ON \( *\alpha \)-CLOSED SETS AND \( *g\alpha \)-CONTINUOUS MAPS

DR. M. SUDHA\(^1\) AND M. MUTHULAKSHMI\(^2\)

\(^1\)Associate Professor, Department of Mathematics, Sri Sarada College for Women (Autonomous), Salem-16, India.

\(^2\)Research Scholar, Department of Mathematics, Sri Sarada College for Women (Autonomous), Salem-16, India.

E-mail: sudhashlm05@yahoo.com\(^1\) and muthulakshminunusamy95@gmail.com\(^2\).

**ABSTRACT**

The purpose of this paper is to introduce the concept of \( *g\alpha \)-closed set. Besides studying some properties, the interrelations of the set introduced with other related sets are studied with necessary counter examples. Introducing \( *g\alpha \)-continuous map, some interesting properties and characterizations are discussed.

**Keywords:** \( \alpha \)-semi-open set, \( \dot{g}\alpha \)-closed set, \( *g\alpha \)-closed set, \( *g\alpha \)-continuous map, \( *g\alpha \)-closure, \( T_{\#rg} \)-space.

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1. **INTRODUCTION**

\( \alpha \)-open set was introduced and investigated by O. Njastad [12]. Cameron [4], Veerakumar. M. K. R. S. [17], Jafari. S, Noiri. T, Rajesh. N and Thivagar. M. L. [8] and Syed Ali Fathima and Mariasingam [14, 15, 16] introduced and investigated regular semi-open set, \( \dot{g}\)-closed set, \( *g\)-closed set, \( \#rg \)-closed set, \( \#rg \)-closure, \( \#rg \)-continuous map and \( T_{\#rg} \)-space. We introduce a new class of set called\( *g\alpha \)-closed set and study some of its properties. The concept of \( *g\alpha \)-continuous map is introduced and some basic properties are characterized.

2. **PRELIMINARIES**

Throughout this paper \( X, Y \) and \( Z \) denote the topological spaces \((X, T),(Y, S),(Z, R)\) respectively on which no separation axioms are assumed unless otherwise mentioned. For a subset \( A \) of a topological space \( X \), the closure of \( A \), interior of \( A \), semi-closure of \( A \), semipre-closure of \( A \), the complement of \( A \) and \( \#rg \)-closure of \( A \) are denoted by \( \text{cl}(A), \text{int}(A), \text{scl}(A), \text{spcl}(A), X\setminus A \) and \( \#rg\text{-cl}(A) \) respectively. We recall the following definitions and results.

**Definition 2.1**

A subset \( A \) of a space \( X \) is called

1. a semi-open set [10] if \( A \subseteq \text{clint}(A) \) and a semi-closed set if \( \text{intcl}(A) \subseteq A \).
2. a regular open set [13] if \( A = \text{intcl}(A) \) and a regular closed set if \( A = \text{clint}(A) \).
3. a \( \pi \)-open set [18] if \( A \) is a finite union of regular open sets.
4. a \( \alpha \)-open set [12] if \( A \subseteq \text{cl}(\text{int}(A)) \) and a \( \alpha \)-closed set if \( \text{cl}(\text{int}(A)) \subseteq A \).
5. a regular semi-open set [4] if there is a regular open \( U \) such that \( U \subseteq A \subseteq \text{cl}(U) \).
6. a semi-preopen set [1] if \( A \subseteq \text{cl}(\text{int}(A)) \) and a semi-preclosed set if \( \text{int}(\text{cl}(A)) \subseteq A \).

**Definition 2.2**

A subset \( A \) of a space \( X \) is called

1. a generalized closed set (briefly, \( g \)-closed) [9] if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \( X \).
2. a weakly generalized closed set (briefly, \( wg \)-closed) [11] if \( \text{cl}(\text{int}(A)) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \( X \).
3. an \( \alpha \)-generalized closed set (briefly, \( \pi \)-\( \alpha \)-closed) [7] if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \pi \)-open in \( X \).
4. an \( \#rg \)-closed set [17] if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is semi-open in \( X \).
5. an \( a\#g \)-closed set [8] if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \#g \)-open in \( X \).
6. a generalized semi-closed set (briefly, \( gs \)-closed) [2] if \( \text{sc}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \( X \).
7. a generalized semi-pre-closed set (briefly, \( gsp \)-closed) [6] if \( \text{spcl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \( X \).
Definition 2.3 [15]: For a subset $A$ of a space $X$, $\#_{rg}$-$cl(A) = \bigcap \{F: A \subseteq F, F$ is $\#_{rg}$ closed in $X\}$ is called the $\#_{rg}$-closure of $A$.

Definition 2.4: A map $f: (X, T) \rightarrow (Y, S)$ is called
(1) continuous [3] if $f^{-1}(V)$ is closed set in $X$ for every closed subset $V$ of $Y$.
(2) $\pi$-continuous [7] if $f^{-1}(V)$ is $\pi$-closed set in $X$ for every closed subset $V$ of $Y$.
(3) $\pi_{rg}$-continuous [7] if $f^{-1}(V)$ is $\pi_{rg}$-closed set in $X$ for every closed subset $V$ of $Y$.
(5) $gs$-continuous [5] if $f^{-1}(V)$ is $gs$-closed set in $X$ for every closed subset $V$ of $Y$.
(6) $gsp$-continuous [6] if $f^{-1}(V)$ is $gsp$-closed set in $X$ for every closed subset $V$ of $Y$.
(7) $\#_{rg}$-continuous [16] if $f^{-1}(V)$ is $\#_{rg}$-closed set in $X$ for every closed subset $V$ of $Y$.

Definition 2.5: A space $X$ is called
(1) $T_{1/2}$-space [9] if every $g$-closed set is closed.
(2) $T_{#_{rg}}$-space [14] if every $#_{rg}$-closed set in it is closed.

Definition 2.6 [15]: Let $(X, T)$ be a topological space and $T_{#_{rg}} = \{V \subseteq X : #_{rg}$-$cl(X \setminus V) = X \setminus V\}$.

3 *$g\alpha$-CLOSED SETS AND THEIR BASIC PROPERTIES

In this section we introduce and study *$g\alpha$-closed sets.

Definition 3.1: A subset $A$ of a space $X$ is called a $\alpha$-semi-open set if there is a $\alpha$-open $U$ such that $U \subseteq A \subseteq cl(U)$.

Definition 3.2: A subset $A$ of a space $X$ is called a $\hat{g}\alpha$-closed set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\hat{g}\alpha$-open in $X$.

Definition 3.3: A subset $A$ of a space $X$ is called a *$g\alpha$-closed set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g\alpha$-open in $X$.

The interrelations among the set introduced and other related sets are exhibited below:

Diagram-1

The converse of the above interrelations need not be true as seen from the following examples.

Example 3.1: Let $X = \{a, b, c, d\}$ and $T = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then
(i) The set $A = \{a, d\}$ is $*g\alpha$-closed but not closed in $X$.
(ii) The set $A = \{a, d\}$ is $*g\alpha$-closed but not $\pi$-closed in $X$.
(iii) The set $A = \{a, c\}$ is $\pi_{rg}$-closed but not $*g\alpha$-closed in $X$.
(iv) The set $A = \{c\}$ is $wg$-closed but not $*g\alpha$-closed in $X$.
(v) The set $A = \{a\}$ is $gs$-closed but not $*g\alpha$-closed in $X$.
(vi) The set $A = \{b\}$ is $gsp$-closed but not $*g\alpha$-closed in $X$.

Proposition 3.1: The union of two *$g\alpha$-closed subsets of $X$ is also a *$g\alpha$-closed subset of $X$.

Proof: Assume that $A$ and $B$ are *$g\alpha$-closed sets in $X$. Let $A \cup B \subseteq U$ and $U$ be $g\alpha$-open in $X$. Then $A \subseteq U$ and $B \subseteq U$ and $U$ is $g\alpha$-open in $X$. Since $A$ and $B$ are *$g\alpha$-closed sets in $X$, $cl(A) \subseteq U$ and $cl(B) \subseteq U$. Hence, $cl(A \cup B) = cl(A) \cup cl(B) \subseteq U$. Therefore, $A \cup B$ is a *$g\alpha$-closed set in $X$.

Proposition 3.2: The intersection of two *$g\alpha$-closed subsets of $X$ is also a *$g\alpha$-closed subset of $X$.

Proposition 3.3: Let $A$ be a *$g\alpha$-closed set in $X$. Then $cl(A) \setminus A$ does not contain any non-empty $g\alpha$-closed set in $X$.

Proof: Let $U$ be a non-empty $g\alpha$-closed subset of $cl(A) \setminus A$. Now $A \subseteq X \setminus U$ and $X \setminus U$ is $g\alpha$-open in $X$. Since $A$ is *$g\alpha$-closed, $cl(A) \subseteq X \setminus U$. Then $U \subseteq X \setminus cl(A)$. This is a contradiction since by assumption, $U \subseteq cl(A)$.

Proposition 3.4: Let $A$ be a *$g\alpha$-closed set in $X$. Then $A$ is closed iff $cl(A) \setminus A$ is $g\alpha$-closed.

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Proposition 3.5: For every point x of a space X, X\{x} is *gα-closed (or) ĝα-open.

Proposition 3.6: Let A be a *gα-closed subset of (X, T) such that A ⊆ B ⊆ cl(A). Then B is also a *gα-closed subset of (X, T).

**Proof:** Let B ⊆ U and U be ĝα-open in (X, T). Since A ⊆ B, A ⊆ U and U is a ĝα-open set in (X, T). Since A is *gα-closed, cl(A) ⊆ U. Therefore, cl(B) ⊆ U. Hence, B is *gα-closed.

The converse of the above Proposition need not be true as seen from the following Example.

**Example 3.2:** Let X = {a, b, c, d} and T = {ϕ, X, {a}, {b}, {a, b}, {a, b, c}}. Let A = {c, d} and B = {a, c, d}. Then A and B are *gα-closed sets in (X, T). But A = {c, d} ⊆ B = {a, c, d} ⊆ cl(A) = {a, c, d}.

**Proposition 3.7:** If a subset A of a topological space X is both ĝα-open and *gα-closed, then A is a closed set.

**Proposition 3.8:** Let A be ĝα-open and *gα-closed in X. Suppose that F is closed in X. Then A ∩ F is a *gα-closed set in X.

**Proof:** Let A be a ĝα-open and *gα-closed set in X and let F be a closed set in X. By Proposition 3.7, A is closed and so A ∩ F is closed. Since every closed set is *gα-closed, A ∩ F is a *gα-closed set in X. Hence, A ∩ F is a *gα-closed set in X.

**Remark 3.1:** If a subset A of a topological space X is
(i) open and g-closed, then A is *gα-closed.
(ii) π-open and πg-closed, then A is *gα-closed.
(iii) semi-open and ĝ-closed, then A is *gα-closed.
(iv) semi-open and ĝα-closed, then A is *gα-closed.
(v) open and wg-closed, then A is *gα-closed.

**Definition 3.4:** A space X is called a T_{gα}-space if every *gα-closed set in it is closed.

**Proposition 3.9:** Every T_{1/2}-space is T_{gα}-space.

**4*gα-CONTINUOUS MAPS**

In this section we introduce and study *gα-continuous maps.

**Definition 4.1:** For a subset A of a space X, *gα-cl(A) = ∩ {F : A ⊆ F, F is *gα-closed in X} is called the *gα-closure of A.

**Definition 4.2:** Let (X, T) be a topological space and T_{gα} = {V ⊆ X : *gα-cl(X\V) = X\V}.

**Definition 4.3:** A map f : (X, T)→ (Y, S) is called *gα-continuous if f^{-1}(V) is *gα-closed set in X for every closed subset V of Y.

**Remark 4.1:** Let A and B be subsets of (X, T). Then
1) *gα-cl(ϕ) = ϕ and *gα-cl(X) = X.
2) If A ⊆ B, then *gα-cl (A) ⊆ *gα-cl (B).
3) A ⊆ *gα-cl(A).
4) If A is *gα-closed, then *gα-cl(A) = A.

**Proposition 4.1:** Suppose T_{gα} is a topology. If A is *gα-closed in (X, T), then A is closed in (X, T_{gα}).

**Proposition 4.2:** A set A ⊆ X is *gα-open iff F ⊆ int(A) whenever F ⊆ A and F is ĝα-closed.

**Proof:** Let A be a ĝα-open set in X. Let F ⊆ A and F be ĝα-closed. Now X\A ⊆ X\F and X\F is ĝα-open. Since X\A is *gα-closed, cl(X\A) ⊆ X\F. Therefore, X\int(A) ⊆ X\F. Hence, F ⊆ int(A).

Suppose F ⊆ int(A) whenever F ⊆ A and F is ĝα-closed. Let X\F ⊆ U where U is ĝα-open. Then X\U ⊆ F ⊆ A, where X\U is ĝα-closed. By hypothesis, X\U ⊆ U. This implies X\int(A) ⊆ U and so cl(X\A) ⊆ U, which implies X\A is *gα-closed. Therefore, A is *gα-open.
Proposition 4.3: Let X be a space in which every singleton set is $\alpha$-closed. Then $f : (X, T) \rightarrow (Y, S)$ is $\alpha$-continuous if and only if $x \in \text{int}(f^{-1}(V))$ for every open subset $V$ of $Y$ which contains $f(x)$.

Proof: Suppose $f : (X, T) \rightarrow (Y, S)$ is $\alpha$-continuous. Fix $x \in X$ and an open set $V$ in $Y$ such that $f(x) \in V$. Then $f^{-1}(V)$ is $\alpha$-open. Since $x \in f^{-1}(V)$ and $\{x\}$ is $\alpha$-closed, $x \in \text{int}(f^{-1}(V))$ by Proposition 4.2.

Suppose $x \in \text{int}(f^{-1}(V))$ for every open subset $V$ of $Y$ which contains $f(x)$. Let $V$ be an open set in $Y$. Suppose $F \subseteq f^{-1}(V)$ and $F$ is $\alpha$-closed. Let $x \in F$. Then $f(x) \in V$ so that $x \in \text{int}(f^{-1}(V))$. This implies $F \subseteq \text{int}(f^{-1}(V))$. By Proposition 4.2, $f^{-1}(V)$ is $\alpha$-open. Hence, $f$ is $\alpha$-continuous.

Proposition 4.4: Let $f : (X, T) \rightarrow (Y, S)$ be a map. Let $(X, T)$ and $(Y, S)$ be any two spaces such that $T_{\text{top}}$ is a topology on $X$. Then the following statements are equivalent:

(i) For every subset $A$ of $X$, $\alpha \cdot \alpha$-cl$(A) \subseteq \alpha \cdot \text{cl}(f(A))$.

(ii) $f : (X, T_{\alpha}) \rightarrow (Y, S)$ is continuous.

Proof:

(i) $\Rightarrow$ (ii): Suppose (i) holds. Let $A$ be a closed set in $Y$. By (i), $\alpha \cdot \alpha$-cl$(f^{-1}(A)) \subseteq \alpha \cdot \text{cl}(f^{-1}(A)) \subseteq \alpha \cdot \text{cl}(A) = A$ and so $\alpha \cdot \alpha$-cl$(f^{-1}(A)) \subseteq f^{-1}(A)$. Also $f^{-1}(A) \subseteq \alpha \cdot \alpha$-cl$(f^{-1}(A))$. Hence, $\alpha \cdot \alpha$-cl$(f^{-1}(A)) = f^{-1}(A)$. This implies $(f^{-1}(A)) \subseteq T_{\alpha}$. Thus, $f^{-1}(A)$ is closed in $(X, T_{\alpha})$. Hence, $f$ is continuous.

(ii) $\Rightarrow$ (i): Suppose (ii) holds. Let $A$ be a subset of $X$. Then $\text{cl}(f(A))$ is closed in $Y$. Since $f : (X, T_{\alpha}) \rightarrow (Y, S)$ is continuous, $\text{cl}(f^{-1}(A))$ is closed in $(X, T_{\alpha})$. By Definition 4.2, $\alpha \cdot \alpha$-cl$(f^{-1}(A)) = f^{-1}(\text{cl}(f(A)))$. Now $A \subseteq f^{-1}(\text{cl}(f(A))) \subseteq f^{-1}(\text{cl}(f(A)))$, which implies $\alpha \cdot \alpha$-cl$(A) \subseteq \alpha \cdot \alpha$-cl$(f^{-1}(\text{cl}(f(A)))) = f^{-1}(\text{cl}(f(A)))$. Therefore, $f^{-1}(\alpha \cdot \alpha$-cl$(A)) \subseteq f^{-1}(\text{cl}(f(A)))$.

Proposition 4.5: The composition of two $\alpha$-continuous maps need not be a $\alpha$-continuous map in general as seen from the following Example.

Example 4.1: Let $X = Y = Z = \{a, b, c, d\}$, $T = \{\emptyset, X, \{c\}, \{a, b\}, \{a, b, c\}\}$, $S = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}$ and $R = \{\emptyset, Z, \{a\}, \{b\}, \{a, b\}\}$. Let $f : (X, T) \rightarrow (Y, S)$ be the identity map. Then clearly, $f$ is $\alpha$-continuous. Let $g : (Y, S) \rightarrow (Z, R)$ be defined by $g(a) = a$, $g(b) = b$, $g(c) = d$ and $g(d) = c$. Then clearly, $g$ is $\alpha$-continuous. But their composition $g \circ f : (X, T) \rightarrow (Z, R)$ is not $\alpha$-continuous, since for closed set $\{d\}$ in $(Z, R)$, $(g \circ f)^{-1}(\{d\}) = f^{-1}(g^{-1}(\{d\})) = f^{-1}(\{c\}) = \{c\}$, which is not $\alpha$-closed in $(X, T)$.

Proposition 4.6: Let $(X, T)$, $(Y, S)$ and $(Z, R)$ be topological spaces such that $S_{\text{top}} = S$. Let $f : (X, T) \rightarrow (Y, S)$ and $g : (Y, S) \rightarrow (Z, R)$ be $\alpha$-continuous. Then their composition $g \circ f : (X, T) \rightarrow (Z, R)$ is also $\alpha$-continuous.

The interrelations among the map introduced and other related maps are exhibited below:

Diagram-2

Proposition 4.7: Let $f : (X, T) \rightarrow (Y, S)$ be a map. Then the following are equivalent:

(i) $f$ is $\alpha$-continuous.

(ii) The inverse image of each open set in $(Y, S)$ is $\alpha$-open in $(X, T)$.

(iii) The inverse image of each closed set in $(Y, S)$ is $\alpha$-closed in $(X, T)$.

Proof:

(i) $\Rightarrow$ (ii): Suppose (i) holds. Let $V$ be open in $Y$. Then $Y \setminus V$ is closed in $Y$. Since $f$ is $\alpha$-continuous, $f^{-1}(Y \setminus V)$ is $\alpha$-closed in $X$. But $f^{-1}(Y \setminus V) = f^{-1}(V)^c$ which is $\alpha$-open in $X$. Therefore, $f^{-1}(V)$ is $\alpha$-open in $X$. Hence, the inverse image of each open set in $(Y, S)$ is $\alpha$-open in $(X, T)$.

(ii) $\Rightarrow$ (iii): Suppose (ii) holds. Let $V$ be a closed set in $Y$. Then $Y \setminus V$ is open in $Y$. Since the inverse image of each open set in $(Y, S)$ is $\alpha$-open in $(X, T)$, $f^{-1}(Y \setminus V)$ is $\alpha$-open. But $f^{-1}(Y \setminus V) = f^{-1}(V)^c$ which is $\alpha$-open. Therefore, $f^{-1}(V)$ is $\alpha$-closed in $X$. Hence, the inverse image of each closed set in $(Y, S)$ is $\alpha$-closed in $(X, T)$.

(iii) $\Rightarrow$ (i): Suppose (iii) holds. Let $V$ be a closed set in $Y$. Since, the inverse image of each closed set in $(Y, S)$ is $\alpha$-closed in $(X, T)$, $f^{-1}(V)$ is $\alpha$-closed in $X$. Hence, $f$ is $\alpha$-continuous.

Proposition 4.8: If a map $f : (X, T) \rightarrow (Y, S)$ is $\alpha$-continuous, then $\overline{\alpha \cdot \alpha \cdot \text{cl}(A)} \subseteq \text{cl}(f(A))$ for every subset $A$ of $X$. 

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5. REFERENCES
