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STOCHASTIC DIFFERENTIAL EQUATIONS WITH FRACTIONAL BROWNIAN MOTION

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ABSTRACT

In this article, we introduce the concept of stochastic differential equations (SDEs) with fractional Brownian motion. Using the concept of dynamic process under multitime scale in sciences and engineering, a mathematical model described by a system of multi-time scale SDEs is formulated.

Keywords: *Stochastic fractional differential equations, fractional brownian motion, dynamic process, multi-time scales.*

1. INTRODUCTION

We study the concept of dynamic process operating under multi-time scales in sciences and engineering. A mathematical model described by a system of multi-time scale stochastic differential equations is formulated. The classical Picard-Lindelf successive approximations scheme is applied to the model validation problem, viz existence and uniqueness of solution process. This leads to the problem of finding closed form solutions of both linear and nonlinear multi-time scale stochastic differential equations of Ito-Doob type. We also discuss the solutions of multi-time scale fractional stochastic differential equations driven by fractional Brownian motions. The rest of the paper is structured as follows: In Section 2, we give the preliminaries. Fractional Stochastic Differential Equations driven by fractional Brownian motion are presented in Section 3. Results related to the Fractional Stochastic Differential Equations are presented here. Finally conclusion is given in Section 4.

2. PRELIMINARIES

In this paper, we present some known concepts and results in the fields of fractional and stochastic differential equations.

Definition 2.1: Let $0 < \alpha < 1$ and $f \in L^1[a,b](L^1[a,b] = L^1\big[[a,b]\mathbb{R}^n\big] = \{y|y:[a,b] \to \mathbb{R}^n$ and y is Lebesgue integrable$\}$). The left-sided and right-sided Riemann –Liouville fractional integrals of order α are defined for almost all $t \in (a,b)$ by

$$(I_{a+}^{\alpha})(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{(\alpha-1)} f(s)ds, \quad t > a \tag{1}$$

$$(I_{b-}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (t-s)^{\alpha-1} f(s)ds, \quad t < b \tag{2}$$

respectively, where $\Gamma(\alpha) = \int_0^\infty r^{\alpha-1}e^{-r}dr$ is the Euler function.

Definition 2.2: A function f is said to be absolutely continuous on an interval J, if for $\varepsilon > 0$ there exists a $\delta > 0$ such that for any pair wise nonintersecting intervals $[a_k, b_k] \subset J$, $k = 1, \dots, n$ such that $\sum_{k=1}^n (b_k - a_k) < \delta$, the inequality $\sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon$ holds.

It is known that the space $AC[a,b]$ of absolutely continuous functions on $[a,b]$ coincides with the Lebesgue summable functions.

$$f \in AC[a,b] \Leftrightarrow f(x) = c + \int_a^x \varphi(t)dt \qquad (\varphi \in \mathrm{L}^1[a,b]). \tag{3}$$

Definition 2.3: Let f be defined and absolutely continuous on an interval $[a,b]$, and let $0<\alpha<1$,

$$(D_{a+}^{\alpha})(t)=\frac{1}{\Gamma(1-\alpha)}\left[\frac{f(a)}{(t-a)^{\alpha}}+\int_a^t (t-s)^{-\alpha}f'(s)ds\right] \tag{4}$$

$$(D_{b-}^{\alpha})(t)=\frac{1}{\Gamma(1-\alpha)}\left[\frac{f(b)}{(b-t)^{\alpha}}-\int_t^b (s-t)^{-\alpha}f'(s)ds\right] \tag{5}$$

are called left-sided and right-sided Riemann-Liouville fractional derivatives, respectively.

Definition 2.4: Let $a,b\in\mathbb{R}$ and $0<\alpha<1$. The L_∞^1 space is defined as follows

$$L_\infty^1[a,b]:=\{y\in L^1[a,b]: D_{a+}^{\alpha}y\in L^1[a,b]\}, \tag{6}$$

where $L^1[a,b]$ is the space of summable or integrable functions in a finite interval $[a,b]$ of the real line $\mathbb{R}$.

Definition 2.5: For $0<\alpha<1$, the left-hand caputo derivatives of order α, denoted by ${}^{c}D_{a+}^{\alpha}f$, is defined as the Riemann-Liouville type fractional derivatives

$$({}^{c}D_{a+}^{\alpha}f)(t)=D_{a+}^{\alpha}[f(t)-f(a)]=\frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha}f'(s)ds \tag{7}$$

and hence

$$({}^{c}D_{a+}^{\alpha}f)(t)=(D_{a+}^{\alpha}f)(t)-(D_{a+}^{\alpha}f)(a)=(D_{a+}^{\alpha}f)(t)-\frac{f(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha} \tag{8}$$

Definition 2.6: A dynamic process is said to be operating under multi-time scales if the effects of certain intra structural and external environmental perturbations are characterized by a set of linearly independent time scales monitored by the classical time.

Definition 2.7: Let $\{T_1(t)=t, T_2(t)=B(t), T_3(t)=t\hat{}\alpha\}$ be the set of linearly independent time scales. A random process $x=\{x(t), t_0<t<t_0+T\}$ is called solution of the Initial Value Problem (IVP), if the composite function $x(x(t)\equiv x(t,w(t),t^{\alpha}))$ is sample continuous with respect to each of the time scales $T_j, j=1,2,3$.

Definition 2.8: A one dimensional fractional Brownian motion (fBm) $B_H=\{B_H(t), t\in[0,T]\}$ of Hurst index $H\in(0,1)$ on $[0,T]$ is a continuous and centered Gaussian process on some probability space (Ω,F,P) with covariance function

$$E[B_H(t)B_H(s)]=\frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H}),\quad t,s\in[0,T]$$

If $H=\frac{1}{2}$, then the corresponding fBm is the usual standard Brownian motion. If $H>\frac{1}{2}$, then the process fBm exhibits a long-rang dependence

Definition 2.9: Let $\alpha>0$. Then the Riemann-Liouville fractional integral of order α with respect to t is defined as

$$I_t^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau,\qquad t>0 \tag{9}$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.10: Let $f\in C([0,T])$ and $m-1<\alpha\le m$, where $m\in\mathbb{N}^+$. The Riemann-Liouville fractional derivative of order α with respect to t is defined as

$$D_t^{\alpha}f(t)=\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t (t-\tau)^{m-\alpha-1}f(\tau)d\tau,\quad t>0 \tag{10}$$

There exists the following relationship between the Riemann-Liouville fractional integral and the Riemann-Liouville fractional derivative.

Definition 2.11: Suppose that the Laplacian $(-\Delta)$ has a complete set of orthonormal eigenfunctions φ_n corresponding to eigenvalues λ_n^2 on a bounded region D; i.e., $(-\Delta)\varphi_n=\lambda_n^2\varphi_n$ on D; $B(\varphi_n)=0$ on ∂D , where $B(\varphi_n)$ is one of the standard three homogenous boundary conditions. Let

$$G=\left\{g=\sum_{n=1}^{\infty}c_n\varphi_n,\ \ c_n=\langle g,\varphi_n\rangle, \sum_{n}^{\infty}|c_n|^2|\lambda_n|^{\alpha}<\infty\right\}, \tag{11}$$

then for any $g\in G$, $(-\Delta)^{\frac{\alpha}{2}}$ is defined by ,

$$(-\Delta)^{\frac{\alpha}{2}}g=\sum_{n=1}^{\infty}c_n\lambda_n^{\alpha}\varphi_n. \tag{12}$$

3. FRACTIONAL BROWNIAN MOTION

In this section, we use some theorems and results based on fractional stochastic differential equations driven by fractional Brownian motion.

Theorem 3.1: (Existence and Uniqueness). Assume that, for $(t,x) \in [t_0, t_0+T] \times \mathbb{R}^n, \alpha \in \left(\frac{1}{2,1}\right), b, \sigma_2 \in C\left[[t_0, t_0+T] \times \mathbb{R}^n, \mathbb{R}^n\right], \sigma_1 \in C[[t_0, t_0+T] \times \mathbb{R}^n, \mathbb{R}^{nm}]$and $B = \{B(t), t \geq 0\}$ is a m-dimensional Brownian motion on a complete probability space $\Omega \equiv (\Omega, F, P)$ the following inequalities hold:

$$|b(t,x)|^2 + |\sigma_1(t,x)|^2 + |\sigma_2(t,x)|^2 \leq K^2(1+|x|^2); \text{(linear growth bound)} \tag{13}$$

for some constant $K > 0$, the lipchitze condition

$$|b(t,x) - b(t,y)| + |\sigma_1(t,x) - \sigma_1(t,y)| + |\sigma_2(t,x) - \sigma_2(t,y)| \leq L|x-y| \tag{14}$$

for some constant $L > 0$. Let x_0 be a random variable defined on (Ω, F, P) and it is independent of the σ −algebra $F_s^t \subseteq F$ generated by $\{B(s), t \geq s \geq 0\}$ and such that $E|x_0|^2 < \infty$.
Then the IVP $dx = b(t,x)dt + \sigma_1(t,x)dB(t) + \sigma_2(t,x)(dt)^\alpha, \quad x(t_0) = x_0$ has a unique solution which is t-continuous with the property that $x(t,\omega)$ is adapted to the filtration $F_t^{x_0}$ generated by x_0 and $\{B(s)(\cdot), s \leq t\}$, and

$$E\left[\int_{t_0}^{T} |x(t,\omega)|^2 < \infty.\right] \tag{15}$$

Remark: The Existence and uniqueness theorem was motivated by the long range delay dependent dynamic process. However, we propose to investigate the proposed problem for not only $0 < \alpha < 1$, but also more general set of linearly independent multi time-scales.

Lemma 3.2: $X(t)$ satisfies

$$dX(t) = u(t)dt + v(t)dB_H(t) \tag{16}$$

where u, v are given functions, Furthermore, let $f \in C^2(\mathbb{R})$, and assume that $f'(X)$ and $f''(X)$ exists and are continuous for $X \in \mathbb{R}$. Then, it has

$$df\big(X(t)\big) = \Big(f'\big(X(t)\big)u(t) + Hf''\big(X(t)\big)t^{2H-1}v^2(t)\Big)\,dt + f'\big(X(t)\big)v(t)dB_H(t) \tag{17}$$

It is intersecting to note that if $H = \frac{1}{2}$ is formally substituted in the equation (17), then the well-known in to formula for classical Brownian motion is obtained.

Lemma 3.3: Suppose that the one-dimensional Laplacian $(-\Delta)$ defined with Dirichlet boundary conditions at $x = 0$ and $x = L$ has a complete set of orthonormal eigenfuntions φ_n corresponding to eigenvalues λ_n^2 on a bounded region $[o, L]$. If $(-\Delta)\varphi_n = \lambda_n^2 \varphi_n$ on $[0, L]$, and $\varphi_n(0) = \varphi_n(L) = 0$, then, the eigen values are given by $\lambda_n^2 = \frac{n^2\pi^2}{L^2}$, and the corresponding eigen functions are $\varphi_n(x) = \sin(n\pi x/L), n = 1,2, \ldots$

Lemma 3.4: Let $\frac{1}{2} < H < 1$ and $\sigma \in C([0,T])$. Then the solution of the equation

$$dY^s(t) = \sigma(t)Y^s(t)dB_H(t), \quad Y^s(0) = y_0^s$$

$$Y^s(t) = y_0^s exp\left(-H\int_0^t \tau^{2H-1}\sigma^2(\tau)d\tau + \int_0^t \sigma(\tau)dB_H(\tau)\right) \tag{18}$$

Theorem 3.5: Let $a, b, \sigma \in C([0,T]), 0 < \alpha < 1$ and $\frac{1}{2} < H < 1$. Then the solution of equation

$$dY(t) = \frac{1}{\Gamma(2-\alpha)}a(t)Y(t)(dt)^{1-\alpha} + b(t)Y(t)dt + \sigma(t)Y(t)dB_H(t), Y(0) = y_0 \tag{19}$$

is given by

$$Y(t) = exp\left(\int_0^t b(\tau)d\tau - H\int_0^t \tau^{2H-1}\sigma^2(\tau)d\tau + \int_0^t \sigma(\tau)dB_H(\tau)\right)\sum_{i=0}^{\infty} R_a^i y_0 \tag{20}$$

where R_a is defined as the equation,

$$(R_a\varphi)(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}a(\tau)\varphi(\tau)d\tau \tag{21}$$

and R_a^i denotes the i- times composition operator of R_a.

We denote

$$\Phi(t) = \exp\left(\int_0^t \big(b(\tau) - H\tau^{2H-1}\sigma^2(\tau)\big)d\tau + \int_0^t \sigma(\tau)dB_H(\tau)\right)\sum_{i=0}^{\infty} R_a^i. \tag{22}$$

One knows that Φ is the fundamental solution of equation (19). In the following, we will show that Φ is invertible on $[0,T]$ in an algebraic sense.

Theorem 3.6: Let Φ be the fundamental solution of equation (19). Then Φ is invertible on $[0, T]$, and its inverse is

$$\Phi^{-1} = \exp\left(-\int_0^t \left(b(\tau) - H\tau^{2H-1}\sigma^2(\tau)\right)d\tau - \int_0^t \sigma(\tau)dB_H(\tau)\right)\sum_{i=0}^{\infty}(-1)^i R_a^i, \quad (23)$$

where R_a is the operator defined as the equation (21).

Theorem 3.7: Let $a, p, q, v, \sigma \in C[0, T]$, $0 < \alpha < 1$ and $\frac{1}{2} < H < 1$. Then the solution of equation (19) is given by

$$Y(t) = \Phi(t) + \int_0^t \Phi(t,\tau)p(\tau)(d\tau)^{1-\alpha} + \int_0^t \Phi(t,\tau)(q(t) - 2H\tau^{2H-1}v(\tau)\sigma(\tau)d\tau) + \int_0^t \Phi(t,\tau)v(\tau)dB_H(\tau)$$

where, $\Phi(t,\tau) = \Phi(t)\Phi^{-1}(\tau)$, Φ and Φ^{-1} are defined as the equations (22) and (23) respectively.

4. CONCLUSION

We introduced the concept of stochastic differential equations (SDEs) with fractional Brownian motion. Using the concept of dynamic process under multitime scale in sciences and engineering, a mathematical model described by a system of multi-time scale SDEs is formulated.

REFERENCE

1. Applebaum D. Lvy processes and stochastic calculus. Cambridge University Press; 2004.
2. Arnold, L. Stochastic Differential Equations: Theory and Applications; Wiley: NewYork, NY, USA, 1974
3. Arnold L. Stochastic differential equations: theory and applications. Wiley; 1974.
4. Balan, R.M., Tudor, C.A.: Stochastic heat equation with multiplicative fractional colored noise. J. Theor. Probab. 23, 834-870 (2010)
5. Balan, R.M.: The Stochastic wave equation with multiplicative fractional noise: a Malliavin calculus approach. Potential Anal. 36, 1-34 (2012)
6. Balan, R.M.: Some linear SPDEs driven by a fractional noise with Hurst index greater that 1/2. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 15, 1250023 (2012).
7. Balan, R.M.: Linear SPDEs driven by stationary random distributions. J. Fourier Anal. Appl.18, 1113-1145 (2012)
8. Biagini F, Hu Y, ksendal B, Zhang T. Stochastic calculus for fractional Brownian motion and applications. Springer; 2008.
9. Da Prato G, Zabczyk J. Stochastic differential equations in infinite dimensions.Cambridge University Press; 1992.
10. Junfeng Liu.Litan Yan; Solving a nonlinear fractional stochastic partial differential equation with fractional noise; Springer New york 2014
11. Jean-C.Pedjeu, Gangaram S.Ladde; Stochastic fractional differential equations: Modeling method and analysis: Chaos Solitons fractals 2012, 45, 279-293
12. Xia-Li Ding and Juan J.Nieto; Analytical Solutions for Multi-time scale fractional stochastic differential equations driven by fractional brownian motion and their applications: Entropy 2018, 20,63, doi;10.3390/e20010063

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