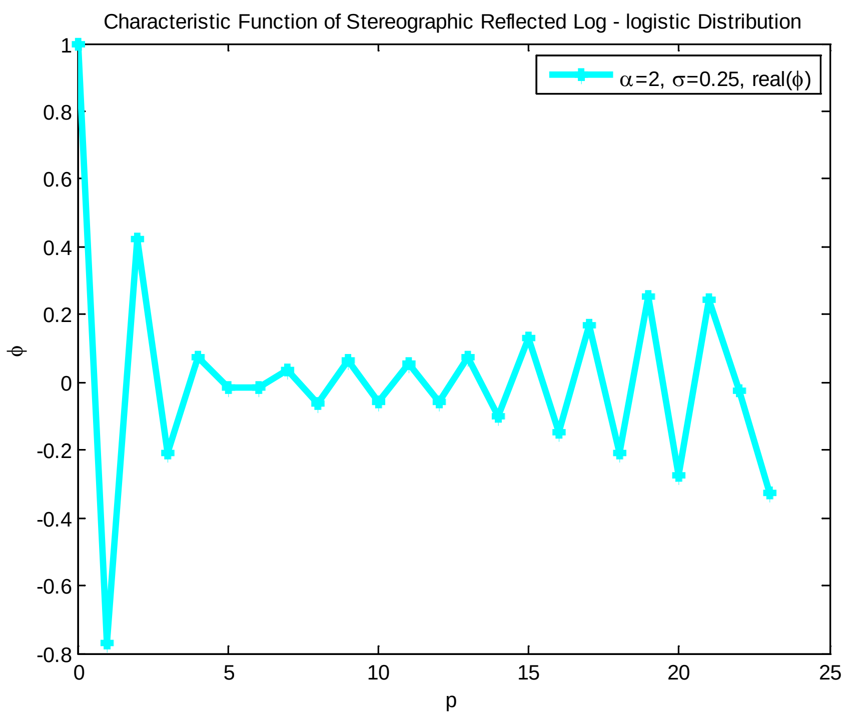
$$= \int_{-\pi}^{\pi} \cos p\theta \frac{\alpha\sigma \sec^2\left(\frac{\theta}{2}\right)\left[\sigma\left|\tan\left(\frac{\theta}{2}\right)\right|\right]^{\alpha-1}}{4\left(1+\left(\sigma\left|\tan\left(\frac{\theta}{2}\right)\right|\right)^{\alpha}\right)^2} d\theta + i\int_{-\pi}^{\pi} \sin p\theta \frac{\alpha\sigma \sec^2\left(\frac{\theta}{2}\right)\left[\sigma\left|\tan\left(\frac{\theta}{2}\right)\right|\right]^{\alpha-1}}{4\left(1+\left(\sigma\left|\tan\left(\frac{\theta}{2}\right)\right|\right)^{\alpha}\right)^2} d\theta$$

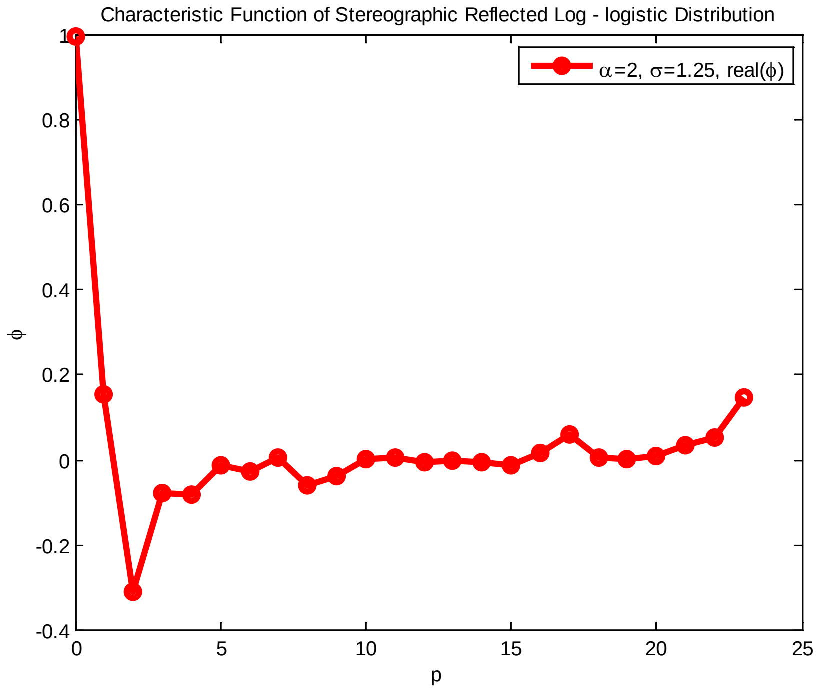
$$= \int_{0}^{\pi} \cos p\theta \frac{\alpha\sigma \sec^2\left(\frac{\theta}{2}\right)\left[\sigma\left|\tan\left(\frac{\theta}{2}\right)\right|\right]^{\alpha-1}}{2\left(1+\left(\sigma\left|\tan\left(\frac{\theta}{2}\right)\right|\right)^{\alpha}\right)^2} d\theta \text{ , since } \sin p\theta \text{ is odd}$$

Trigonometric moments of the Stereographic Reflected Log-Logistic Model

The trigonometric moments of the distribution are given by $\{\varphi_p : p = \pm 1, \pm 2, \pm 3, ...\}$, where $\varphi_p = \alpha_p + i\beta_p$, with $\alpha_p = E(\cos p\theta)$ and $\beta_p = E(\sin p\theta)$ being the p^{th} order cosine and sine moments of the random angle θ, respectively. Because the **Stereographic Reflected Log-Logistic** distribution is symmetric, it follows that the sine moments are zero. Thus $\varphi_p = \alpha_p$.

The graphs of real part of the characteristic function are plotted here.





Mardia and Jupp (2000) gave expressions of mean direction, resultant length, circular variance, circular standard deviation, central trigonometric moments, skewness and kurtosis for circular distributions in terms of first four moments. These characteristics for the Stereographic Reflected Log - logistic model are also based on their respective trigonometric moments. These can be expressed in terms of trigonometric moments α_p and β_p and are presented here. If one has to evaluate the characteristic function for all integral values of p, numerical integration of Weddle's rule is to be used.

Table-4.1: Population Characteristics of SRLGLGD for $\alpha = 2$

	$\sigma = 0.25$	$\sigma = 0.5$	$\sigma = 0.75$	$\sigma = 1.25$
Trigonometric Moments				
α_1	-0.7457	-0.4382	-0.1685	0.1548
α_2	0.3666	-0.0979	-0.3210	-0.3102
Resultant Length				
ρ_1	0.7457	0.4382	0.1685	0.1548
ρ_2	0.3666	0.0979	0.3210	0.3102
Mean Direction				
μ_0	0	0	0	0
Circular Variance				
V_0	0.2543	0.5618	0.8315	0.8452
Circular Standard Deviation				
σ_0	0.7661	1.2846	1.8874	1.9317
	1.4167	2.1558	1.5076	1.5302
Central Trigonometric Moments				
α_1^*	0.7457	0.4382	0.1685	0.1548
α_2^*	0.3666	0.3479	0.3210	0.3102
Kurtosis				
γ_2^0	0.8872	0.5934	0.4630	0.4333

5. STEREOGRAPHIC *l* - AXIAL REFLECTED LOG – LOGISTIC DISTRIBUTION

We extend the above **Stereographic Reflected Log – logistic distribution** to the l -axial distribution, which is applicable to any arc of arbitrary length say $2\pi/l$ for $l=1,2,...,$ so it is desirable to extend the **StereographicReflected Log – logistic** distribution to construct the **Stereographic** l **-axial Reflected Log – logistic** distribution, we consider the density function of **Stereographic Reflected Log – logistic** distribution and use the transformation $\phi=\dfrac{2\theta}{l}$, $l=1,2,...,$. The probability density function of ϕ is given by

$$g(\phi)=\frac{l\alpha\sigma\sec^2\left(\frac{l\phi}{2}\right)\left(\sigma\left|\tan\left(\frac{l\phi}{2}\right)\right|\right)^{\alpha-1}}{4\left(1+\left(\sigma\left|\tan\left(\frac{l\phi}{2}\right)\right|\right)^{\alpha}\right)^2}\text{, where } \alpha,\sigma>0\,, \frac{-\pi}{l}<\phi<\frac{\pi}{l} \tag{5.1}$$

We call it as **Stereographic -** l **-axial Reflected Log – logistic distribution**

Case-(1): When $l=1$, in the probability density function (5.1), we get the density function

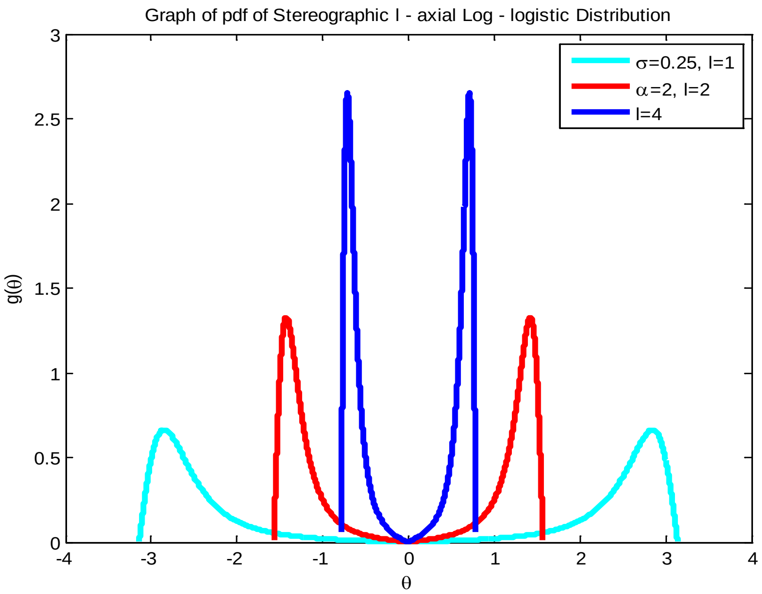
$$g(\phi)=\frac{\alpha\sigma\sec^2\left(\frac{\phi}{2}\right)\left(\sigma\left|\tan\left(\frac{\phi}{2}\right)\right|\right)^{\alpha-1}}{4\left(1+\left(\sigma\left|\tan\left(\frac{\phi}{2}\right)\right|\right)^{\alpha}\right)^2}\text{, where } \alpha,\sigma>0\,, -\pi<\phi<\pi \tag{5.2}$$

Which is the density function of **Stereographic Circular Reflected Log – logistic distribution**.

Case-(2): When $l=2$, the probability density function (5.1) is the same as that of **Stereographic Semicircular Reflected Log – logistic Distribution**

$$g(\phi)=\frac{2\alpha\sigma\sec^2(\phi)\left(\sigma\left|\tan(\phi)\right|\right)^{\alpha-1}}{4\left(1+\left(\sigma\left|\tan(\phi)\right|\right)^{\alpha}\right)^2}\text{, where } \alpha,\sigma>0\,, \frac{-\pi}{2}<\phi<\frac{\pi}{2} \tag{5.3}$$

The graph of the pdf of **Stereographic** *l* **- axial Reflected Log – logistic Distribution** for various values of l is plotted.



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