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# FIXED POINT THEOREMS THROUGH CONTRACTIVETYPE MAPPINGS

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***ABSTRACT***

*In this research article, some fixed point theorems in 2- metric spaces are established. It also introduces contraction type mappings in 2-metric spaces. The theorems are generalizations of some fixed point theorems of Pal and Maiti [2].*

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***Key Words:*** *Fixed point, 2-metric space, Contractive mapping, Generated Contractive mapping.*

## INTRODUCTION

The notion of 2-metric space was introduced by *Gahler* [1] in 1963 as a generalizationof area function for Euclidean triangles. Fixed point theory was first studied by *Poincare* and developed by many mathematicians *Brouwer, Banach, Schauder, Rhoades* [4], etc. Let X be a non-empty set. A function $f : X \to X$ from X into itself is called a self-map on X. A point $z \in X$ is called a fixed point of self-map $f : X \to X$ if $f(z) = z$, Theorems concerning fixed points of self-maps are known as fixed point theorems. Most of the fixed point theorems were proved for contraction mappings. It is well known that every contraction on a metric space is continuous. The converse is not necessarily true. The identity mapping on [0,1] simply serves the counter example.

In this present work, some of the fixed point theoremsof *Pal* and *Maiti* are extended to a more generalized 2-metric space setting.

In what follows $X$ stands for a 2-metric space.

## 1. PRELIMINARIES

This section is devoted to some basic definitions which are needed for the further study of this Article.

**Definition 1.1:** Let $X$ be a non-empty set and $\lambda : X \times X \times X \to R$. For all $x, y, z$ and $u$ in $X$, if $\lambda$ satisfies the following conditions

(a) $\lambda(x, y, z) = 0$ if atleast two of $x, y, z$ are equal

(b) $\lambda(x, y, z) = \lambda(x, z, y) = \lambda(y, z, x) = ...$

(c) $\lambda(x, y, z) \le \lambda(x, y, u) + \lambda(x, u, z) + \lambda(u, y, z)$

Then $d$ is called a 2-*metric*on $X$ and the pair $(X, \lambda)$ is called a *2-metric space.*

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**Definition 1.2:** Let $(X,\lambda)$ be a 2-metric space. A mapping $T:X\rightarrow X$ is said to be a *Contractive* if

$$\lambda(Tx,Ty,a)<\lambda(x,y,a) \text{ for all } x,y,a \text{ in } X.$$

**Remark 1.3:** It is well known that a contractive mapping on a complete metric space has a unique fixed point in $X$. But a contractive mapping on a complete 2-metric space $(X,\lambda)$ neednot have a fixed point. It can be seen from the following example.

**Example- 1.4:**Let $X=\{x\in R/x\geq 2\}=[2,\infty)$ with 2-metric defined by

$$\lambda(x,y,z)=\min\{|x-y|,|y-z|,|z-x|\}.$$

Define $f:X\rightarrow X$ by $f(x)=\frac{1}{x}\forall\ x\in X$.

$$\begin{aligned}\lambda(fx,fy,a)&=\lambda\left(\frac{1}{x},\frac{1}{y},a\right)\\&=\min\left\{\left|\frac{1}{x}-\frac{1}{y}\right|,\left|\frac{1}{y}-a\right|,\left|\frac{1}{x}-a\right|\right\}\\&<\min\{|x-y|,|y-a|,|a-x|\}\\&=\lambda(x,y,a)\end{aligned}$$

$\Rightarrow$ $f$ is a contractive mapping on $X$ ,but it has no fixed point in $X$.

**Definition 1.5:** Let $(X,\lambda)$ be a 2-metric space. A mapping $T:X\rightarrow X$ is said to be a *Generated Contractive* if

$$\lambda(Tx,Ty,a)<\max\left\{\lambda(x,y,a),\lambda(x,Tx,a),\lambda(y,Ty,a),\frac{1}{2}[\lambda(x,Ty,a)+\lambda(y,Tx,a)]\right\} \text{ for all } x,y,a \text{ in } X.$$

**Remark 1.6:** A Generated contractive mapping is need not be contractive mapping. Itcan be seen from the following example.

**Example 1.7:** Let $X=[0,6]$ with 2-metric defined as $\lambda(x,y,z)=\min\{|x-y|,|y-z|,|z-x|\}$ for all $x,y,z$ in $X$.

We define a map $T:[0,6]\rightarrow[0,6]$ by $T(x)=\begin{cases}3x & if \quad x\in[0,3]\\ 3x-1 & if \quad x\in(3,6]\end{cases}$

Clearly $T$ is generated contractive mapping.

But $\lambda(T(3),T(5),0)=\lambda(9,14,0)=5>\lambda(3,5,0)$

$\Rightarrow$ $T$ is not a contractive mapping.

**Definition 1.8:** A 2-metric space $(X,\lambda)$ is said to be *2-compact*if every sequence in $X$ has a convergent subsequence.

## 2. FIXED POINT THEOREMS FOR GENERATED CONTRACTIVE MAPPINGS

In this section we proved fixed point theorems for Generated Contractive type mappings.

The following Theorem-2.1 is a generalization of *Pal* and *Maiti* [2] in 2-metric space setting.

**Theorem 2.1:** Let $T$ be a continuous generated contractive self-mapping on a compact 2-metric space $X$ such that $\lambda(x,Ty,a)+\lambda(y,Tx,a)<2\lambda(x,y,a)$ for all $x,y,a$ in $X$. Then $T$ has a unique fixed point in $X$.

**Proof:** Let $s_0 \in X$..

Define a sequence $\{s_n\}$ in $X$ such that $s_n = T^n s_0$ when $n = 1, 2, 3, \ldots$

Consider $\lambda(x, Ty, a) + \lambda(y, Tx, a) < 2\lambda(x, y, a)$

Put $x = s_{n+1}$ , $y = s_n$

$$\lambda(s_{n+1}, Ts_n, a) + \lambda(s_n, Ts_{n+1}, a) < 2\lambda(s_{n+1}, s_n, a)$$

$$\Rightarrow \lambda(s_{n+1}, s_{n+1}, a) + \lambda(s_n, s_{n+2}, a) < 2\lambda(s_{n+1}, s_n, a)$$

$$\Rightarrow \lambda(s_n, s_{n+2}, a) < 2\lambda(s_{n+1}, s_n, a)$$

$$\Rightarrow \frac{1}{2}\lambda(s_n, s_{n+2}, a) < \lambda(s_{n+1}, s_n, a) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

.Put $c_n = d(s_n, s_{n+1}, a)$

Since $T$ is generated contractive,

$$\lambda(Tx, Ty, a) < \max\left\{\lambda(x, y, a), \lambda(x, Tx, a), \lambda(y, Ty, a), \frac{1}{2}\left[\lambda(x, Ty, a) + \lambda(y, Tx, a)\right]\right\}$$

Put $x = s_{n+1}$ , $y = s_n$

$$\lambda(Ts_{n+1}, Ts_n, a)$$

$$< \max\left\{\lambda(s_{n+1}, s_n, a), \lambda(s_{n+1}, Ts_{n+1}, a), \lambda(s_n, Ts_n, a), \frac{1}{2}\left[\lambda(s_{n+1}, Ts_n, a) + \lambda(s_n, Ts_{n+1}, a)\right]\right\}$$

$$= \max\left\{\lambda(s_{n+1}, s_n, a), \lambda(s_{n+1}, s_{n+2}, a), \lambda(s_n, s_{n+1}, a), \frac{1}{2}\left[\lambda(s_{n+1}, s_{n+1}, a) + \lambda(s_n, s_{n+2}, a)\right]\right\}$$

$$\Rightarrow \lambda(s_{n+2}, s_{n+1}, a) < \max\left\{\lambda(s_{n+1}, s_n, a), \lambda(s_{n+1}, s_{n+2}, a), \frac{1}{2}\lambda(s_n, s_{n+2}, a)\right\}$$

Let $M = \lambda(s_{n+1}, s_{n+2}, a)$

Then $\lambda(s_{n+1}, s_{n+2}, a) < \lambda(s_{n+1}, s_{n+2}, a)$

This is a contradiction.

So $M \neq \lambda(s_{n+1}, s_{n+2}, a)$

By $(1)$, $M = \lambda(s_n, s_{n+1}, a)$

So we get $\lambda(s_{n+1}, s_{n+2}, a) < \lambda(s_n, s_{n+1}, a)$

$$\Rightarrow c_{n+1} < c_n$$

Similarly $c_{n+1} < c_n < c_{n-1} < \ldots < c_0$

$\Rightarrow \{c_n\}$ is a monotonically decreasing and bounded sequence of nonnegative real numbers.

$\Rightarrow \{c_n\}$ Converges.

Suppose that $\lim_{n \to \infty} c_n = l$

Since $X$ is compact, $\{s_n\}$ has a convergent subsequence, say, $\{p_m\}$.

Let $\lim_{m\to\infty} p_m = u,$ where $u \in X$.

Since $T$ is continuous and $\lim_{m\to\infty} p_m = u,$ we have $\lim_{m\to\infty} Tp_m = Tu$

$$\Rightarrow \lim_{m\to\infty} Tp_{m+1} = Tu$$

Now we show that $u$ is a fixed point of $T$ Assume that $u \neq Tu$

Then $\lambda(u,Tu,a) \le \lambda(u,Tu,p_m) + \lambda(u,p_m,a) + \lambda(p_m,Tu,a)$

$$= \lambda(u,Tu,p_m) + \lambda(u,p_m,a) + \lambda(Tp_{m-1},Tu,a)$$

$$< \lambda(u,Tu,p_m) + \lambda(u,p_m,a)$$

$$+ \max\left\{\lambda(p_{m-1},u,a), \lambda(p_{m-1},Tp_{m-1},a), \lambda(u,Tu,a), \frac{1}{2}\left[\lambda(p_{m-1},Tu,a) + \lambda(u,Tp_{m-1},a)\right]\right\}$$

$$= \lambda(u,Tu,p_m) + \lambda(u,p_m,a)$$

$$+ \max\left\{\lambda(p_{m-1},u,a), \lambda(p_{m-1},p_m,a), \lambda(u,Tu,a), \frac{1}{2}\left[\lambda(p_{m-1},Tu,a) + \lambda(u,p_m,a)\right]\right\}$$

Letting $m \to \infty$

$$\lambda(u,Tu,a) < \max\left\{\lambda(u,Tu,a), \frac{1}{2}\lambda(u,Tu,a)\right\}$$

$$\lambda(u,Tu,a) < \lambda(u,Tu,a)$$

This is a contradiction.

Therefore our assumption is false.

Hence $Tu = u$.

$\Rightarrow u$ is a fixed point of $T$ in $X$ .

Now we show that it is unique.

Let $v$ be another fixed point of $T$

i.e, $Tv = v$

Assume that $u \neq v$

Then $\lambda(u,v,a) = \lambda(Tu,Tv,a)$

$$< \max\left\{\lambda(u,v,a), \lambda(u,Tu,a), \lambda(v,Tv,a), \frac{1}{2}\left[\lambda(u,Tv,a) + \lambda(v,Tu,a)\right]\right\}$$

$$= \max\left\{\lambda(u,v,a), \lambda(u,u,a), \lambda(v,v,a), \frac{1}{2}\left[\lambda(u,v,a) + \lambda(v,u,a)\right]\right\}$$

$$= \lambda(u,v,a)$$

$$\Rightarrow \lambda(u,v,a) < \lambda(u,v,a)$$

This is a contradiction.

Hence $u = v$.

$\Rightarrow T$ has a unique fixed point in $X$ .

**Theorem- 2.2:** Let $(X,\lambda)$ be a compact 2-metric space. Suppose that $S$ and $T$ are two continuous self-maps on a $X$ such that

$$(1)\ \lambda(x,Ty,a)+\lambda(y,Sx,a)<2\lambda(x,y,a)$$

$$(2)\ \lambda(Sx,Ty,a)<\max\left\{\lambda(x,y,a),\lambda(x,Sx,a),\lambda(y,Ty,a),\frac{1}{2}\left[\lambda(x,Ty,a)+\lambda(y,Sx,a)\right]\right\}$$

for all $x,y,a$ in $X$.

Then $S$ and $T$ have a unique common fixed point in $X$.

**Proof:** Let $p_0 \in X$.

Define a sequence $\{p_n\}$ in $X$ such that $p_{2n+1}=Sp_{2n}$, where $n=0,1,2,3,\ldots$

And $p_{2m}=Tp_{2m-1}$, where $m=1,2,3,\ldots$

Suppose that $\lambda_{2n}=\lambda(p_{2n},p_{2n+1},a)$, where $n=0,1,2,3,\ldots$

Consider $\lambda(x,Ty,a)+\lambda(y,Sx,a)<2\lambda(x,y,a)$

Put $x=p_{2n}$ and $y=p_{2n-1}$

$$\lambda(p_{2n},Tp_{2n-1},a)+\lambda(p_{2n-1},Sp_{2n},a)<2\lambda(p_{2n},p_{2n-1},a)$$

$$\Rightarrow \lambda(p_{2n},p_{2n},a)+\lambda(p_{2n-1},p_{2n+1},a)<2\lambda(p_{2n},p_{2n-1},a)$$

$$\Rightarrow \lambda(p_{2n-1},p_{2n+1},a)<2\lambda(p_{2n},p_{2n-1},a)$$

$$\Rightarrow \frac{1}{2}\lambda(p_{2n-1},p_{2n+1},a)<\lambda(p_{2n-1},p_{2n},a)\ (1)$$

Now, consider $\lambda(Sx,Ty,a)<\max\left\{\lambda(x,y,a),\lambda(x,Sx,a),\lambda(y,Ty,a),\frac{1}{2}\left[\lambda(x,Ty,a)+\lambda(y,Sx,a)\right]\right\}$

Put $x=p_{2n}$ and $y=p_{2n-1}$

$$\lambda(Sp_{2n},Tp_{2n-1},a)$$

$$<\max\left\{\lambda(p_{2n},p_{2n-1},a),\lambda(p_{2n},Sp_{2n},a),\lambda(p_{2n-1},Tp_{2n-1},a),\frac{1}{2}\left[\lambda(p_{2n},Tp_{2n-1},a)+\lambda(p_{2n-1},Sp_{2n},a)\right]\right\}$$

$$=\max\left\{\lambda(p_{2n},p_{2n-1},a),\lambda(p_{2n},p_{2n+1},a),\lambda(p_{2n-1},p_{2n},a),\frac{1}{2}\left[\lambda(p_{2n},p_{2n},a)+\lambda(p_{2n-1},p_{2n+1},a)\right]\right\}$$

$$\Rightarrow \lambda(p_{2n+1},p_{2n},a)<\max\left\{\lambda(p_{2n},p_{2n-1},a),\lambda(p_{2n},p_{2n+1},a),\frac{1}{2}\lambda(p_{2n-1},p_{2n+1},a)\right\}$$

If $\max\left\{\lambda(p_{2n},p_{2n-1},a),\lambda(p_{2n},p_{2n+1},a),\frac{1}{2}\lambda(p_{2n-1},p_{2n+1},a)\right\}=\lambda(p_{2n},p_{2n+1},a)$

Then $\lambda(p_{2n},p_{2n+1},a)<\lambda(p_{2n},p_{2n+1},a)$

This is a contradiction.

So $\max\left\{\lambda(p_{2n},p_{2n-1},a),\lambda(p_{2n},p_{2n+1},a),\frac{1}{2}\lambda(p_{2n-1},p_{2n+1},a)\right\}\neq\lambda(p_{2n},p_{2n+1},a)$

 

By $(1)$, we get

$$\max\left\{\lambda\left(p_{2n},p_{2n-1},a\right),\lambda\left(p_{2n},p_{2n+1},a\right),\frac{1}{2}\lambda\left(p_{2n-1},p_{2n+1},a\right)\right\}=\lambda\left(p_{2n-1},p_{2n},a\right)$$

$$\Rightarrow d_{2n}<d_{2n-1}$$

Similarly $d_{2n}<d_{2n-1}<d_{2n-2}<...<d_0$

$\Rightarrow\{d_{2n}\}$ is a monotonically decreasing and bounded sequence of nonnegative real numbers.

$\Rightarrow\{d_{2n}\}$ Converges.

Suppose that $\lim\limits_{n\to\infty} d_{2n}=l$

Since $X$ is compact, every sequence in $X$ has a convergent subsequence

So $\{p_n\}$ has a convergent subsequencein $X$ .

Suppose that $\{p_{2n}\}$ is a convergent subsequence of $\{p_n\}$ in $X$ and let $\lim\limits_{n\to\infty} p_{2n}=u,$ where $u\in X$.

Now we prove that $u$ is acommon fixed point of $S$ and $T$ .
Assume that $u\neq Su$

Since $S$ is continuous on $X$ and $\lim\limits_{n\to\infty} p_{2n}=u,$ we have $\lim\limits_{n\to\infty} Sp_{2n}=Su$

$$\Rightarrow \lim_{n\to\infty} p_{2n+1}=Su$$

Since $T$ is continuous on $X$ and $\lim\limits_{n\to\infty} p_{2n+1}=Su,$ we have $\lim\limits_{n\to\infty} Tp_{2n+1}=TSu$

$$\Rightarrow \lim_{n\to\infty} Tp_{2n+1}=TSu$$

$\Rightarrow \lim\limits_{n\to\infty} p_{2n+2}=TSu$ because $Tp_{2n+1}=p_{2n+2}$

Then $\lambda(u,Su,a)\le\lambda(u,Su,p_{2n})+\lambda(u,p_{2n},a)+\lambda(p_{2n},Su,a)$

$$=\lambda(u,Su,p_{2n})+\lambda(u,p_{2n},a)+\lambda(Tp_{2n-1},Su,a)$$

$$<\lambda(u,S\ up_{2n})+\lambda(u,p_{2n},a)$$

$$+\max\left\{\lambda(u,p_{2n-1},a),\lambda(u,Su,a),\lambda\left(p_{2n-1},Tp_{2n-1},a\right),\frac{1}{2}\left[\lambda(u,Tp_{2n-1},a)+\lambda\left(p_{2n-1},Su,a\right)\right]\right\}$$

$$=\lambda(u,S\ up_{2n})+\lambda(u,p_{2n},a)$$

$$+\max\left\{\lambda(u,p_{2n-1},a),\lambda(u,Su,a),\lambda\left(p_{2n-1},p_{2n},a\right),\frac{1}{2}\left[\lambda(u,p_{2n},a)+\lambda\left(p_{2n-1},Su,a\right)\right]\right\}$$

Letting $n\to\infty$

$$\lambda(u,Su,a)<\max\left\{\lambda(u,Su,a),\frac{1}{2}\lambda(u,Su,a)\right\}$$

$$\Rightarrow\lambda(u,Su,a)<\lambda(u,Su,a)$$

This is a contradiction.Soour assumption is false.

Hence $Su=u$ .

$\Rightarrow u$ is a fixed point of $S$ in $X$ .

Consider $\lambda(x,Ty,a)+\lambda(y,Sx,a)<2\lambda(x,y,a)$

Put $x=y=u$

$$\lambda(u,Tu,a)+\lambda(u,Su,a)<2\lambda(u,u,a)$$

Since $Su=u$, $\lambda(u,Tu,a)<0$.

But $\lambda(u,Tu,a)\geq 0$.

$$\Rightarrow \lambda(u,Tu,a)=0$$
$$\Rightarrow Tu=u$$

Hence $u$ is a common fixed point of $S$ and $T$ in $X$.

Now we show that it is unique.

Let $v$ be another fixed point of $T$
i.e, $Sv=Tv=v$

Assume that $u\neq v$
Then $\lambda(u,v,a)=\lambda(Su,Tv,a)$

$$<\max\left\{\lambda(u,v,a),\lambda(u,Su,a),\lambda(v,Tv,a),\frac{1}{2}\left[\lambda(u,Tv,a)+\lambda(v,Su,a)\right]\right\}$$
$$=\max\left\{\lambda(u,v,a),\lambda(u,u,a),\lambda(v,v,a),\frac{1}{2}\left[\lambda(u,v,a)+\lambda(v,u,a)\right]\right\}$$
$$=\lambda(u,v,a)$$
$$\Rightarrow \lambda(u,v,a)<\lambda(u,v,a)$$

This is a contradiction.

Hence $u=v$.

Thus $S$ and $T$ have a unique common fixed point in $X$.

**Remark-2.3:** If we take $S=T$ in Theorem-2.2 then we obtain Theorem-2.1. So 2.2 is a further generalization of Theorem-2.1. A point is a unique fixed point of $T:X\to X$ iff it is unique fixed point of any positive power of $T$. This fact leads us to the following theorem and proof of the following is similar to the proof of previous theorem.

**Theorem- 2.4:** Let $(X,\lambda)$ be a compact 2-metric space. Suppose that $S$ and $T$ are two continuous self-maps on $X$ such that

(1) $\lambda(x,T^q y,a)+\lambda(y,S^p x,a)<2\lambda(x,y,a)$

(2) $\lambda(S^p x,T^q y,a)<\max\left\{\lambda(x,y,a),\lambda(x,S^p x,a),\lambda(y,T^q y,a),\frac{1}{2}\left[\lambda(x,T^q y,a)+\lambda(y,S^p x,a)\right]\right\}$

For all $x,y,a$ in $X$ and for all $p,q\in\mathbb{N}$.
Then $S$ and $T$ have a unique common fixed point in $X$.

 

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