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OPTIMAL METHODS OF EIGHTH-ORDER FOR SOLVING NONLINEAR EQUATIONS

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ABSTRACT

In this article, methods of optimal eighth-order iterative methods are presented. The new methods are developed by combining special case of Al-Subaihi's method of fourth-order and adding Newton's method as a third step. Using the forward divided difference and three real-valued functions in the third step to increase the convergence order and reduce the number of function evaluations to be optimal. Numerical examples are provided to show the good performance of the new method.

Keywords: *Efficiency index; Nonlinear equations; Optimal eighth-order; Iterative methods; Convergence order.*

1. INTRODUCTION

Solving nonlinear equations is one and old important problems in science and engineering. In this article, we construct iterative methods to find a simple root of a nonlinear equation, $f\ (\gamma) = 0$, where $f: D \subset \mathrm{R} \to \mathrm{R}$ for an open interval D.

One the most famous method for solving nonlinear equation is Newton method (NM), [3].

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1}$$

which converges quadratically. In the last years, many researchers worked to develop iterative methods for solving nonlinear equations. For example using weighted functions in iterative methods to find a simple root has been presented, for example, [5, 10]. Three-step methods with eighth-order convergence developed, for example, [1, 4].

In this article, we present to a new method which use Al-Subaihi's method [2], in the first two steps of fourth order convergence (SM1).

$$\begin{aligned} w_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \left(\frac{\left(f(x_n)\right)^2}{f'(x_n)\left(f(x_n) - f(w_n)\right)}\right)\left(1 + \left(\frac{f(w_n)}{f(x_n)}\right)^2 + \left(\frac{f(w_n)}{f(x_n)}\right)^3\right). \end{aligned} \tag{2}$$

The efficiency index (EI) is defined by $E = p^{1/\delta}$, where p is the order of convergence and δ is the number of function or derivative evaluations per iteration [6]. The optimal order of any multipoint iterative method is given by $2^{\delta-1}$ [7]. So, the efficiency index of Newton method, (NM), (1), is $2^{1/2} \approx 1.4142$ and the efficiency index of the optimal fourth order Al-Subaihi's method, (SM1), (2), is $4^{1/3} \approx 1.5874$.

Theorem 1[9]: Let $\beta_1(x), \beta_2(x), \ldots, \beta_s(x)$ be iterative functions with the orders $p_1,\ p_2 \ldots, p_s,$ respectively. Then the composition of iterative functions $\beta\left(\beta_2\left(\ldots\left(\beta_s(x)\right)\ldots\right)\right)$, defines the iterative method of the order $p_1 p_2 \ldots p_s$.

Using theorem 1, adding the Newton's method as a third step as follows, (SM1).

$$\begin{aligned} w_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \left(\frac{\left(f(x_n)\right)^2}{f'(x_n)\left(f(x_n) - f(w_n)\right)}\right)\left(1 + \left(\frac{f(w_n)}{f(x_n)}\right)^2 + \left(\frac{f(w_n)}{f(x_n)}\right)^3\right), \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)}. \end{aligned} \tag{3}$$

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The method (3) has $\mathrm{EI} = 8^{1/5} \approx 1.5157$, and is not optimal. To reduce the number of functions evaluation of method (3) to four, by replacing $f'(z_n)$ to $\frac{f[x_n,z_n]f[w_n,z_n]}{f[x_n,w_n]}$ using the divided difference [11] to develop a family of optimal eight-order of convergence methods.

2. DEVELOPMENT OF METHOD AND CONVERGENCE ANALYSIS

The order of convergence of the proposed method (3) is eight but it is not optimal. To construct an optimal eighth-order method without using more evaluations, we present a new family of optimal eighth-order as follows, (SSM1).

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = x_n - \left(\frac{\left(f(x_n)\right)^2}{f'(x_n)\left(f(x_n) - f(w_n)\right)}\right) \cdot \left(1 + \left(\frac{f(w_n)}{f(x_n)}\right)^2 + \left(\frac{f(w_n)}{f(x_n)}\right)^3\right),$$
$$x_{n+1} = z_n - \frac{f(z_n)f[x_n,w_n]}{f[x_n,z_n]f[w_n,z_n]}\{H(s_1) \cdot K(s_2) \cdot B(s_3)\}, \tag{4}$$

where $\boldsymbol{H(s_1)}, \boldsymbol{K(s_2)}, \boldsymbol{B(s_3)}$, are three real-valued weight functions, and

$$s_1 = \frac{f(w_n)}{f(x_n)}, s_2 = \frac{f(z_n)}{f(w_n)}, s_3 = \frac{f(z_n)}{f(x_n)}. \tag{5}$$

The weight functions $H(s_1), K(s_2)$ and $B(s_3)$ have be chosen such that the order of convergence of method (4) comes at an optimal level of eight. In the next theorem, we prove that method (4) has an optimal eighth-order of convergence under conditions for the weighted functions.

Theorem 2: Let γ in D be a simple root of a sufficiently differentiable function f: D ⊆ R → R. If x_0 is sufficiently close to γ then the family of iterative methods (4) has an optimal eighth-order of convergence when

$$H(0)=1, H'(0) = H''(0) = 0\,, H'''(0) = -6,\ \left|H^{(4)}(0)\right| < \infty,$$
$$K(0) = 1, K'(0) = 0, |K''(0)| < \infty\,,$$
$$B(0) = 1, B'(0) = 1.$$

Proof: Let $e_n = x_n - \gamma$ be the error at the nth iteration, by Taylor expansion, we have

$$f(x) = f'(\gamma)[e + c_2e^2 + c_3e^3 + c_4e^4 + c_5e^5 + c_6e^6 + c_7e^7 + c_8e^8 + O(e^9)]. \tag{6}$$

Where, $c_k = \frac{f^{(k)}(\gamma)}{k!f'(\gamma)}, k = 2,3, \ldots$.

$$f'(x) = f'(\gamma)[1 + 2c_2e + 3c_3e^2 + 4c_4e^3 + 5c_5e^4 + 6c_6e^5 + 7c_7e^6 + 8c_8e^7 + 9c_9e^8 + O(e^9)]. \tag{7}$$

Dividing (6) by (7), gives us

$$\frac{f(x)}{f'(x)} = e - c_2e^2 + (2c_2^2 - 2c_3)e^3 + \ldots$$
$$+\begin{pmatrix} -64c_2^7 + 304c_2^5c_3 - 176c_2^4c_4 - 408c_2^3 + 92c_2^3c_5 + 348c_2^2c_3c_4 + 135c_2c_3^3 - \\ 44c_2^2c_6 - 118c_2c_3c_5 - 64c_2c_4^2 - 75c_3^2c_4 + 19c_2c_7 + 27c_3c_6 + 31c_4c_5 - 7c_8 \end{pmatrix} e^8 + O(e^9). \tag{8}$$

From (8), we have

$$w_n = \gamma + c_2e^2 + (-2c_2^2 + 2c_3)e^3 + \ldots$$
$$+\begin{pmatrix} 64c_2^7 - 304c_2^5c_3 + 176c_2^4c_4 + 408c_2^3c_3^2 - 92c_2^3c_5 - 48c_2^2c_3c_4 - 135c_2c_3^3 \\ +44c_2^2c_6 + 118c_2c_3c_5 + 64c_2c_4^2 + 75c_3^2c_4 - 19c_2c_7 - 7c_3c_6 - 31c_4c_5 + 7c_8 \end{pmatrix} e^8 + O(e^9). \tag{9}$$

Expanding $f(w_n)$ about γ to get

$$f(w_n) = f'(\gamma)[c_2e^2 + (-2c_2^2 + 2c_3)e^3 + \cdots$$
$$+\begin{pmatrix} 144c_2^7 - 552c_2^5c_3 + 297c_2^4c_4 + 582c_2^3c_3^2 - 134c_2^3c_5 - 455c_2^2c_3c_4 - 147c_2c_3^3 \\ +54c_2^2c_6 + 134c_2c_3c_5 + 73c_2c_4^2 + 75c_3^2c_4 - 19c_2c_7 - 27c_3c_6 - 31c_4c_5 + 7c_8 \end{pmatrix} e^8] + O(e^9) \tag{10}$$

$$z_n = \gamma + (2c_2^3 - c_2c_3)e^4 + (-9c_2^4 + 14c_2^2c_3 - 2c_2c_4 - 2c_3^2)e^5 + \ldots$$
$$+\begin{pmatrix} -204c_2^7 + 99c_2^5c_3 + 182c_2^4c_4 + 385c_2^3c_3^2 - 113c_2^3c_5 - 443c_2^2c_3c_4 - 179c_2c_3^3 \\ +35c_2^2c_6 + 116c_2c_3c_5 + 64c_2c_4^2 + 86c_3^2c_4 - 5c_2c_7 - 13c_3c_6 - 17c_4c_5 \end{pmatrix} e^8 + O(e^9). \tag{11}$$

From (11), we get

$$f(z_n) = f'(\gamma)[(2c_2^3 - c_2c_3)e^4 + (-2c_2c_4 - 2c_3^2 + 14c_3c_2^2 - 9c_2^4)e^5 + \ldots$$
$$+\begin{pmatrix} -200c_2^7 + 95c_2^5c_3 + 182c_2^4c_4 + 386c_2^3c_3^2 - 113c_2^3c_5 - 443c_2^2c_3c_4 - 179c_2c_3^3 \\ +35c_2^2c_6 + 116c_2c_3c_5 + 64c_2c_4^2 + 86c_3^2c_4 - 5c_2c_7 - 13c_3c_6 - 17c_4c_5 \end{pmatrix} e^8] + O(e^9). \tag{12}$$

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From (6), (10) and (12), it can be easily determine that

$$f[x_n,w_n] = f'(\gamma)[1 + c_2 e + (c_2^2 + c_3)e^2 + . . . + \begin{pmatrix} -313c_2^3c_3c_4 + 116c_2^2c_3c_5 + 56c_2c_3^2c_4 - 36c_2c_3c_6 - 40c_2c_4c_5 - 256c_2^6c_3 \\ + 184c_2^5c_4 + 264c_2^4c_3^2 - 93c_2^4c_5 - 42c_2^2c_3^3 + 45c_2^3c_6 + 69c_2^2c_4^2 - 20c_2^2c_7 - 8c_3^2c_5 \\ -8c_3c_4^2 + 8c_2c_8 + 8c_3c_7 + 8c_4c_6 + 4c_2^8 - 6c_3^4 + 4c_5^2 + c_9 \end{pmatrix} e^8] + O(e^9). \quad (13)$$

$$f[w_n,z_n] = f'(\gamma)[1 + c_2^2e^2 + (-2c_2^3 + 2c_2c_3)e^3 + (6c_2^4 - 7c_2^2c_3 + 3c_2c_4)e^4 + (30c_2^3c_3 - 12c_2^2c_4 - 4c_2c_3^2 + 4c_2c_5 - 17c_2^5)e^5] + O(e^6). \quad (14)$$

$$f[x_n,z_n] = f'(\gamma)[1 + c_2e + c_3e^2 + c_4e^3 + . . . +(-10c_2c_3c_4 - 73c_2^4c_3 + 23c_2^3c_4 44c_2^2c_3^2 - 3c_2^2c_5 + 20c_2^6 - 2c_3^3 + c_7)e^6] + O(e^7). \quad (15)$$

By expanding $H(s_1), K(s_2), B(s_3)$ using Taylor series expansion, we have

$$H(s_1) = H(0) + H'(0)s_1 + \frac{1}{2!}H''(0){s_1}^2 + \frac{1}{3!}H'''(0){s_1}^3 + . . . + O({s_1}^9), \quad (16)$$

$$K(s_2) = K(0) + K'(0)s_2 + \frac{1}{2!}K''(0){s_2}^2 + . . . + O({s_2}^9), \quad (17)$$

$$B(s_3) = B(0) + B'(0)s_3 + . . . + O({s_3}^9). \quad (18)$$

Finally, using (11) - (15) and the conditions

$H(0)=1,\ H'(0) = H''(0) = 0,\ H'''(0) = -6,\ |H^{(4)}(0)| < \infty,$
$K(0) = 1, K'(0) = 0,\ |K''(0)| < \infty,$
$B(0) = 1, B'(0) = 1.$

We obtain the error expression

$$e_{n+1} = \gamma + \Big(\frac{1}{24}H^{(4)}(0)c_2^5c_3 + 6c_2^5K''(0)c_3 - 3c_2^3K''(0)c_3^2 + \frac{1}{2}c_2c_3^3K''(0) - 4c_2^7K''(0) - 9c_2^5c_3 + 2c_2^4c_4 +$$

4c23c32+2c27−c22c3c4−112H40c27e8+0e9. (19)

Method1: choosing

$H(s_1) = 1 - {s_1}^3 + (t{s_1}^a)\ , a \geq 3\ ,\ a, t \in R$
$K(s_2) = 1 + ({s_2}^\mu), \mu > 1\ , \mu \in R$
$B(s_3) = 1 + s_3.$

Then the method can be written as (SSM1).

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = x_n - \Big(\frac{(f(x_n))^2}{f'(x_n)(f(x_n) - f(w_n))}\Big) \cdot \Big(1 + \big(\frac{f(w_n)}{f(x_n)}\big)^2 + \big(\frac{f(w_n)}{f(x_n)}\big)^3\Big),$$

$$x_{n+1} = z_n - \Big\{\{t\, {s_1}^a {s_2}^\mu ({s_2}^\mu + 1) + ({s_2}^\mu(-{s_1}^3 + 1)) - {s_1}^3\}\{s_3 + 1\}\Big\} \frac{f(z_n)f[x_n,w_n]}{f[x_n,z_n]f[w_n,z_n]}. \quad (20)$$

Method2: choosing

$H(s_1) = 1 - (s_1)^3,$
$K(s_2) = 1 + {s_2}^\mu, \mu > 1, \mu \in R$
$B(s_3) = 1 + \sin(s_3).$

Then, (SSM2) will be as

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = x_n - \Big(\frac{(f(x_n))^2}{f'(x_n)(f(x_n) - f(w_n))}\Big) \cdot \Big(1 + \big(\frac{f(w_n)}{f(x_n)}\big)^2 + \big(\frac{f(w_n)}{f(x_n)}\big)^3\Big),$$

$$x_{n+1} = z_n - \{\{1 - (s_1)^3\}\{1 + {s_1}^\mu\}\{1 + \sin(s_3)\}\} \frac{f(z_n)f[x_n,w_n]}{f[x_n,z_n]f[w_n,z_n]}. \quad (21)$$

2. NUMERICAL RESULTS

In this section, we present the numerical results obtained by employing the presented methods in (SSM1, SSM2) to solve some nonlinear equations, Newton's method (NM), (1), Al-Subaihi's method, (SM1), (2), and some optimal eighth-order methods, as well as the Sharma method **[11].**

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = w_n - \frac{f(x_n)}{f(x_n) - 2f(w_n)} \cdot \frac{f(w_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - \Big(1 + \frac{f(z_n)}{f(x_n)} + \big(\frac{f(z_n)}{f(x_n)}\big)^2\Big) \frac{f(z_n)f[x_n,w_n]}{f[x_n,z_n]f[w_n,z_n]},$$

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Wang and Liu method (BM8) [8].

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$z_n = w_n - \frac{f(x_n)}{f(x_n)-2f(w_n)} \cdot \frac{f(w_n)}{f'(x_n)},$$
$$x_{n+1} = z_n - \frac{f(z_n)}{2f[x_n,z_n]+f[w_n,z_n]-2f[x_n,w_n]+(w_n-z_n)f[w_n,x_n,x_n]},$$

Tables 2, show the numerical performances of the methods. The number of iterations (IT) required satisfying the stopping criterion$|x_n - \gamma| \leq 10^{-200}, |f(x_n)| \leq 10^{-200}$.

In addition, the computational order of convergence (COC) are also shown in Tables 2, the COC is defined by [12].

$$\rho = \frac{ln|(x_{n+1} - \alpha)/(x_n - \alpha)|}{ln|(x_n - \alpha)/(x_{n-1} - \alpha)|}.$$

Remark: The weighted function in (SSM1), $a = 4, \mu = 4, t = 1$ are chosen, and $\mu = 6$ for (SSM2).

3. CONCLUSION

In this work, we presented a new family of eighth–order methods based on a special case of Al-Subaihi's method. The method has been developed by replacing $f'(z)$ using the divided difference and equivalent construction of weighted functions to reduce the numbers of functions evaluation to four. Finally, other methods using numerical examples are compared to explain the convergence of the new methods.

Table-1: Numerical properties of the methods are checked through five test examples

Functions	**Roots**
$f_1(x) = xe^{x^2} - sin^2(x) + 3cos(x) + 5$	γ =-1.20764782713092
$f_2(x) = x^2 - (1-x)^{25}$	γ =0.143739259299754
$f_3(x) = xe^x + lo\,g(1 + x + x^4)$	γ =0.0
$f_4(x) = x^5 + x^4 + 4x^2 - 15$,	γ
$f_5(x) = x^3 + log(1 + x)$,	$\gamma = 0.0$

Table-2: Comparison of various iterative methods

Method	IT	$\lvert f(x_n)\rvert$	$\lvert x_n - \gamma\rvert$	COC
$f_1(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5, x_0 = -1.1$				
NM	8	1.17628e-200	5.79239e-202	2
SM1	4	4.60169e-205	2.26601e-206	4
BM8	3	4.43873e-494	2.18577e-495	8
SHM	-	fails	-	-
SSM1	3	1.11912e-348	5.51091e-350	8
SSM2	3	8.07803e-361	3.97787e-362	8
$f_2(x) = x^2 - (1-x)^{25}, x_0 = 0.35$				
NM	10	3.32259e-325	3.73026e-325	2
SM1	5	1.27907e-210	1.43602e-210	4
BM8	-	fails	-	-
SHM	4	0	0	6.4216
SSM1	4	3.48993e-663	3.91814e-663	8
SSM2	4	1.2724e-801	1.42852e-801	8
$f_3(x) = log\,(x^4 + x + 1) + xe^x, x_0 = 0.25$				
NM	8	6.57994e-276	3.28997e-276	2
SM1	4	2.30177e-252	1.15088e-252	4
BM8	3	4.79076e-442	2.39538e-442	8
SHM	3	9.33255e-441	4.66628e-441	8
SSM1	3	1.52326e-454	7.61632e-455	8
SSM2	3	8.45612e-457	4.22806e-457	8
$f_4(x) = x^5 + x^4 + 4x^2 - 15, x_0 = 1.6$				
NM	9	4.61266e-319	1.24511e-320	2
SM1	5	1.09129e-588	2.94576e-590	4
BM8	3	5.68394e-350	1.53429e-351	8
SHM	3	1.90069e-340	5.13061e-342	8
SSM1	3	3.87619e-317	1.04632e-318	8
SSM2	3	2.40633e-309	6.49549e-311	8

$f_5(x) = x^3 + log\ (1+x), x_0 = 0.25$				
NM	8	6.0375e-307	6.0375e-307	2
SM1	4	6.11663e-212	6.11663e-212	4
BM8	3	1.29369e-444	1.29369e-444	8
SHM	3	1.66015e-423	1.66015e-423	8
SSM1	3	5.98477e-428	5.98477e-428	8
SSM2	3	9.75445e-425	9.75445e-425	8

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