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ON SEMIPRIME NEAR-RINGS WITH GENERALIZED DERIVATIONS

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ABSTRACT

In this paper, we study semiprime near-ring using a map $F: N \to N$, generalized derivation and a map $H: N \to N$, right centralizer, under some conditions. Inspired by the work of Ali et al [1] and Khan [7], we also study similar situations admitting generalized derivation on a semiprime near-ring.

Keywords: *Semiprime near-ring, distributive near-ring, derivation, generalized derivation, right centralizer, left Ideal.*

ASM Classification Number: *16Y30, 16W25.*

1. INTRODUCTION

The idea of generalized derivation was introduced in 1991 by Daif [3]. Ali *et al.* [1] proceed it further by taking some more identities admitting generalized derivation in prime and semiprime rings. The study of derivations of near-rings was initiated by H. E. Bell and G.Mason in 1987[2]. Generalized derivations have been primarily studied on operator algebras. Therefore any investigation from the algebraic point of view might be interesting. Recently, there has been a great deal of work concerning commutativity of prime and semiprime rings admitting suitably constrained derivations and generalized derivations [11]. In this paper, we have proved comparable results of [4, 5] for near-rings.

2. PRELIMINARIES

In this section, we collect all basic concepts and results in near-rings mostly from H. E. Heatherly [6], Mehsin Jabel Atteya, Dalal Jbrahee Rasen [8], Nurcan Argac [9], G. Pilz [10] and M. Samman, L. Outkhtite, A. Boua [11] which are required for our study.

Definition 2.1: [10: 7] A **left near-ring (resp. right near-ring)** is a set N together with two binary operations "$+$" and "**.**" such that

a) $(N, +)$ is a group (not necessarily abelian);
b) $(N,\cdot)$ is a semigroup and
c) $\forall\, n_1, n_2, n_3 \in N: n_1.(n_2 + n_3) = n_1 \cdot n_2 + n_1 \cdot n_3$ ("left distributive law")

Definition 2.2: [6: 63] A **distributive near-ring** is a near-ring satisfying both distributive laws.

Definition 2.3: [11: 407] An additive mapping $d: N \to N$ is said to be a **derivation** on N if $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$ or equivalently, $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$.

Definition 2.4: [11: 407] An additive mapping $F: N \to N$ is said to be a right (resp., left) generalized derivation with associated derivation d if $F(xy) = F(x)y + xd(y)$ (resp., $F(xy) = d(x)y + xF(y)$) for all $x, y \in N$, and F is said to be a **generalized derivation** with associated derivation d on N if it is both a right and left generalized derivation on N with associated derivation d.

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Definition 2.5: [8: 37] A near-ring N is said to be **semiprime** if $xNx = 0$ for $x \in N$ implies that $x = 0$.

Definition 2.6: [8: 38] For any $x, y \in N$, $[x, y] = xy - yx$ will denote the **commutator** and $(x \circ y) = xy + yx$ will denote the **anti-commutator**.

For any $x, y, z \in N$, the following identities hold:
i) $[x, yz] = y[x, z] + [x, y]z$
ii) $[xy, z] = x[y, z] + [x, z]y$

Definition 2.7: [11: 407] An additive mapping $F: N \to N$ satisfying $F(xy) = F(x)y$ for all $x, y \in N$ is called **left multiplier**.

Definition 2.8: [10: 15-16] A normal subgroup I of $(N, +)$ is called **ideal** of N ($I \trianglelefteq N$) if
α) $IN \subseteq I$
β) $\forall\, n, n' \in N \;\; \forall\, i \in I : n(n' + i) - nn' \in I$.

Normal subgroups R of $(N, +)$ with α) are called right ideals of N ($R \trianglelefteq_r N$), while normal subgroups L of $(N, +)$ with β) are said to be left ideals of N ($L \trianglelefteq_l N$).

Definition 2.9: [2: 31] The derivation D will be called **commuting** if $[x, D(x)] = 0$ for all $x \in N$.

3. GENERALIZED DERIVATIONS ON SEMIPRIME NEAR-RINGS

We need the following Lemmas to prove the main Theorems of this section.

Definition 3.1: An additive mapping $H: N \to N$ satisfying $H(xy) = xH(y)$ for all $x, y \in N$ is called right multiplier H is said to be a **multiplier** if it is both a right and left multiplier.

Lemma 3.2: Let N be a semiprime distributive near-ring. If F is a left generalized derivation associated with the map f, then f is a derivation, that is, $f(xz) = xf(z) + f(x)z$ for all $x, y \in N$.

Proof: Since F is a generalized derivation, we have
$$F(xy) = xF(y) + f(x)y \text{ for all } x, y \in N.$$

Replace x by xz,
$$F\big((xz)y\big) = xzF(y) + f(xz)y \text{ for all } x, y, z \in N \text{ and}$$
$$F\big(x(zy)\big) = xF(zy) + f(x)zy = xzF(y) + xf(z)y + f(x)zy \text{ for all } x, y, z \in N.$$

By the associativity, we get
$$f(xz)y = xf(z)y + f(x)zy.$$

Since N is distributive near-ring,
$$\big(f(xz) - xf(z) - f(x)z\big)y = 0 \Longrightarrow f(xz) - xf(z) - f(x)z = 0$$
$$\Longrightarrow f(xz) = xf(z) + f(x)z \text{ for all } x, y, z \in N.$$

Lemma 3.3: Let N be a semiprime near-ring and F be a left generalized derivation associated with f. If $F(xy) = 0$ holds for all $x, y \in N$, then $F = 0$.

Proof: By the hypothesis, we have
$$F(xy) = 0 \text{ for all } x, y \in N.$$

If we replace y by yz with $z \in N$, we get
$$F\big(x(yz)\big) = 0 \text{ for all } x, y, z \in N.$$

Since F is a left generalized derivation, we get
$$xF(yz) + f(x)yz = 0 \text{ for all } x, y, z \in N.$$

Using the hypothesis,
$$f(x)yz = 0 \text{ for all } x, y, z \in N$$
$$\Longrightarrow f(x)z = 0 \text{ for all } x, z \in N \Longrightarrow f = 0.$$

Thus $F(xy) = xF(y)$ for all $x, y \in N$. By the hypothesis,

$$xF(y) = 0 \text{ for all } x, y \in N \Longrightarrow F = 0.$$

Lemma 3.4: Let N be a semiprime near-ring and F be a left generalized derivation associated with f and H be a right multiplier. If the map $G: N \to N$ is defined as $G(x) = F(x) \mp H(x)$ for all $x \in N$. Then G is a left generalized derivation associated with f.

Proof: For all $x \in N$, by the hypothesis

$$\begin{aligned} G(xy) &= F(xy) \mp H(xy) = xF(y) + f(x)y \mp xH(y) \\ &= x\left(F(y) \mp H(y)\right) + f(x)y \\ &= x\,G(y) + f(x)y \text{ for all } x, y \in N \end{aligned}$$

Hence G is a left generalized derivation associated with f.

Theorem 3.5: Let N be a semiprime near-ring and $F: N \to N$ be a left generalized derivation associated with f and $H: N \to N$ be a right multiplier. If $F(xy) \mp H(xy) = 0$ holds for all $x, y \in N$, then $f = 0$. Moreover, $F(xy) = xF(y)$ holds for all $x, y \in N$ and for all $x, y \in N$ and for all $F = \pm H$.

Proof: By the hypothesis, we have

$$F(xy) - H(xy) = 0 \text{ for all } x, y \in N$$

$$G(xy) = 0 \text{ for all } x, y \in N, \text{ by Lemma 3.3}$$

Where Using Lemma 3.3, $G = 0$

Thus $$F = H \qquad (1)$$

Using the definition of F and (1) in the hypothesis, we get

$$\begin{aligned} 0 = F(xy) - H(xy) &= xF(y) + f(x)y - xH(y) \\ &= f(x)y \text{ for all } x, y \in N \end{aligned}$$

We obtain $f = 0$. Thus $F(xy) = xF(y)$ for all $x, y \in N$.

By using the similar argument in the case of $F(xy) + H(xy) = 0$ for all $x, y \in N$, we get $F = -H$ and $f = 0$.

Hence $F = \pm H$.

Theorem 3.6: Let N be a semiprime near-ring, $F: N \to N$ be a left generalized derivation associated with f and $H: N \to N$ be a right multiplier. If $F(x)F(y) \mp H(xy) = 0$ holds for all $x, y \in N$, then $f = 0$. Moreover, $F(xy) = xF(y)$ for all $x, y \in N$.

Proof: By the hypothesis, we have

$$F(x)F(y) - H(xy) = 0 \text{ for all } x, y \in N \qquad (2)$$

Replacing x by xz with $z \in N$

$$F(xz)F(y) - H\big((xz)y\big) = 0 \text{ for all } x, y, z \in N$$

Since F is a left generalized derivation, we have

$$x\big(F(z)\,F(y) - H(zy)\big) + f(x)z\,F(y) = 0$$

Using equation (2), we get

$$f(x)z\,F(y) = 0 \text{ for all } x, y, z \in N \qquad (3)$$

Replacing y by uy with $u \in N$ in (3) and using (3), from the definition of F, we obtain

$$f(x)z\,f(u)y = 0 \text{ for all } u, x, y, z \in N$$

In the last equation replacing z by $zr, r \in N$ and using N is a semiprime near-ring, we get $f = 0$.

Thus $F(xy) = xF(y)$ for all $x, y \in N$.

By the similar argument in the case of $F(x)F(y) + H(xy) = 0$ for all $x, y \in N$, we get $f = 0$. Thus $F(xy) = xF(y)$ for all $x, y \in N$.

Theorem 3.7: Let N be a semiprime distributive near-ring and I a nonzero left ideal of N. If $F: N \to N$ is a generalized derivation associated with a map $f: N \to N$ such that $F[x,y] \pm xy = 0$ for all $x, y \in I$. Then $I[x, f(x)] = 0$ for all $y \in I$.

Proof: Assume that

$$F[x,y] \pm xy = 0 \text{ for all } x, y \in I \tag{4}$$

Replace x by yx and using equation (4), we obtain $F[yx,y] \pm (yx)y = 0$ implies that

$$f(y)[x,y] = 0 \text{ for all } x, y \in I \tag{5}$$

Substituting $yf(x)$ for y,

$$f(y)[x, yf(x)] = 0 \text{ for all } x, y \in I$$
$$f(x)[x, yf(x)] = 0 \text{ for all } x, y \in I$$
$$f(x)y[x, f(x)] = 0 \text{ for all } x, y \in I \tag{6}$$

On replacing y by xy, we obtain

$$f(x)xy[x, f(x)] = 0 \text{ for all } x, y \in I. \tag{7}$$

Left multiply (6) by x and subtract (7),

$$[x, f(x)]y[x, f(x)] = 0 \text{ for all } x, y \in I$$

Replacing y by ry,

$$[x, f(x)]ry[x, f(x)] = 0 \text{ for all } x, y \in I$$

Left multiply by y,

$$y[x, f(x)]Ny[x, f(x)] = 0 \text{ for all } x, y \in I$$

Since the semiprime of N yields that,

$$y[x, f(x)] = 0 \text{ for all } x, y \in I$$

Therefore, $I[x, f(x)] = 0$ for all $x \in I$.

Theorem 3.8: Let N be a semiprime distributive near-ring and I a nonzero left ideal of N. If $F: N \to N$ is a generalized derivation associated with a map $f: N \to N$ such that $F[x,y] \pm yx = 0$ for all $x, y \in I$. Then $I[x, f(x)] = 0$ for all $y \in I$.

Proof: Given that

$$F[x,y] \pm yx = 0 \text{ for all } x, y \in I \tag{8}$$

On replacing x by yx and using equation (8), we obtain $F[yx,y] \pm y(yx) = 0$ implies that

$$f(y)[x,y] = 0 \text{ for all } x, y \in I$$

Further, proceed as Theorem 3.7 after the equation (5). Hence $I[x, f(x)] = 0$ for all $y \in I$.

Theorem 3.9: Let N be a semiprime distributive near-ring and I a nonzero left ideal of N. If $F: N \to N$ is a generalized derivation associated with a map $f: N \to N$ such that $F(x \circ y) \pm xy = 0$ for all $x, y \in I$. Then $I[x, f(x)] = 0$ for all $y \in I$.

Proof: We assume that

$$F(x \circ y) \pm xy = 0 \text{ for all } x, y \in I \tag{9}$$

Replace x by yx and using (9),

$$F(yx \circ y) \pm (yx)y = 0 \text{ for all } x, y \in I$$
$$\Rightarrow f(y)(x \circ y) = 0 \text{ for all } x, y \in I. \tag{10}$$

Replacing y by $yf(x)$, we have

$$f(y)\big(x \circ yf(x)\big) = 0 \text{ for all } x, y \in I$$

Using equation (10),

$$f(x)y[x, f(x)] = 0 \text{ for all } x, y \in I. \tag{11}$$

On replacing y by xy, we obtain

$$f(x)xy[x,f(x)] = 0 \text{ for all } x,y \in I. \tag{12}$$

Left multiply (11) by x and subtract (12) we get

$$[x,f(x)]y[x,f(x)] = 0 \text{ for all } x,y \in I$$

Replacing y by ry,

$$[x,f(x)]ry[x,f(x)] = 0 \text{ for all } x,y \in I$$

Left multiply by y,

$$y[x,f(x)]Ny[x,f(x)] = 0 \text{ for all } x,y \in I$$

Since the semiprime of N yields that,

$$y[x,f(x)] = 0 \text{ for all } x,y \in I$$

Therefore, $I[x,f(x)] = 0$ for all $x \in I$.

Theorem 3.10: Let N be a semiprime distributive near-ring and I a nonzero left ideal of N. If $F: N \to N$ is a generalized derivation associated with a map $f: N \to N$ such that $F(x \circ y) \pm yx = 0$ for all $x,y \in I$. Then $I[x,f(x)] = 0$ for all $y \in I$.

Proof: Given that

$$F(x \circ y) \pm yx = 0 \text{ for all } x,y \in I \tag{13}$$

On replacing x by yx and using equation (13), we obtain $F(yx \circ y) \pm y(yx) = 0$ implies that $f(y)[x,y] = 0$ for all $x,y \in I$

Further, proceed as Theorem 3.9 after the equation (10). Hence $I[x,f(x)] = 0$ for all $y \in I$.

Notation: Denote $\big[[f(y),y],f(y)\big]$ by $[f(y),y]_2$.

Theorem 3.11: Let N be a semiprime distributive near-ring and I a nonzero left ideal of N. If $F: N \to N$ is a generalized derivation associated with a map $f: N \to N$ such that $F(x)f(y) \pm xy = 0$ for all $x,y \in I$. Then $I[f(y),y]_2 = 0$ for all $y \in I$.

Proof: We assume that

$$F(x)f(y) \pm xy = 0 \text{ for all } x,y \in I \tag{14}$$

Replace x by yx,

$$F(yx)f(y) \pm (yx)y = 0 \text{ for all } x,y \in I$$

$$\Longrightarrow f(y)xf(y) = 0 \text{ for all } x,y \in I. \tag{15}$$

Substituting $x[f(y),y]$ for x in (14), we get

$$f(y)x[f(y),y]f(y) = 0. \tag{16}$$

Right multiply (15) by $[f(y),y]$ and subtract from (16), we

$$f(y)x\big[[f(y),y],f(y)\big] = 0 \text{ for all } x,y \in I. \tag{17}$$

Replace x by yx,

$$f(y)yx\big[[f(y),y],f(y)\big] = 0 \text{ for all } x,y \in I$$

Since N is distributive near-ring. Left multiply (17) by y and subtract (16),

$$[f(y),y]x\big[[f(y),y],f(y)\big] = 0.$$

Left multiply by $f(y)$,

$$f(y)[f(y),y]x\big[[f(y),y],f(y)\big] = 0 \tag{18}$$

Left multiply by $[f(y),y]$ in (16) and subtract (17), we get

$$\big[[f(y),y],f(y)\big]x\big[[f(y),y],f(y)\big] = 0$$

Replacing x by rx,

$$\big[[f(y),y],f(y)\big]rx\big[[f(y),y],f(y)\big] = 0 \text{ for all } x,y \in I \text{ and } r \in N.$$

Left multiply by x,

$$x\big[[f(y),y],f(y)\big]Nx\big[[f(y),y],f(y)\big]=0 \text{ for all } x,y\in I \text{ and } r\in N.$$

Since N is semiprime,

$$x\big[[f(y),y],f(y)\big]=0 \text{ for all } x,y\in I.$$

Then $$I\big[[f(y),y],f(y)\big]=0 \text{ for all } y\in I.$$

Therefore, $I[f(y),y]_2=0$ for all $y\in I$.

Theorem 3.12: Let N be a semiprime distributive near-ring and I a nonzero left ideal of N. If $F: N\to N$ is a generalized derivation associated with a map $f: N\to N$ such that $F(x)f(y)\pm yx=0$ for all $x,y\in I$. Then $I[f(y),y]_2=0$ for all $y\in I$.

Proof: Given that

$$F(x)f(y)\pm yx=0 \text{ for all } x,y\in I \tag{19}$$

On replacing x by yx and using equation (18), we obtain $F(yx)f(y)\pm y(yx)=0$ implies that

$$f(y)xf(y)=0 \text{ for all } x,y\in I.$$

Further, proceed as Theorem 3.11 after the equation (15). Hence we get $I[f(y),y]_2=0$ for all $y\in I$.

Corollary 3.13: Let N be a semiprime near-ring and I a nonzero left ideal of N. If $F: N\to N$ is a generalized derivation associated with a map $f: N\to N$. If I satisfies any one of the identities $F(x)f(y)\pm xy=0$ and $F(x)f(y)\pm yx=0$ for all $x,y\in I$, then f is commuting on I.

Proof: Using equation (15) in Theorem 3.11, we have

$$f(y)xf(y)=0 \text{ for all } x,y\in I.$$

Therefore, $$[f(y),y]x[f(y),y]=0 \text{ for all } x,y\in I.$$

$$\Rightarrow [f(y),y]I[f(y),y]=0 \text{ for all } y\in I.$$

Since I is an ideal of a semiprime near-ring, $[f(y),y]=0$ for all $y\in I$.

Thus, f is commuting on I.

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