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OPTIMAL REPLACEMENT MODEL FOR A DEGENERATING FAILURE SYSTEM

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ABSTRACT

In this article, we study the optimal replacement model for degenerating failure systems. We obtained optimal replacement model N^* by minimizing the average cost rate $C(N)$. We show that uniqueness of the optimal replacement model N^*.

Keywords: *Replacement, Poisson process, Geometric repair process, δ-Shock models, Maintenance, α-Series process, Monotone process.*

1. INTRODUCTION

A critical machine or facility failure may interrupt the production of a manufacturing firm. Such an interruption will have negative impacts on a firm's performance such as the revenue and customer service. To minimize the negative effects of machine failures, practitioners are interested in finding the appropriate machine maintenance and replacement models. Usually, a machine, also called a system, experience two stages consecutively before its replacement an operating stage (productive) and a repair stage (non-productive). We model both stages and consider a threshold replacement model for the system subject to random shocks.

There are extensive studies on the maintenance problems for the system with operating and repair stages. This is mainly because that some classical assumptions are not realistic in modeling the real systems. Barlow and Proschan (1983) introduced an imperfect repair model, where the repair is prefect with probability p and minimal with probability $1-p$. Other studies along this line include Block *et al.* (1985), Kijima (1989), Makis and Jardine (1992), Dekker (1996), Moustafa *et al.* (2004), Sheu *et al*. (2006), Wang and Zhang (2009), Zhang and Wang (2011), and Yuan and Xu (2011).

We also consider the non-zero repair and replacement times in contrast to a classical assumption that that repairs are instantaneous. In most practical situations, to reflect the aging process of the system, the consecutive repair times are assumed to become longer and longer till the system is replaced with a new one according to some replacement rule. Lam (1988) first introduced the geometric processes (GP) to study the maintenance for such a deteriorating system.

The rest of the paper is structured as follows: In Section 2 we give the preliminaries. Develops the average cost per unit time $C(N)$ and determines the optimal replacement model N^* in Section 3. Finally conclusion is given in section 4.

2. PRELIMINARIES

Definition 2.1: A stochastic process $\{\xi_n, n = 1,2,\ldots\}$ is called a geometric increasing or decreasing process if there exists a real number$(0 < a \leq 1 \ or \ a > 1)$, thus, $\{a^{n-1}\xi_n, \ n = 1,2,\ldots.\}$ forms a new renewal process. The real a is called the ratio of the geometric process (GP). Letting $E(\xi_n) = \tau$ and $Var(\xi_1) = \sigma^2$, we have $E(\xi_n) = \frac{\tau}{a^{n-1}}$ and $Var(\xi_1) = \frac{\sigma^2}{a^{2(n-1)}}$. Therefore, a geometric process (GP) has three parameters of a, τ and σ^2.

Definition 2.2: Consider a continuous positive random variable X having distribution function F and density f. The failure (or hazard) rate function $r(t)$ is defined by

$$r(t) = \frac{f(t)}{1-F(t)}$$

Definition 2.3: A stochastic process $\{X_n, n = 1,2, \ldots.\}$ is called an α-series process, if there exists a real, such that $\{n^\alpha X_n, n = 1,2,3 \ldots\}$ forms a renewal process. The real α is called the exponent of the process. Obviously, for an α -series process, if the distribution function of X_1 is F, then the distribution function of X_n will be F_n with $F_n(t) = F(n^\alpha t), n = 1,2,3 \ldots$ If $\alpha > 0$, the α-series process $\{X_n, n = 1, 2, \ldots\}$ is stochastically decreasing; and when $\alpha < 0$, the α -series process is stochastically increasing; and when $\alpha = 0$, the α -series process reduces to the renewal process.

If $E(X_1) = \lambda$ then we have the $E(X_n) = \frac{\lambda}{n^\alpha}$

Assumption 2.1: At t = 0, a new system is installed. Whenever the system fails, it is either repaired or replaced with a new one.

Assumption 2.2: Shocks arrive according to a Poisson process with rate λ_1 or $EX_i = \frac{1}{\lambda}$ where X_i is the i-th inter-arrival time of two consecutive shocks. Let δ_i be another exponentially distributed random variable associated with X_i. We assume that the sequence $\{\delta_i, i = 1,2 \ldots.\}$ forms an increasing geometric process with $0 < a \leq 1$. Then δ_i has cumulative distribution function $Q(a^{i-1}x)$, where $Q(x)$ is the cumulative distribution function of δ_i. $\{X_i, \delta_i\}$ follows a δ-shock model if the system fails at i-th shock which satisfies $X_i \leq \delta_i$, and then the life time or equivalently the operating time is the sum of all X_i until the one satisfying the above condition. Further, we assume that X_i is independent of δ_i.

Assumption 2.3: Let T_n be the operating time after the $(n-1)$-th repair. $\{T_n, n = 1,2, \ldots\}$ is a stochastically decreasing random variable sequence induced by the δ-shock model.

Assumption 2.4: Let Y_n be the repair time after the n-th failure and forms an increasing geometric process with $0 < b \leq 1$. Then Y_n has cumulative distribution function $G(b^{n-1}y)$, where G(y) is the cumulative distribution function of Y_1 with $EY_1 = \mu > 0$.

Assumption 2.5: T_n and Y_n, $n = 1,2,3 \ldots.$ are two independent sequences.

Assumption 2.6: Assume that the repair cost rate is c, the operating reward rate is r, and the replacement cost consists of fixed cost R and variable cost $v = r_p Z$, where Z is the replacement time and r_p is the rate of cost per time unit during replacement. Let $E(Z) = t$.

Assumption 2.7: A threshold N replacement models is adopted. Under such a model, the system will be replaced with a new one after it fails for N times.

3. LONG-RUN AVERAGE COST PER UNIT TIME C(N)

Now we develop the average cost function of the N -replacement model under the imposed cost structure. According to renewal reward theorem.

$$C(N) = \frac{\text{Expected cost incurred in a renewal cycle}}{\text{Expected length in a renewal cycle}} \tag{1}$$

Now, let W be the length of a renewal cycle under N-replacement model. Thus, we have

$$W = \sum_{n=1}^{N} T_n + \sum_{n=1}^{N-1} Y_n + Z \tag{2}$$

To evaluate the expected cost in a cycle, we first calculate $E(T_n)$, the expected operating time of the system after the $(n-1)$-th failure. Let l_{ni} be the inter-arrival time between the $(i-1)$-th and i-th shock following the $(n-1)$-th repair, where $i = 1,2,3 \ldots$ Define

$$M_n = \min\{m | l_{n1} > a^{n-1}\delta_1, \ldots, l_{n(m-1)} > a^{n-1}\delta_1, l_{nm} < a^{n-1}\delta_1\}$$

And

$$T_n = \sum_{j=1}^{M_n} l_{nj}$$

Thus M_n denotes the number of shocks till the first deadly shock occurs. Obviously, M_n has a geometric distribution, with

$$P(M_n = k) = q_n^{k-1} p_n, k = 1,2,3..$$

Where p_n is the probability of a shock, following the $(n-1)$-th repair and $q_n = 1 - p_n$. Therefore, we have

$$E(M_n) = \frac{1}{p_n}.$$

As M_n is a stopping time with respect to the random sequence $\{l_{nj}, j = 1,2,3 \dots\}$ which are independent identically distributed random variables. Using Wald equation (1983), we have

$$E(T_n) = E\left(\sum_{j=1}^{M_n} l_{nj}\right) = El_{n1}EM_n = \frac{El_{n1}}{p_n}$$

According to **Assumption 2.2**, as $F(x)$ and $Q(x)$ are all exponentially distributed, we have

$$F(x) = 1 - e^{-\lambda_1 x}, x \geq 0, \quad Q(a^{n-1}\, x) = 1 - e^{a^{n-1}\lambda_2 x}, x \geq 0 \tag{3}$$

and

$$El_{n1} = \int_0^\infty x dF(x) = \int_0^\infty x d\left(1 - e^{-\lambda_1 x}\right) = \frac{1}{\lambda_1}$$

Furthermore, as l_{nj} and $\delta_n(a^{n-1}\delta_1)$ are independent and have the marginal exponential distributions with means of $\frac{1}{\lambda_1}$ and $\frac{1}{a^{n-1}\lambda_2}$, respectively. Therefore, we obtain

$$p_n = P\left(l_{nj} < \delta_n\right) = \int_0^\infty e^{-a^{n-1}\lambda_2 x}\lambda_1\, e^{-\lambda_1 x} dx = \lambda_1 \int_0^\infty e^{-(a^{n-1}\lambda_2 + \lambda_1)x} dx = \frac{\lambda_1}{\lambda_1 + a^{n-1}\,\lambda_2} \tag{4}$$

and

$$\zeta_n = E(T_n) = \frac{\lambda_1 + a^{n-1}\lambda_2}{\lambda_1^2} \tag{5}$$

Consequently,

$$E\left(\sum_{n=1}^{N} T_n\right) = \sum_{n=1}^{N} E(T_n) = \sum_{n=1}^{N} \frac{\lambda_1 + a^{n-1}\lambda_2}{\lambda_1^2} \tag{6}$$

On the other hand, since Y_n, $n = 1,2,3 \dots$ is an increasing geometric processes (GP). with ratio $0 < b \leq 1$, we have

$$E(Y_n) = \frac{\mu}{b^{n-1}} \tag{7}$$

Then, by the equation (1), the long-run average cost $C(N)$ of the system under the models N is given by

$$C(N) = \frac{E\left(c\sum_{n=1}^{N-1} Y_n - r\sum_{n=1}^{N} T_n + R + r_p Z\right)}{E(\sum_{n=1}^{N} T_n + \sum_{n=1}^{N-1} Y_n + Z)}$$

$$= \frac{c\sum_{n=1}^{N-1} E(Y_n) - r\sum_{n=1}^{N} E(T_n) + ER + E(r_p Z)}{\sum_{n=1}^{N} E(T_n) + \sum_{n=1}^{N-1} E(Y_n) + E(Z)}$$

$$= \frac{c\sum_{n=1}^{N-1} \frac{\mu}{b^{n-1}} - r\sum_{n=1}^{N} \frac{\lambda_1 + a^{n-1}\lambda_2}{\lambda_1^2} + R + r_p t}{\sum_{n=1}^{N} \frac{\lambda_1 + a^{n-1}\lambda_2}{\lambda_1^2} + \sum_{n=1}^{N-1} \frac{\mu}{b^{n-1}} + t} \tag{8}$$

The optimal replacement models N^* can be determined by minimizing $C(N)$. To determine the optimal N^*, the equation (8) can be re-written as.

$$C(N) = \frac{(c + r)\sum_{n=1}^{N-1} \frac{\mu}{b^{n-1}} + R + (r_p + r)t}{\sum_{n=1}^{N} \frac{\lambda_1 + a^{n-1}\lambda_2}{\lambda_1^2} + \sum_{n=1}^{N-1} \frac{\mu}{b^{n-1}} + t} - r \tag{9}$$

Thus, to minimize $C(N)$ is equivalent to minimize the first term of the equation (9) denoted by $B(N)$

$$B(N) = \frac{(c + r)\sum_{n=1}^{N-1} \frac{\mu}{b^{n-1}} + R + (r_p + r)t}{\sum_{n=1}^{N} \frac{\lambda_1 + a^{n-1}\lambda_2}{\lambda_1^2} + \sum_{n=1}^{N-1} \frac{\mu}{b^{n-1}} + t} \tag{10}$$

Now, we study the difference between $B(N + 1)$ and $B(N)$. Let

$$f(N) = \sum_{n=1}^{N} \frac{\lambda_1 + a^{n-1}\lambda_2}{\lambda_1^2} \sum_{n=1}^{N-1} \frac{\mu}{b^{n-1}}$$

Then

$$B(N+1)-B(N)=\frac{(c+r)\sum_{n=1}^{N}\frac{\mu}{b^{n-1}}+R+(r_p+r)t}{\sum_{n=1}^{N+1}\zeta_n+\sum_{n=1}^{N}\frac{\mu}{b^{n-1}}+t}-\frac{(c+r)\sum_{n=1}^{N-1}\frac{\mu}{b^{n-1}}+R+(r_p+r)t}{\sum_{n=1}^{N}\zeta_n+\sum_{n=1}^{N-1}\frac{\mu}{b^{n-1}}+t}$$

$$=\frac{1}{f(N+1)f(N)}\left\{\left[(c+r)\sum_{n=1}^{N}\frac{\mu}{b^{n-1}}+R+(r_p+r)t\right]\left(\sum_{n=1}^{N}\zeta_n+\sum_{n=1}^{N-1}\frac{\mu}{b^{n-1}}+t\right)\right.$$
$$\left.-\left[(c+r)\sum_{n=1}^{N-1}\frac{\mu}{b^{n-1}}+R+(r_p+r)t\right]\left(\sum_{n=1}^{N+1}\zeta_n+\sum_{n=1}^{N}\frac{\mu}{b^{n-1}}+t\right)\right\}$$

$$=\frac{1}{f(N+1)f(N)}\left\{(c+r)\mu\sum_{n=1}^{N}\frac{1}{b^{n-1}}\sum_{n=1}^{N}\zeta_n+(c+r)\mu^2\sum_{n=1}^{N}\frac{1}{b^{n-1}}\sum_{n=1}^{N-1}\frac{1}{b^{n-1}}\right.$$
$$+(c+r)\mu t\sum_{n=1}^{N}\frac{\mu}{b^{n-1}}+[R+(r_p+r)t]\left(\sum_{n=1}^{N}\zeta_n+\sum_{n=1}^{N-1}\frac{\mu}{b^{n-1}}+t\right)$$
$$-(c+r)\mu\sum_{n=1}^{N-1}\frac{1}{b^{n-1}}(\sum_{n=1}^{N}\zeta_n+\zeta_{N+1})-(c+r)\mu^2\sum_{n=1}^{N}\frac{1}{b^{n-1}}\sum_{n=1}^{N-1}\frac{1}{b^{n-1}}$$
$$\left.-(c+r)\mu t\sum_{n=1}^{N-1}\frac{1}{b^{n-1}}-[R+(r_p+r)t]\left[\left(\sum_{n=1}^{N-1}\zeta_n+\zeta_{N+1}\right)+\left(\sum_{n=1}^{N-1}\frac{\mu}{b^{n-1}}+\frac{\mu}{b^{N-1}}+t\right)+t\right]\right\}$$
$$=\frac{1}{b^{N-1}f(N+1)f(N)}\left\{(c+r)\mu\left(\sum_{n=1}^{N}\zeta_n-\zeta_{N+1}\sum_{n=1}^{N}b^{N-n}+t\right)-[R+(r_p+r)t]\right.$$
$$\left.(\zeta_{N+1}b^{N-1}+\mu)\right\} \tag{11}$$

Define the auxiliary function, $A(N)$ as follow,

$$A(N)=\frac{(c+r)\mu(\sum_{n=1}^{N}\zeta_n-\zeta_{N+1}\sum_{n=1}^{N-1}b^{N-n}+t)}{[R+(r_p+r)t](\zeta_{N+1}b^{N-1}+\mu)} \tag{12}$$

As the denominator of $B(N+1)-B(N)$ is always positive, it is clear that the sign of $B(N+1)-B(N)$ is the same as that of its numerator. Thus, we have

$$B(N+1)>(=,<)B(N)\Leftrightarrow A(N)>(=,<)1$$

Furthermore, it is clearly that,

$$A(N+1)-A(N)=\frac{(c+r)\mu(\sum_{n=1}^{N+1}\zeta_n-\zeta_{N+2}\sum_{n=1}^{N}b^{N+1-n}+t)}{[R+(r_p+r)t](\zeta_{N+2}b^{N}+\mu)}-\frac{(c+r)\mu(\sum_{n=1}^{N}\zeta_n-\zeta_{N+1}\sum_{n=1}^{N-1}b^{N-n}+t)}{[R+(r_p+r)t](\zeta_{N+1}b^{N-1}+\mu)}$$
$$=\frac{(c+r)\mu(\zeta_{N+1}-b\zeta_{N+2})(\sum_{n=1}^{N+1}\zeta_n+\mu\sum_{n=1}^{N}b^{1-n}+t)}{[R+(r_p+r)t](\zeta_{N+2}b^{N}+\mu)(\zeta_{N+1}b^{N-1}+\mu)} \tag{13}$$

$0<a\leq 1$ and the equation (5) imply that ζ_n is a decreasing function (or non-increasing function) in n. Meanwhile, $0<b\leq 1$, so $\zeta_{N+1}\geq\zeta_{N+2}\geq b\zeta_{N+2}$. Then, we have $A(N+1)\geq A(N)$. Therefore, for any integer N, this indicates that $A(N)$ is an increasing function (or non-decreasing function). Then the optimal N^* can be determined by

$$N^*=min\{N|A(N)\geq 1\} \tag{14}$$

Furthermore, if $A(N^*)>1$ for certain N^*, then the optimal N^* is unique. Because $A(N)$ is non-decreasing in N, there exists an integer N^*, thus

$$A(N)\geq 1\Leftrightarrow N\geq N^*$$

and $$A(N)<1\Leftrightarrow N<N^*$$

4. CONCLUSION

In this paper, we studied the optimal replacement model for degenerating failure systems. We obtained optimal replacement model N^* by minimizing the average cost rate $C(N)$. We show that uniqueness of the optimal replacement model N^*.

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