International Journal of Mathematical Archive-9(8), 2018, 7-13
Available online through www.ijma.info ISSN 2229 – 5046

SOME FIXED POINT THEOREMS IN QUASI 2-BANACH SPACE UNDER QUASI WEAK CONTRACTIONS

B. STEPHEN JOHN[1] AND LEENA NELSON S.N.[2*]

[1]Associate Professor, Dept. of Mathematics,
Annai Velankanni College, Tholayavattom, K. K. Dist. Tamilnadu, India.

[2]Assistant Professor, Dept. of Mathematics,
Women's Christian College, Nagercoil, K. K. Dist. Tamilnadu, India.

(Received On: 30-12-17; Revised & Accepted On: 10-08-18)

ABSTRACT

C. Park [13] introduced the term of a quasi 2-normed space. Also he proved some properties of quasi 2-norm and M. Kir and M. Acikgoz[9] elaborated the procedure for completing the quasi 2-normed space. In this paper, we prove some fixed point theorems for self mappings in quasi-2-Banach space via φ'-contraction mapping and Quasi weak contractions.

Keywords: *Quasi 2-Banach space, fixed point, nonlinear contraction, comparison function.*

I. INTRODUCTION

The theory of 2-normed space and 2-Banach space were initiated by Gahler. These new spaces have subsequently been studied by several mathematicians. In 2006, Park introduced the concepts of quasi-2-normed space and quasi-(2,p)-normed spaces. One of the most interesting area in fixed point theorems is to introduce and proved some theorems under contractions of type almost contraction.

The organization of the paper is as follows: Section II comprises some preliminary definitions and results, which are used in this paper. In Section III, we establish fixed point theorems in Quasi 2-Banach space.

II. SOME PRELIMINARY RESULTS

Definition 2.1 Let X be a linear space and $\|.,.\|$ be a real valued function defined on X satisfying the following conditions:

(i) $\|x,y\| = 0$ if and only if x and y are linearly dependent.
(ii) $\|x,y\| = \|y,x\|$ for all x,y$\in X$
(iii) $\|x,y+z\| \leq \|x,y\| + \|x,z\| \; for\ all\ x,y,z \in X$

$\|.,.\|$ is called a 2-norm and the pair $(X,\|.,.\|)$ is called a linear 2-normed space.

Definition 2.2: Let X be a linear space. A quasi-2-norm is a real-valued function on $X \times X$ satisfying the following conditions:

(iv) $\|x,y\| = 0$ if and only if x and y are linearly dependent.
(i) $\|x,y\| = \|y,x\|$ for all $x,y \in X$
(ii) $\|\alpha x,y\| \leq |\alpha|\|x,y\|$
(iii) There is a constant $k \geq 1$ such that $\|x+y,z\| \leq k\|x,z\| + K\|y,z\|$ for all $x,y,z \in X$.

The pair $(X,\|.,.\|)$ is called a quasi-2-normed space if $\|.,.\|$ is a quasi-2-norm on X. The smallest possible K is called the modulus of concavity of $\|.,.\|$.

Corresponding Author: Leena Nelson S.N.[3*]
[3]Assistant Professor, Dept. of Mathematics,
Women's Christian College, Nagercoil, K. K. Dist. Tamilnadu, India.

Definition 2.3: Let X be a quasi-2-normed space. A sequence $\{x_n\}$ in X is called a Cauchy sequence if

$$\lim_{m,n\to\infty} \|x_m - x_n, z\| = 0 \text{ for all } z \in X.$$

Definition 2.4: A sequence $\{x_n\}$ in X is said to be a convergent if there is a point $x \in X$ such that

$$\lim_{n\to\infty} \|x_n - x, z\| = 0 \text{ for all } z \in X.$$

Definition 2.5: A quasi-2-normed space in which every Cauchy sequence converges is called complete.

Definition 2.6: A complete quasi-2-normed space is called a quasi-2-Banach space.

Example 2.7: Let E_3 denotes the Euclidean vector three spaces. Let $x = ai + bj + ck$ and $y = di + ej + fk$. Define

$$\|x, y\| = |x \times y| = abs \begin{vmatrix} i & j & k \\ a & b & c \\ d & e & f \end{vmatrix}$$
$$= (|i(bf - ce)^2 + j(cd - af)^2 + k(ae - bd)^2|)^{1/2}$$

Then $(E_3, \|.,.\|)$ quasi 2-Banach space.

Example 2.8: Let X be a linear space with $dimX \geq 2$, and let $\|.,.\|$ be 2-norm on X. Define $\|x, y\|_q = 2\|x, y\|$ is a quasi 2-norm on X and $(X, \|.,.\|_q)$ is a quasi-2-normed space.

Solution:

(i) $\|x, y\|_q = 0$ if and only if $2\|x, y\| = 0$ if and only if $\|x, y\| = 0$ if and only if x and y are linearly dependent.

(ii) $\|x, y\|_q = 2\|x, y\| = 2\|y, x\| = \|y, x\|_q$

(iii) $\|\alpha x, y\|_q = 2\|\alpha x, y\| = |\alpha|(2\|x, y\|) = |\alpha|\|x, y\|_q$

(iv) For all $x, y, z \in X$, $\|x + y, z\|_q = 2\|x + y, z\| \leq 2[\|x, z\| + \|y, z\|]$
$$= 2\|x, z\| + 2\|y, z\| = \|x, z\|_q + \|y, z\|_q$$

Thus, $(X, \|.,.\|_q)$ is a quasi 2-normed space.

Definition 2.9: Let $(X, \|.,.\|_X)$ and $(Y, \|.,.\|_Y)$ be quasi-2-normed spaces. A mapping $T: X \to Y$ is said to be isometric or isometry, if for all $x, y \in X$, $\|Tx, Ty\|_Y = \|x, y\|_X$.

The space X is said to be isometry with the space Y if there exists a bijective isometry of X onto Y. The spaces X and Y are called isometric spaces.

Definition 2.10: The function $\varphi: [0, \infty) \to [0, \infty)$ is called a c-comparison function if the following properties are satisfied:

(i) φ is monotonic increasing
(ii) $\sum_{n=0}^{\infty} \varphi^n(t) < \infty$ for all $t \geq 0$.

Clearly φ is a c-comparison function then $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$.

Definition 2.11: Let $(X, \|.,.\|)$ be a quasi-2-Banach space. A self mapping $T: X \to X$ is called a quasi (φ,L)-weak contraction if there exist a c-comparison function φ and some $L \geq 0$ such that for all $x, y \in X$,

$$\|Tx - Ty, u\| \leq \varphi\|x - y, u\| + L\, min\{\|x - Ty, u\|_m, \|y - Tx, u\|_m, \|x - Tx, u\|_m. \quad (1)$$

Definition 2.12: Let $(X, \|.,.\|)$ be a quasi-2-Banach space. A self mapping $T: X \to X$ is called a quasi (α,L)-weak contraction if there exist $\alpha \in [0,1)$ and some $L \geq 0$ such that for all $x, y \in X$, the function T satisfies the following condition:

$$\|Tx - Ty, u\| \leq \alpha\, max\left\{\|x - y, u\|, \|x - Tx, u\|, \|y - Ty, u\|, \frac{\|x-Ty,u\|+\|y-Tx,u\|}{2}\right\}+$$
$$L\, min\{\|x - Ty, u\|_m, \|y - Tx, u\|_m, \|x - Tx, u\|_m. \quad (2)$$

III. NEW RESULTS AND THEOREMS

Theorem 3.1: Let $(X, \|.,.\|)$ be a complete quasi-2-Banach space and $T: X \to X$ be a self mapping such that T is a quasi (φ, L)-weak contraction. Then T has a unique fixed point in X.

Proof: Let $x_0 \in X$ and define a sequence (x_n) in X by taking $x_{n+1} = Tx_n, n \geq 0$.

If there exists $r \in N$ such that $x_r = x_{r+1}$, then x_r is a fixed point of T.

Suppose that $x_n \neq x_{n+1}$, for all $n \in N$.

Substitute $x = x_{n-1}$ and $y = x_n$ in (1)

$$\begin{aligned}\|x_n - x_{n+1}, u\| &= \|Tx_{n-1} - Tx_n, u\| \\ &\le \varphi\|x_{n-1} - x_n, u\| + L min\{\|x_{n-1} - Tx_n, u\|_m, \|x_n - Tx_{n-1}, u\|_m, \|x_{n-1} - Tx_{n-1}, u\|_m\} \\ &= \varphi\|x_{n-1} - x_n, u\| + L min\{\|x_{n-1} - x_{n+1}, u\|_m, \|x_n - x_n, u\|_m, \|x_{n-1} - x_{n+1}, u\|_m\} \\ &= \varphi\|x_{n-1} - x_n, u\| + L min\{\|x_{n-1} - x_{n+1}, u\|_m, 0, \|x_{n-1} - x_{n+1}, u\|_m\} \\ &= \varphi\|x_{n-1} - x_n, u\|\end{aligned}$$

Therefore, $\|x_n - x_{n+1}, u\| \le \varphi\|x_{n-1} - x_n, u\|$

Since φ is increasing, we have
$$\begin{aligned}\|x_n - x_{n+1}, u\| &\le \varphi\|x_{n-1} - x_n, u\| \\ &\le \varphi^2\|x_{n-2} - x_{n-1}, u\| \\ &\dots \dots \dots \dots \dots \dots \\ &\dots \dots \dots \dots \dots \dots \\ &\le \varphi^n\|x_0 - x_1, u\|\end{aligned}$$

Thus, $\|x_n - x_{n+1}, u\| \le \varphi^n\|x_0 - x_1, u\|$

Similarly, we can deduce that $\|x_{n+1} - x_n, u\| \le \varphi^n\|x_1 - x_0, u\|$

Now we can prove that (x_n) is a Cauchy sequence.

To prove this, we have to prove that (x_n) is left-Cauchy and right-Cauchy sequence.

Let $n, m \in N$ such that $n > m$.

Then by the triangle inequality, we have

$$\begin{aligned}\|x_n - x_m, u\| &\le \|x_n - x_{n-1}, u\| + \|x_{n-1} - x_{n-2}, u\| + \cdots \dots + \|x_{m+1} - x_m, u\| \\ &= \textstyle\sum_{i=m}^{n-1} \|x_{i+1} - x_i, u\| \\ &= \textstyle\sum_{i=m}^{n-1} \varphi^{i+1}(\|x_1 - x_0, u\|)\end{aligned}$$

Since φ is c-comparison function, then $\sum_{i=m}^{\infty} \varphi^{i+1}(\|x_1 - x_0, u\|)$ is convergent.

Therefore, $\sum_{i=m}^{\infty} \varphi^{i+1}(\|x_1 - x_0, u\|) < \in$, for any $\in > 0$ and for all $m \ge N$.

Hence, for $n, m \ge N$, we have
$$\begin{aligned}\|x_n - x_m, u\| &\le \textstyle\sum_{i=m}^{n-1} \varphi^{i+1}(\|x_1 - x_0, u\|) \\ &\le \textstyle\sum_{i=m}^{\infty} \varphi^{i+1}(\|x_1 - x_0, u\|) \\ &< \in\end{aligned}$$

Thus, $\|x_n - x_m, u\| < \in$.

In the same way, we can show that (x_n) is a right-Cauchy sequence.

Thus, the sequence (x_n) is a Cauchy sequence in the space $(X, \|.,.\|)$.

Since, $(X, \|.,.\|)$ is complete, we have $\lim_{n\to\infty}\|x_n - x_{n+1}, u\| = 0$.

Let $\lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n x_n = \alpha \in X$.

The limit α is unique.

Assume that $\alpha' \ne \alpha$ and $\alpha' = \lim_{n\to\infty} x_n$.

Now, we have

$$\begin{aligned}\|x_n - T\alpha, u\| &= \|Tx_{n-1} - T\alpha, u\| \\ &\le \varphi\|x_{n-1} - \alpha, u\| + L\, min\left\{\begin{matrix}\|x_{n-1} - T\alpha, u\|_m, \|\alpha - x_n, u\|_m, \\ \|x_{n-1} - x_n, u\|_m\end{matrix}\right\} \\ &< \|x_{n-1} - \alpha, u\| + L\, min\left\{\begin{matrix}\|x_{n-1} - T\alpha, u\|_m, \|\alpha - x_n, u\|_m, \\ \|x_{n-1} - x_n, u\|_m\end{matrix}\right\}\end{aligned}$$

Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \|x_n - T\alpha, u\| = 0$$

That is, $\|\alpha - T\alpha, u\| = 0$.

Also, $\|T\alpha - x_n, u\| = \|T\alpha - Tx_{n-1}, u\|$

$$\leq \varphi\|\alpha - x_{n-1}, u\| + L\, min\left\{\begin{matrix}\|\alpha - x_n, u\|_m, \|x_{n-1} - T\alpha, u\|_m, \\ \|\alpha - T\alpha, u\|_m\end{matrix}\right\}$$

$$< \|\alpha - x_{n-1}, u\| + L\, min\left\{\begin{matrix}\|\alpha - x_n, u\|_m, \|x_{n-1} - T\alpha, u\|_m, \\ \|\alpha - T\alpha, u\|_m\end{matrix}\right\}$$

Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \|T\alpha - x_n, u\| = 0$$

That is, $\|T\alpha - x_n, u\| = 0$.

Thus we have, $\|\alpha - T\alpha, u\| = 0$ for all $u \in X$.

By uniqueness of the limit we have $T\alpha = \alpha$.

So, α is a fixed point of T.

Finally, we show that the fixed point is unique.

Assume that $\alpha' \neq \alpha$ is also a fixed point of T.

That is, $T\alpha = \alpha$ and $T\alpha' = \alpha'$.

Now, $\|\alpha - \alpha', u\| = \|T\alpha - T\alpha', u\|$

$$\begin{aligned}
&\leq \varphi\|\alpha - \alpha', u\| + L \min\{\|\alpha - \alpha', u\|_m, \|\alpha' - \alpha, u\|_m, \|\alpha - T\alpha, u\|_m\} \\
&= \varphi\|\alpha - \alpha', u\| + L \min\{\|\alpha - \alpha', u\|_m, \|\alpha' - \alpha, u\|_m, \|\alpha - \alpha, u\|_m\} \\
&= \varphi\|\alpha - \alpha', u\| + L \min\{\|\alpha - \alpha', u\|_m, \|\alpha' - \alpha, u\|_m, 0\} \\
&= \varphi\|\alpha - \alpha', u\|
\end{aligned}$$

If $\|\alpha - \alpha', u\| > 0$ then $\|\alpha - \alpha', u\| \leq \varphi\|\alpha - \alpha', u\| < \|\alpha - \alpha', u\|$

which is a contradiction.

Therefore, $\|\alpha - \alpha', u\| = 0$.

Thus we get $\alpha = \alpha'$.

This completes the proof of the theorem.

Theorem 3.2: Let $(X, \|.,.\|)$ be a complete quasi-2-Banach space and $T: X \to X$ be a self mapping such that T is a quasi (α, L)-weak contraction. Then T has a unique fixed point in X.

Proof: Let $x_0 \in X$ and define a sequence (x_n) in X by taking $x_{n+1} = Tx_n, n \geq 0$.

If there exists $r \in N$ such that $x_r = x_{r+1}$, then x_r is a fixed point of T.

Suppose that $x_n \neq x_{n+1}$, for all $n \in N$.

Substitute $x = x_{n-1}$ and $y = x_n$ in (2)

$$\begin{aligned}
\|Tx_{n-1} - Tx_n, u\| &\leq \alpha max\left\{\begin{matrix}\|x_{n-1} - x_n, u\|, \|x_{n-1} - Tx_{n-1}, u\|, \|x_n - Tx_n, u\|, \\ \frac{\|x_{n-1} - Tx_n, u\| + \|x_n - Tx_{n-1}, u\|}{2}\end{matrix}\right\} + \\
&\quad L\, min\{\|x_{n-1} - Tx_n, u\|_m, \|x_n - Tx_{n-1}, u\|_m, \ \|x_{n-1} - Tx_{n-1}, u\|_m\} \\
&= \alpha max\left\{\begin{matrix}\|x_{n-1} - x_n, u\|, \|x_{n-1} - x_n, u\|, \|x_n - x_{n+1}, u\|, \\ \frac{\|x_{n-1} - x_{n+1}, u\| + \|x_n - x_n, u\|}{2}\end{matrix}\right\} + \\
&\quad L\, min\{\|x_{n-1} - x_{n+1}, u\|_m, \|x_n - x_n, u\|_m, \|x_{n-1} - x_n, u\|_m\} \\
&= \alpha max\left\{\|x_{n-1} - x_n, u\|, \ \|x_n - x_{n+1}, u\|, \ \frac{\|x_{n-1} - x_{n+1}, u\| + 0}{2}\right\} + \\
&\quad L\, min\{\|x_{n-1} - x_{n+1}, u\|_m, 0, \|x_{n-1} - x_n, u\|_m\}
\end{aligned}$$

$$= \alpha max\left\{\|x_{n-1}-x_n,u\|,\ \|x_n-x_{n+1},u\|,\ \frac{\|x_{n-1}-x_{n+1},u\|}{2}\right\}+L\times 0$$
$$= \alpha max\left\{\|x_{n-1}-x_n,u\|,\ \|x_n-x_{n+1},u\|,\ \frac{\|x_{n-1}-x_{n+1},u\|}{2}\right\}$$
$$\leq \alpha max\left\{\|x_{n-1}-x_n,u\|, \|x_n-x_{n+1},u\|, \frac{\|x_{n-1}-x_n,u\|+\|x_n-x_{n+1},u\|}{2}\right\}$$
$$= \alpha max\{\|x_{n-1}-x_n,u\|, \|x_n-x_{n+1},u\|\}$$

If $max\{\|x_{n-1}-x_n,u\|, \|x_n-x_{n+1},u\|\} = \|x_n-x_{n+1},u\|$ then
$$\|x_n-x_{n+1},u\| \leq \alpha\|x_n-x_{n+1},u\|$$

which is a contradiction.

Therefore, $max\{\|x_{n-1}-x_n,u\|, \|x_n-x_{n+1},u\|\} = \|x_{n-1}-x_n,u\|$.

So, $\|x_n-x_{n+1},u\| \leq \alpha\|x_{n-1}-x_n,u\|$. (3)

Similarly, by taking $x = x_n$ and $y = x_{n-1}$ in (2) we get,
$$\|Tx_n-Tx_{n-1},u\| \leq \alpha\ max\left\{\begin{matrix}\|x_n-x_{n-1},u\|, \|x_n-Tx_n,u\|, \|x_{n-1}-Tx_{n-1},u\|,\\ \frac{\|x_n-Tx_{n-1},u\|+\|x_{n-1}-Tx_n,u\|}{2}\end{matrix}\right\}+$$
$$L\ min\{\|x_n-Tx_{n-1},u\|_m, \|x_{n-1}-Tx_n,u\|_m, \|x_n-Tx_n,u\|_m$$

That is,
$$\|x_{n+1}-x_n,u\| \leq \alpha\ max\left\{\begin{matrix}\|x_n-x_{n-1},u\|, \|x_n-x_{n+1},u\|, \|x_{n-1}-x_n,u\|,\\ \frac{\|x_n-x_n,u\|+\|x_{n-1}-x_{n+1},u\|}{2}\end{matrix}\right\}+$$
$$L\ min\{\|x_n-x_n,u\|_m, \|x_{n-1}-x_{n+1},u\|_m, \|x_n-x_{n+1},u\|_m$$

$$= \alpha\ max\left\{\|x_n-x_{n-1},u\|, \|x_n-x_{n+1},u\|, \|x_{n-1}-x_n,u\|,\ \frac{0+\|x_{n-1}-x_{n+1},u\|}{2}\right\}+$$
$$L\ min\{0, \|x_{n-1}-x_{n+1},u\|_m, \|x_n-x_{n+1},u\|_m$$
$$= \alpha\ max\left\{\|x_n-x_{n-1},u\|, \|x_n-x_{n+1},u\|, \|x_{n-1}-x_n,u\|,\ \frac{\|x_{n-1}-x_{n+1},u\|}{2}\right\}+L\times 0$$
$$= \alpha\ max\left\{\|x_n-x_{n-1},u\|, \|x_n-x_{n+1},u\|, \|x_{n-1}-x_n,u\|, \frac{\|x_{n-1}-x_{n+1},u\|}{2}\right\}$$
$$\leq \alpha\ max\left\{\|x_n-x_{n-1},u\|, \|x_n-x_{n+1},u\|, \|x_{n-1}-x_n,u\|, \frac{\|x_{n-1}-x_n,u\|+\|x_n-x_{n+1},u\|}{2}\right\}$$
$$= \alpha\ max\{\|x_n-x_{n-1},u\|, \|x_n-x_{n+1},u\|, \|x_{n-1}-x_n,u\|\}$$

From (3) we have, $\|x_{n+1}-x_n,u\| \leq \alpha\ max\{\|x_{n-1}-x_n,u\|, \|x_n-x_{n-1},u\|\}$ (4)

Also, from (3) we have, $\|x_n-x_{n+1},u\| \leq \alpha\ \|x_{n-1}-x_n,u\|$
$$\leq \alpha\ max\{\|x_{n-1}-x_n,u\|, \|x_n-x_{n-1},u\|\} \quad (5)$$

From (4) and (5) we have,
$$max\{\|x_{n+1}-x_n,u\|, \|x_n-x_{n+1},u\|\} \leq \alpha\ max\{\|x_{n-1}-x_n,u\|, \|x_n-x_{n-1},u\|\}$$

By repeating the previous steps $(n-1)$ times, we get
$$max\{\|x_{n+1}-x_n,u\|, \|x_n-x_{n+1},u\|\} \leq \alpha^n\ max\{\|x_1-x_0,u\|, \|x_0-x_1,u\|\}$$

Let $max\{\|x_1-x_0,u\|, \|x_0-x_1,u\|\} = M$.

Then $max\{\|x_{n+1}-x_n,u\|, \|x_n-x_{n+1},u\|\} \leq \alpha^n M$

Hence, we have $\|x_n-x_{n+1},u\| \leq \alpha^n M$. (6)
and $\|x_{n+1}-x_n,u\| \leq \alpha^n M$. (7)

Now, we show that (x_n) is a Cauchy sequence.

For that we can show that (x_n) is left-Cauchy and right-Cauchy.

Let $m, n \in N$ such that $n > m$.

Then by using the triangle inequality and (7) we have,

$$\begin{aligned}\|x_n - x_m, u\| &\le \|x_n - x_{n-1}, u\| + \|x_{n-1} - x_{n-2}, u\| + \ \dots\dots\dots + \|x_{m+1} - x_m, u\| \\ &= \sum_{i=m}^{n-1} \|x_{i+1} - x_i, u\| \\ &\le \sum_{i=m}^{n-1} \alpha^{i+1} M \\ &\le \sum_{i=m}^{\infty} \alpha^{i+1} M \\ &= M \sum_{i=m}^{\infty} \alpha^{i+1} \\ &= \frac{\alpha^{m+1}}{1-\alpha} M\end{aligned}$$

Since, $\alpha < 1$, then for any $\epsilon > 0$ such that $\frac{\alpha^{m+1}}{1-\alpha} < \frac{\epsilon}{M}, for\ all\ m \ge N.$

Thus, for $n, m \in N$ with $n > m \ge N$, we have, $\|x_n - x_m, u\| \le \frac{\alpha^{m+1}}{1-\alpha} M < \frac{\epsilon}{M} M = \epsilon.$

Therefore, $\|x_n - x_m, u\| < \epsilon.$

Hence, (x_n) is a left-Cauchy sequence.

Similarly, we can show that (x_n) is a right-Cauchy sequence.

Thus, the sequence (x_n) is a Cauchy sequence in the space $(X, \|.,.\|)$.

Since $(X, \|.,.\|)$ is complete, there exists $a \in X$ such that

$$\lim_{n\to\infty} \|x_n - a, u\| = \lim_{n\to\infty} \|a - x_n, u\| = 0.$$

Now, by (2) we have $\|x_n - Ta, u\| = \|Tx_{n-1} - Ta, u\|$

$$\le \alpha max \left\{ \|x_{n-1} - a, u\|, \|x_{n-1} - x_n, u\|, \|a - Ta, u\|, \frac{\|x_{n-1} - Ta, u\| + \|a - x_n, u\|}{2} \right\} + Lmin\{\|x_{n-1} - Ta, u\|_m, \|a - x_n, u\|_m, \|x_{n-1} - x_n, u\|_m\}$$

By taking the limit as $n \to \infty$, we get $\|a - Ta, u\| \le \alpha \|a - Ta, u\|.$

Since $\alpha < 1$, then $\|a - Ta, u\| = 0.$

Hence $Ta = a.$

Therefore, a is a fixed point of T.

Finally, we show that the fixed point a is unique.

Assume that $b \ne a$ is also a fixed point of T.

Then $Tb = b$.

By (2) we have, $\|a - b, u\| = \|Ta - Tb, u\|$

$$\begin{aligned}&\le \alpha\ max \left\{ \|a - b, u\|, \|a - Ta, u\|, \|b - Tb, u\|, \frac{\|a - Tb, u\| + \|b - Ta, u\|}{2} \right\} \\ &\qquad +L\ min\{\|a - Tb, u\|_m, \|b - Ta, u\|_m, \|a - Ta, u\|_m\} \\ &= \alpha\ max \left\{ \|a - b, u\|, \|a - a, u\|, \|b - b, u\|, \frac{\|a - Tb, u\| + \|b - Ta, u\|}{2} \right\} \\ &\qquad +L\ min\{\|a - Tb, u\|_m, \|b - Ta, u\|_m, \|a - a, u\|_m\}\end{aligned}$$

$$= \alpha\, max\left\{\|a-b,u\|, 0, 0, \frac{\|a-Tb,u\| + \|b-Ta,u\|}{2}\right\}$$
$$+ L\, min\{\|a-Tb,u\|_m, \|b-Ta,u\|_m, 0\}$$
$$= \alpha\, max\left\{\|a-b,u\|, \frac{\|a-Tb,u\| + \|b-Ta,u\|}{2}\right\} + 0$$
$$= \alpha\, max\left\{\|a-b,u\|, \frac{\|a-Tb,u\| + \|b-Ta,u\|}{2}\right\}$$

Therefore, $\|a-b,u\| \le \alpha\, max\left\{\|a-b,u\|, \frac{\|a-Tb,u\|+\|b-Ta,u\|}{2}\right\}$.

By the same argument, we have

$$\|b-a,u\| \le \alpha\, max\left\{\|b-a,u\|, \frac{\|b-Ta,u\| + \|a-Tb,u\|}{2}\right\}$$

Thus $max\{\|a-b,u\|, \|b-a,u\|\} \le \alpha\, max\left\{\|a-b,u\|, \|b-a,u\|, \frac{\|a-Tb,u\|+\|b-Ta,u\|}{2}\right\}$
$$= \alpha\, max\{\|a-b,u\|, \|b-a,u\|\}$$

Since $\alpha < 1$, we get that $\|a-b,u\| = \|b-a,u\| = 0$.

Thus, we have again $a = b$.

This completes the proof of the theorem.

CONCLUSION

In this paper we proved fixed point theorems in quasi-2-Banach space via quasi weak contractions. The results of this paper extend the previously known results for metric space in a quasi-2-Banach space.

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Source of support: Nil, Conflict of interest: None Declared.