

## FACE MAGIC $\mathbb{A}$ - LABELING OF GRAPHS

A. MEENA KUMARI<sup>1</sup> AND S. AROCKIARAJ<sup>2</sup>

<sup>1</sup>Research Scholar, Department of Mathematics,  
Bharathiar University, Coimbatore and Faculty in department of Mathematics,  
Mepco Schlenk Engineering College, Sivakasi - 626 005, Tamil Nadu, India.

<sup>2</sup>Department of Mathematics,  
Government Arts and Science College, Sivakasi – 626 124, Tamil Nadu, India.

(Received On: 18-08-18; Revised & Accepted On: 11-10-18)

---

### ABSTRACT

*In this paper, we introduce the Face magic  $\mathbb{A}$ -labeling of graphs, where the labels are the elements of an arbitrary abelian group  $\mathbb{A}$ , of order  $p + q$  and the weight of each face is the product of the labels of all vertices and edges in the boundary of that face, an assignment of labels to the vertices and edges which induces an assignment of labels to the faces of a graph such that the weight of each face is constant. We discuss the Face magic  $\mathbb{A}$ -labeling of some graphs and the relationship between face, vertex and edge magic constants.*

**Key Words.** Labeling, Face magic  $\mathbb{A}$ -labeling of graphs, Face magic graph over  $\mathbb{A}$ .

AMS(2010): 05C76, 05C78.

---

### 1 INTRODUCTION

By a graph, we mean a finite, connected, undirected planar graph without loops and multiple edges. By a planar graph, we mean that it can be drawn in a plane such that no two edges intersect. The group  $\mathbb{A}$  is always a finite abelian group. Let us consider  $\mathbb{A} = \langle a \rangle$  where  $a^n = 1_{\mathbb{A}}$  or simply 1, the identity element of  $\mathbb{A}$ .

A graph  $G$  is magic if the edges of  $G$  can be labeled by the numbers  $1, 2, 3, \dots, |E(G)|$  so that the sum of the labels of all the edges incident with any vertex is the same [1]. A group distance magic labeling of a graph  $G(V, E)$  with  $|V| = n$  is an injection from  $V$  to an abelian group  $\Gamma$  of order  $n$  such that the sum of labels of all neighbors of every vertex  $x$  in  $V$  is equal to the same element  $\mu$  in  $\Gamma$ . A graph that admits a distance magic labeling is called a distance magic graph [4].

There are some graphs with an embedding in the plane such that all vertices belong to the unbounded face of the embedding. Such planar graph is an outer planar graph.

Path on  $n$  vertices is denoted by  $P_n$  and a cycle on  $n$  vertices is denoted by  $C_n$ . Duplication of a vertex  $v$  of a graph  $G$  by a vertex  $v'$  produces a new graph  $G'$  with  $N(v') = N(v)$  where  $N(v)$  is the set of all neighbors of  $v$ . Duplication of an edge  $e = uv$  of a graph  $G$  by an edge  $e' = u'v'$  produces a new graph  $G'$  where  $N(u)$  and  $N(v)$ . Duplication of an edge  $e = uv$  by a vertex  $v'$  in a graph  $G$  produces a new graph  $G'$  where  $V(G') = V(G) \cup \{v'\}$ ,  $E(G') = E(G) \cup \{uv', v'v\}$ . Duplication of a vertex  $v$  of a graph  $G$  by an edge  $e' = u'v'$  produces a new graph  $G'$  where  $N(u') = \{v, v'\}$  and  $N(v') = \{v, u'\}$ .

The  $(m, n)$  – tadpole graph  $G$  is a graph obtained by identifying a vertex of a cycle of length  $m$  and a vertex of a path of length  $n$ . A Barbell graph  $D_{a,b,c}$  is a graph obtained by connecting two disjoint cycles on  $a$  and  $b$  number of vertices respectively by a path of  $c$  number of vertices.

---

**Corresponding Author: A. Meena Kumari<sup>1</sup>,**

<sup>1</sup>Research Scholar, Department of Mathematics,  
Bharathiar University, Coimbatore and Faculty in department of Mathematics,  
Mepco Schlenk Engineering College, Sivakasi - 626 005, Tamil Nadu, India.

Let  $\lambda$  be a labeling over a group  $\mathbb{A}$ . D.Combe *et al.* [5] define the  $\lambda$ -weight,  $\omega = \omega_\lambda$  of the graph as follows

(i) for every vertex  $x \in V$ , the weight is the product of the label of  $x$  and the labels of the edges incident with  $x$ , that is  $\omega(x) = \lambda(x) \times \prod_{y \in V: xy} \lambda(xy)$ ; (ii) for every edge  $xy \in E$ , the weight is the product of the labels of  $xy$  and the labels of  $x$  and  $y$ , that is  $\omega(xy) = \lambda(x) \times \lambda(xy) \times \lambda(y)$ .

The labeling  $\lambda$  is a vertex magic  $\mathbb{A}$ -labeling of  $G$  if there is an element  $h = h(\lambda)$  of  $\mathbb{A}$  such that for every  $x \in V$ ,  $\omega(x) = h$ . The element  $h$  is the vertex constant. The labeling  $\lambda$  is an edge magic  $\mathbb{A}$ -labeling of  $G$  if there is an element  $k = k(\lambda)$  of  $\mathbb{A}$  such that for every  $xy \in E$ ,  $\omega(xy) = k$ . The element  $k$  is the edge constant.

The labeling  $\lambda$  is a totally magic  $\mathbb{A}$ -labeling of  $G$  if it is both a vertex magic  $\mathbb{A}$ -labeling and an edge magic  $\mathbb{A}$ -labeling. A graph  $G$  is said to be vertex  $\mathbb{A}$ -magic if there exists a vertex magic labeling over  $\mathbb{A}$ . Similarly  $G$  is edge  $\mathbb{A}$ -magic if there exists an edge magic labeling over  $\mathbb{A}$ ; and  $G$  is totally  $\mathbb{A}$ -magic if there exists a totally magic labeling over  $\mathbb{A}$  [5].

Two  $\mathbb{A}$ -labelings  $\lambda_1, \lambda_2$  of  $G$  are translation equivalent, denoted by  $\lambda_1 \sim_T \lambda_2$  if there exists  $a \in \mathbb{A}$  such that  $\forall z \in V \cup E, \lambda_1(z) = a\lambda_2(z)$ . The labeling  $\lambda_1$  is the translation of  $\lambda_2$  by  $a$  and is denoted by  $\lambda_1 = a\lambda_2$ . Clearly, translation equivalence is an equivalence relation on labelings [5].

Motivated by these works, we introduce face magic  $\mathbb{A}$ -labeling as follows:

The labeling  $\psi$  is a face magic  $\mathbb{A}$ -labeling of  $G$  if there is an element  $l = l(\psi)$  of  $\mathbb{A}$  such that for every face  $f \in F$ ,  $\omega(f) = l$ . We define for every  $f \in F$ , the weight is the product of the labels of all vertices and edges in the boundary of the face  $f$ , that is  $\omega(f) = \prod_{x \in \text{inf} f} \psi(x) \times \prod_{xy \in \text{inf} f} \psi(xy)$

The element  $l$  is the face constant. A graph  $G$  is said to be face magic graph over  $\mathbb{A}$  if there exists a face magic labeling over  $\mathbb{A}$ .

In this paper, We introduce face magic  $\mathbb{A}$ -labeling of  $G$  and derive the relation between vertex, edge and face constants of a graph  $G$ . Also we discuss the face magic  $\mathbb{A}$ -labeling of some graphs obtained by duplication of any one of a vertex or an edge of a graph.

## 2 MAIN RESULTS

**Proposition 2.1:** Let  $G$  be a graph having  $\mathbb{A}$ -labeling which is both edge magic  $\mathbb{A}$ -labeling with edge constant  $k$  and a face magic  $\mathbb{A}$ -labeling with face constant  $l$ . Then the product of the vertex labels of any  $s$ -sided face is  $k^s l^{-1}$ .

**Proof:** Let  $f$  be a  $s$ -sided face in  $G$  which admits a  $\mathbb{A}$ -labeling  $\psi$ . Product of all edge constants of all edges in  $f$  is

$$\begin{aligned} &= k^s = \left[ \prod_{x \in f} \psi(x) \right]^2 \cdot \left[ \prod_{xy \in f} \psi(xy) \right] \\ &= \left[ \prod_{x \in f} \psi(x) \right] \cdot \left[ \prod_{xy \in f} \psi(xy) \right] \cdot \left[ \prod_{x \in f} \psi(x) \right] \\ &= l \cdot \prod_{x \in f} \psi(x) \end{aligned}$$

Hence the result follows.

**Proposition 2.2:** Let  $G$  be a graph having  $\mathbb{A}$ -labeling which is both vertex magic  $\mathbb{A}$ -labeling with vertex constant  $h$  and a face magic  $\mathbb{A}$ -labeling with face constant  $l$ . Then the product of all the edge labels incident with every vertex of any  $s$ -sided face is  $h^s l^{-1}$ .

**Proof:** Let  $f$  be a  $s$ -sided face in  $G$  which admits a  $\mathbb{A}$ -labeling  $\psi$ . Product of all vertex constants of all vertices in  $s$ -sided face  $f$  is

$$\begin{aligned} &= h^s = \prod_{x \in f} \omega(x) \\ &= \prod_{x \in f} \left\{ \psi(x) \left[ \prod_{xy \in f} \psi(xy) \right]^2 \left[ \prod_{xz \in f} \psi(xz) \right]^{d_x - 2} \right\} \\ &= \prod_{x \in f} \left\{ \psi(x) \prod_{xy \in f} \psi(xy) \right\} \cdot \prod_{x \in f} \left\{ \prod_{xy \in f} \psi(xy) \prod_{xz \in f} \psi(xz) \right\} \\ &= l \cdot \prod_{x \in f} \left( \prod_{xy \in E(G)} \psi(xy) \right) \end{aligned}$$

Thus the product of all the edge labels incident with every vertex of the  $s$ -sided face is  $h^s l^{-1}$ .

**Proposition 2.3:** Let  $G$  be a planar graph in which all its edges are in the boundary of its exterior face. If  $G$  has a face magic  $\mathbb{A}$ -labeling with face constant  $l$ , then

$$l = \begin{cases} 1_{\mathbb{A}}, & \text{if } p + q \text{ is odd} \\ a^{(p+q)/2}, & \text{if } p + q \text{ is even} \end{cases}$$

**Proof:** Let  $G$  has a face magic  $\mathbb{A}$ -labeling  $\psi$  with face constant  $l$ . Let  $f_0$  be the face in  $G$  which covers all vertices and edges of  $G$ . Then

$$\begin{aligned} l &= \prod_{x \in f_0} \psi(x) \cdot \prod_{xy \in f_0} \psi(xy) \\ &= 1 \cdot a \cdot a^2 \dots a^{p+q-1} \\ &= a^k, \text{ where } k = \frac{(p+q)(p+q-1)}{2} \end{aligned}$$

when  $p + q$  is odd,  $a^k = (a^{p+q})^{\frac{p+q-1}{2}} = 1_{\mathbb{A}}$ . Also,

$$\left(a^{\frac{p+q}{2}}\right)^r = \begin{cases} a^{\frac{p+q}{2}r}, & \text{if } p+q \text{ is even and } r \text{ is odd} \\ 1_{\mathbb{A}}, & \text{if } p+q \text{ is even and } r \text{ is even} \end{cases}$$

So,  $a^k = a^{\frac{p+q}{2}}$  when  $p + q$  is even. Hence the result follows.

Consider the outer planar graph  $G$  isomorphic to  $(m, n)$ -Tadpole graph.

Let  $\{u_1, u_2, \dots, u_m\}$  be the vertices on cycle and  $\{v_0, v_1, v_2, \dots, v_n\}$  be the vertices on path where  $u_1 = v_0$ . In  $G$ ,  $|V(G)| = m + n$ ,  $|E(G)| = m + n$  and  $|V(G)| + |E(G)| = 2m + 2n$ ,  $F(G) = \{f_0, f_1\}$  where  $f_0$  is the outer face having all vertices and edges and  $f_1$  is the cycle  $C_m$ .

Consider any cyclic group  $\langle a \rangle$  of order  $2m + 2n$ . Then  $\langle a \rangle = \{1, a, a^2, \dots, a^{2m+2n-1}\}$  where  $a^{2m+2n} = 1$ . The labels of the path in the sequence vertex - edge - vertex - edge - vertex... are given as  $a^{2m+n}, a^2, a^{2m+n+1}, a^3, a^{2m+n+2}, a^4, \dots, a^{2m+2n-1}$ . The labels for the cycle  $C_n$  starting with identified vertex in the sequence vertex - edge - vertex - edge - vertex... are given as  $1, a^{n+m}, a^{n+1}, a^{n+m+1}, a^{n+2}, \dots, a^{n+m-1}, a^{2m+n-1}$ . The above labels are the labels in the face  $f_1$ . Therefore, Weight of the face  $f_1$  is

$$\begin{aligned} &= 1 \cdot a^{n+m} \cdot a^{n+1} \cdot a^{n+m+1} \cdot a^{n+2} \dots a^{n+m-1} \cdot a^{2m+n-1} \\ &= a^{m(n+m)} \cdot a^{2(1+2+\dots+(m-1))} \cdot a^{n(m-1)} \\ &= a^{m(2n+2m)-(m+n)} \\ &= a^{(m+n)}, \text{ where } p + q = 2n + 2m \text{ is even and } a^{2n+2m} = 1. \end{aligned}$$

Now, Weight of the outer face  $f_0$  is the product of all elements in the cyclic group which is

$$\begin{aligned} &= 1 \cdot a \cdot a^2 \cdot a^3 \dots a^{2m+2n-1} \\ &= a^{(2m+2n)(2m+2n-1)/2} \\ &= a^{m+n}, \text{ where } p + q = 2n + 2m \text{ is even and } a^{2n+2m} = 1. \end{aligned}$$

This example shows that  $(m, n)$ -Tadpole graph is a face magic graph over  $\mathbb{A}$  with face constant  $a^{m+n}$  and illustrates proposition 2.3 when  $p + q$  is even.

Consider the outer planar graph  $G$  isomorphic to a Barbell graph  $D_{n,n,2}$ .

Let  $V(G) = V(C_1) \cup V(C_2) = \{u_1, u_2, \dots, u_n\} \cup \{v_1, v_2, \dots, v_n\}$ ,  $E(G) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_n u_1\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1\} \cup \{u_n v_1\}$  and  $F(G) = \{f_0, f_1, f_2\}$  where  $f_1, f_2$  are cycles  $C_1, C_2$  respectively and  $f_0$  is the face having all the vertices and edges of  $G$ . In  $G$ ,  $p = |V(G)| = 2n$ ,  $q = |E(G)| = 2n + 1$ ,  $\deg(f_i) = n$ ,  $i = 1, 2$ ,  $\deg(f_0) = 4n + 1$ . So,  $p + q = 4n + 1$  which is odd.

Consider any cyclic group  $\langle a \rangle$  of order  $4n + 1$ . Then  $\langle a \rangle = \{1, a, a^2, \dots, a^{4n}\}$  where  $a^{4n+1} = 1$ . Label the bridge of  $G$  by the identity element 1 of  $\langle a \rangle$ . Label the first cycle  $C_1$  starting with vertex  $u_1$  and end with the edge  $u_n u_1$  in the sequence vertex - edge - vertex - edge - vertex...edge as  $a, a^4, a^5 a^8, a^9, a^{12}, \dots, a^{4n-3}, a^{4n}$ . Label the second cycle  $C_2$  starting with vertex  $v_1$  and end with the edge  $v_n v_1$  in the sequence vertex - edge - vertex - edge - vertex...edge as  $a^2, a^3 a^6, a^7, a^{10}, \dots, a^{4n-6}, a^{4n-2}, a^{4n-1}$ .

Now, the weight of the face  $f_1$  is the product of all the labels of vertices and edges in  $C_1$  which is

$$\begin{aligned} &= [a \cdot a^5 \cdot a^9 \dots a^{4n-3}] \cdot [a^4 \cdot a^8 \cdot a^{12} \dots a^{4n}] \\ &= [a^{4(1+2+\dots+n)-3n}] \cdot [a^{4(1+2+\dots+n)}] \\ &= a^{4n^2+n} = a^{n(4n+1)} = 1_{\mathbb{A}}. \end{aligned}$$

Also the weight of the face  $f_2$  is the product of all the labels of vertices and edges in  $C_2$  which is

$$\begin{aligned} &= [a^2 \cdot a^6 \cdot a^{10} \dots a^{4n-2}] \cdot [a^3 \cdot a^7 \dots a^{4n-6} \cdot a^{4n-1}] \\ &= [a^{2(1+3+\dots+(2n-1))}] \cdot [a^{4(1+2+\dots+n)-n}] \\ &= a^{4n^2+n} = a^{n(4n+1)} = 1_{\mathbb{A}}. \end{aligned}$$

Now, the weight of the outer face  $f_0$  is the product of all elements in the cyclic group which is

$$\begin{aligned} &= 1 \cdot a \cdot a^2 \dots a^{4n} \\ &= a^{2n(4n+1)} = 1_{\mathbb{A}}. \end{aligned}$$

This example shows that  $D_{n,n,2}$  is a face magic graph over  $\mathbb{A}$  with face constant  $1_{\mathbb{A}}$ , and illustrates proposition 2.3 when  $p + q$  is odd.

**Corollary 2.4:** Every Tree is a face magic graph over  $\mathbb{A}$  where  $\mathbb{A}$  is a cyclic group of order  $2n - 1$ .

**Proposition 2.5:** If any 2 –connected graph which is totally magic with vertex constant  $h$  and an edge constant  $k$  admits a face magic  $\mathbb{A}$ - labeling  $\psi$  of  $G$  with a face constant  $l_{\mathbb{A}}$ , then the product of all face constants of  $G$  is  $i_{\mathbb{A}}. h^p. k^q$ , where  $p, q$  and  $r$  are the number of vertices, edges and faces of  $G$  respectively and  $i_{\mathbb{A}} = \prod_{g \in \mathbb{A}} g$ .

**Proof:** Product of all face constants of  $G$  is  $l^r$  which is

$$\begin{aligned} &= \prod_{e \in E} \psi(e) \cdot \prod_{v \in V} (\psi(v))^{d_v} \\ &= [\prod_{e \in E} \psi(e) \cdot \prod_{v \in V} \psi(v)] \cdot [\prod_{e \in E} \psi(e)] \cdot [\prod_{v \in V} (\psi(v))^{d_v-1}] \\ &= i_{\mathbb{A}} \cdot h^p \cdot i_{\mathbb{A}} \cdot i_{\mathbb{A}}^{-1} \cdot k^q \\ &= i_{\mathbb{A}} \cdot h^p \cdot k^q \end{aligned}$$

Hence the result follows.

**Proposition 2.6:** If  $\mathbb{A}$  is of even order and  $\psi$  is a face magic  $\mathbb{A}$ -labeling of  $G$ , then there exists an element  $a$  such that  $a\psi$  is also a face magic  $\mathbb{A}$ -labeling of  $G$ .

**Proof:** If  $\mathbb{A}$  is of even order, then there exists  $a$  in  $G$  such that  $a^2 = e$ . For each  $s$ -sided face  $f$ ,  $(a\psi)f = a^{2s}\psi(f) = \psi(f)$ . Hence the result follows.

**Proposition 2.7:** Let  $\mathbb{A}$  be an abelian group in which every element is an involution and  $\psi$  be a face magic  $\mathbb{A}$ -labeling of  $G$ . Let  $T_{\psi}$  be the set of all face magic  $\mathbb{A}$  labelings obtained from the translations of  $\psi$ , that is,  $T_{\psi} = \{a\psi: a \in \mathbb{A}\}$ . Then  $T_{\psi}$  forms an abelian group under the operation defined by  $(a_1\psi * a_2\psi)(x) = (a_1a_2)\psi(x), x \in V \cup E$

**Proof:** The set  $T_{\psi}$  is closed under the operation  $*$ , since if  $a_1\psi, a_2\psi \in T_{\psi}$ ,  $(a_1\psi * a_2\psi)(x) = (a_1a_2)\psi(x) \in T_{\psi}$ , by proposition 2.6.

If  $a_1\psi, a_2\psi \in T_{\psi}$ , then  $(a_1\psi * a_2\psi) * (a_3\psi) = (a_1a_2)\psi * a_3\psi = ((a_1a_2)a_3)\psi = (a_1(a_2a_3))\psi = a_1\psi * (a_2\psi * a_3\psi)$  and hence associativity of  $T_{\psi}$  follows.

Also, the element  $e\psi \in T_{\psi}$ ,  $e \in \mathbb{A}$  is considered as an identity element, since  $(a\psi * e\psi)(x) = (ae)\psi(x) = (a\psi)(x) = ((ea)\psi)(x) = (e\psi * a\psi)(x)$ .

Since  $(a\psi * a\psi)(x) = (a^2)\psi(x) = (e\psi)(x)$ , every element in  $T_{\psi}$  has a self inverse in  $T_{\psi}$ .

As  $(a_1\psi * a_2\psi)(x) = (a_1a_2)\psi(x) = (a_2a_1)\psi(x)$ , by the commutative property in  $\mathbb{A}$ , equals  $(a_2\psi * a_1\psi)(x)$ . Hence  $(T_{\psi}, *)$  is an abelian group.

**Proposition 2.8:** The graph obtained by duplication of any vertex in cycle  $C_n$  by a vertex admits a face magic  $\mathbb{A}$ -labeling with face constant  $1_{\mathbb{A}}$ , for  $n \geq 3$ .

**Proof:** Let  $v_1, v_2, \dots, v_n$  be the vertices of the cycle  $C_n$ , for  $n \geq 3$ . By the graph isomorphism, duplicate the vertex  $v_1$  by a vertex  $v_1'$ . Let the resultant graph be  $G$  whose vertex set is  $V(C_n) \cup \{v_1'\}$ , edge set is  $E(C_n) \cup \{v_1'v_n, v_1'v_2\}$  and face set is  $\{f_0, f_1, f_2\}$  where  $f_1$  is the cycle  $C_n$ ,  $f_2$  is the face bounded by the edges  $v_1v_2, v_2v_1', v_1'v_n, v_nv_1$  and  $f_0$  is the outer face bounded by all the edges of  $G$  except  $v_1v_2, v_nv_1$ .

Consider any cyclic group  $\langle a \rangle$  of order  $p + q = 2n + 3$ .

Define a labeling  $\psi: V(G) \cup E(G) \rightarrow \langle a \rangle$  as follows.

$$\psi(v_i) = \begin{cases} a^n, & i = 1 \\ a^{2n+2}, & i = 2 \\ 1, & i = 3 \\ a^{i-1}, & 4 \leq i \leq n-1 \\ a, & i = n, \end{cases}$$

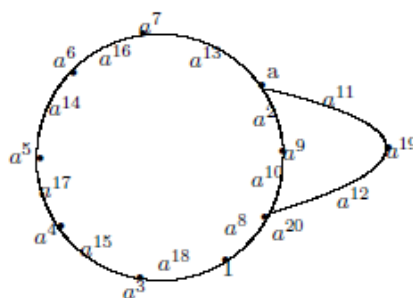
$$\begin{aligned} \psi(v_1') &= a^{2n+1}, \\ \psi(v_1v_n) &= 2, \\ \psi(v_1'v_2) &= a^{n+2}, \\ \psi(v_1'v_n) &= a^{n+3} \text{ and} \\ \psi(v_iv_{i+1}) &= \begin{cases} a^{n+1}, & i = 1 \\ a^{n-1}, & i = 2 \\ n+i+1, & 3 \leq i \leq n-1 \end{cases} \end{aligned}$$

Then the induced face labeling is given by

$$\begin{aligned}
\psi^*(f_1) &= \prod_{i=1}^n \psi(v_i) \cdot \prod_{i=1}^n \psi(v_i v_{i+1}) \cdot \psi(v_n v_1) \\
&= a^n \cdot a^{2n+2} \cdot 1 \cdot a^3 \cdot a^4 \dots a^{n-2} \cdot a \cdot a^{n+1} \cdot a^{n-1} \cdot a^2 \cdot a^{n+4} \cdot a^{n+5} \dots a^{2n} \\
&= [1 \cdot a \cdot a^2 \cdot a^3 \dots a^{n-2} \cdot a^{n-1} \cdot a^n \cdot a^{n+1}] \cdot [a^{n+4} \cdot a^{n+5} \dots a^{2n}] \cdot a^{2n+2} \\
&= a^{[2n(2n+1)/2] - (n+2) - (n+3) + (2n+2)} \\
&= a^{(2n+3)(n-1)} = 1_{\mathbb{A}}, \\
\psi^*(f_0) &= \prod_{i=2}^n \psi(v_i) \cdot \psi(v_1) \cdot \prod_{i=3}^{n-1} \psi(v_i v_{i+1}) \cdot \psi(v_1' v_2) \cdot \psi(v_1' v_n) \cdot \psi(v_2 v_3) \\
&= a^{2n+1} \cdot a^{2n+2} \cdot 1 \cdot a^3 \cdot a^4 \dots a^{n-2} \cdot a^{n+2} \cdot a^{n+3} \cdot a^{n-1} \cdot a^{n+4} \cdot a^{n+5} \dots a^{2n} \cdot a \\
&= [a \cdot a^3 \cdot a^4 \dots a^{n-2} \cdot a^{n-1} \cdot a^{n+2} \cdot a^{n+3}] \cdot [a^{n+4} \cdot a^{n+5} \dots a^{2n}] \cdot a^{2n+1} \cdot a^{2n+2} \\
&= a^{[(2n+2)(2n+3)/2] - 2 - n - (n+1)} \\
&= a^{(2n+3)(n)} = 1_{\mathbb{A}} \text{ and} \\
\psi^*(f_2) &= \psi(v_1) \cdot \psi(v_2) \cdot \psi(v_n) \cdot \psi(v_1' v_2) \cdot \psi(v_1' v_n) \cdot \psi(v_1') \cdot \psi(v_1' v_2) \cdot \psi(v_1' v_n) \cdot \psi(v_1 v_2) \cdot \psi(v_1 v_n) \\
&= a^n \cdot a^{2n+2} \cdot a \cdot a^{n+2} \cdot a^{n+3} \cdot a^{2n+1} \cdot a^{n+1} \cdot a^2 \\
&= a^{8n+12} = a^{(2n+3)4} = 1_{\mathbb{A}}.
\end{aligned}$$

Thus  $\psi$  admits a face magic  $\mathbb{A}$ -labeling with face constant  $1_{\mathbb{A}}$ .

A Face magic  $\mathbb{A}$ -labeling of a vertex duplication of  $C_9$  by a vertex is shown in Figure 1.



**Figure-1:** Face magic  $\mathbb{A}$ -labeling of a vertex duplication of  $C_9$  by a vertex with face constant  $1_{\mathbb{A}}$ .

**Proposition 2.9:** *The graph  $G$  obtained by duplication of any edge in cycle  $C_n$  by an edge admits a face magic A-labeling with face constant  $1_{\mathbb{A}}$  for  $n \geq 3$ .*

**Proof:** Let  $v_1, v_2, \dots, v_n$  be the vertices of the cycle  $C_n$ , for  $n \geq 3$ . By the graph isomorphism, duplicate the edge  $v_1 v_2$  by a vertex  $v_1' v_2'$ . Let the resultant graph be  $G$  whose vertex set is  $V(C_n) \cup \{v_1'\} \cup \{v_2'\}$ , edge set is  $E(C_n) \cup \{v_1' v_n, v_1' v_2', v_2' v_3\}$  and face set is  $\{f_0, f_1, f_2\}$  where  $f_1$  is the cycle  $C_n$ ,  $f_2$  is the face bounded by the edges  $v_1 v_2, v_2 v_3, v_3 v_2', v_2' v_1', v_1' v_n, v_n v_1$  and  $f_0$  is the outer face bounded by all the edges of  $G$  except  $v_1 v_2, v_2 v_3, v_n v_1$ .

Consider any cyclic group  $\langle a \rangle$  of order  $p + q = 2n + 5$ .

Define a labeling  $\psi: V(G) \cup E(G) \rightarrow \langle a \rangle$  as follows.

$$\begin{aligned}\psi(v_i') &= \begin{cases} a^n, & i = 1 \\ a^{n+2}, & i = 2, \end{cases} \\ \psi(v_i) &= \begin{cases} a^{n+3}, & i = 1 \\ a^{n+5}, & i = 2 \\ a^{2n+4}, & i = 3 \\ a^{n+i+3}, & 4 \leq i \leq n-1 \\ a, & i = n, \end{cases} \\ \psi(v_i v_{i+1}) &= \begin{cases} a^{n-1}, & i = 1 \\ a^{n+1}, & i = 2 \\ a^i, & 3 \leq i \leq n-2 \\ 1, & i = n-1, \end{cases} \\ \psi(v_n v_1) &= a^2, \psi(v_1 v_n) = a^{n+4}, \\ \psi(v_1' v_2') &= a^{n+6} \text{ and } \psi(v_2' v_3) = a^{2n+3}. \end{aligned}$$

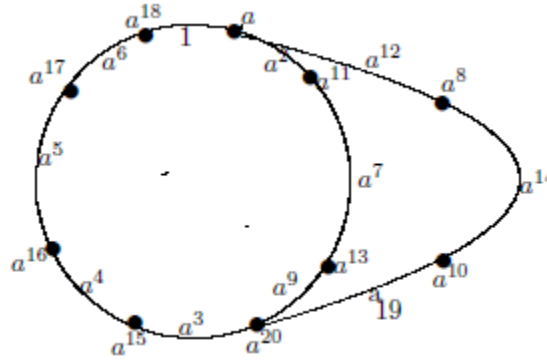
Then the induced face labeling is given by

$$\begin{aligned}\psi^*(f_i) &= \prod_{i=1}^n \psi(v_i) \cdot \prod_{i=1}^{n-1} \psi(v_i v_{i+1}) \cdot \psi(v_n v_1) \\ &= a^{n+3} \cdot a^{n+5} \cdot a^{2n+4} \cdot a^{n+7} \cdot a^{n+8} \dots a^{2n+2} \cdot a \cdot a^{n-1} \cdot a^{n+1} \cdot a^3 \cdot a^4 \dots a^{n-2} \cdot 1 \cdot a^2 \\ &= 1 \cdot a \cdot a^2 \cdot a^3 \dots a^{n-2} \cdot a^{n-1} \cdot a^{n+1} \cdot a^{n+3} \cdot a^{n+5} \cdot a^{n+7} \cdot a^{n+8} \dots a^{2n+2} \cdot a^{2n+4} \\ &= a^{[(2n+4)(2n+5)/2] - n - (n+2) - (n+4) - (n+6) - (2n+3)} \\ &= a^{2n^2+3n-5} = a^{(2n+5)(n-1)} = 1_{\Delta}.\end{aligned}$$

$$\begin{aligned}
 \psi^*(f_0) &= \prod_{i=3}^n \psi(v_i) \cdot \psi(v_1') \cdot \psi(v_2') \cdot \prod_{i=3}^{n-1} \psi(v_i v_{i+1}) \cdot \psi(v_1' v_2) \cdot \psi(v_1' v_n) \cdot \psi(v_2 v_3) \\
 &= a^{2n+4} \cdot a^{n+7} \cdot a^{n+8} \dots a^{2n+2} \cdot a \cdot a^3 \cdot a^4 \dots a^{n-2} \cdot 1 \cdot a^{n+6} \cdot a^{n+4} \cdot a^{2n+3} \cdot a^n \cdot a^{n+2} \\
 &= 1 \cdot a \cdot a^3 \cdot a^4 \dots a^{n-2} \cdot a^n \cdot a^{n+2} \cdot a^{n+4} \cdot a^{n+6} \cdot a^{n+7} \cdot a^{n+8} \dots a^{2n+2} \cdot a^{2n+3} \cdot a^{2n+4} \\
 &= a^{[(2n+4)(2n+5)/2] - (n-1) - (n+1) - (n+3) - (n+5) - 2} \\
 &= a^{(2n+5)(n)} = 1_{\mathbb{A}} \text{ and} \\
 \psi^*(f_2) &= \prod_{i=1}^3 \psi(v_i) \cdot \psi(v_n) \cdot \psi(v_1) \cdot \psi(v_2) \cdot \psi(v_1 v_2) \cdot \psi(v_2 v_3) \cdot \psi(v_1 v_n) \cdot \psi(v_1' v_2) \cdot \psi(v_1' v_n) \cdot \psi(v_2' v_3) \\
 &= a^{n+3} \cdot a^{n+5} \cdot a^{2n+4} \cdot a \cdot a^n \cdot a^{n+2} \cdot a^{n-1} \cdot a^{n+1} \cdot a^2 \cdot a^{n+6} \cdot a^{n+4} \cdot a^{2n+3} \\
 &= a^{12n+50} = a^{(2n+5)6} = 1_{\mathbb{A}}.
 \end{aligned}$$

Thus  $\psi$  admits a face magic  $\mathbb{A}$ -labeling with face constant  $1_{\mathbb{A}}$ .

A face magic  $\mathbb{A}$ -labeling of edge duplication of  $C_8$  by an edge is shown in Figure 2.



**Figure-2:** Face magic  $\mathbb{A}$ -labeling of edge duplication of  $C_8$  by an edge with face constant  $1_{\mathbb{A}}$ .

**Proposition 2.10:** The graph  $G$  obtained by duplication of any edge in cycle  $C_n$  by a vertex admits a face magic  $\mathbb{A}$ -labeling with face constant  $1_{\mathbb{A}}$  for  $n \geq 3$ .

**Proof:** Let  $v_1, v_2, \dots, v_n$  be the vertices of the cycle  $C_n$ , for  $n \geq 3$ . By the graph isomorphism, duplicate the edge  $v_1 v_2$  by a vertex  $u$ . This vertex  $u$  may be placed inside the cycle  $C_n$  or outside the cycle  $C_n$ .

**Case i:** Assume that vertex  $u$  is placed outside the cycle  $C_n$ . Let the resultant graph be  $G$  whose vertex set is  $V(C_n) \cup \{u\}$ , edge set is  $E(G) = E(C_n) \cup \{uv_1, uv_2\}$  and face set is  $\{f_0, f_1, f_2\}$  where  $f_1$  is the cycle  $C_n$ ,  $f_2$  is the face bounded by the edges  $v_1 v_2, uv_1, uv_2$  and  $f_0$  is the outer face bounded by all the edges of  $G$  except  $v_1 v_2$ .

Consider any cyclic group  $\langle a \rangle$  of order  $p + q = 2n + 3$ .

Define a labeling  $\psi: V(G) \cup E(G) \rightarrow \langle a \rangle$  as follows.

$$\begin{aligned}
 \psi(u) &= a^{2n+1}, \\
 \psi(v_i) &= \begin{cases} a, & i = 1 \\ a^{2n+2}, & i = 2 \\ a^{n+1}, & i = 3 \\ a^{n+i}, & 4 \leq i \leq n, \end{cases} \\
 \psi(v_i v_{i+1}) &= \begin{cases} 1, & i = 1 \\ a^i, & 2 \leq i \leq n-1, \end{cases} \\
 \psi(v_n v_1) &= a^n, \\
 \psi(uv_1) &= a^{n+2} \text{ and} \\
 \psi(uv_2) &= a^{n+3}.
 \end{aligned}$$

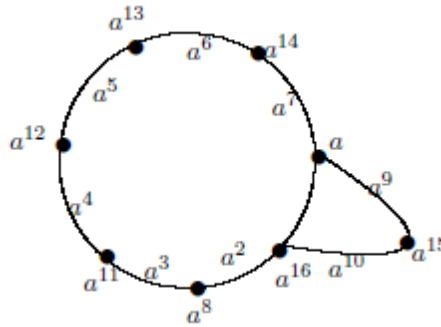
Then the induced face labeling is given by

$$\begin{aligned}
 \psi^*(f_1) &= \prod_{i=1}^n \psi(v_i) \cdot \prod_{i=1}^{n-1} \psi(v_i v_{i+1}) \cdot \psi(v_n v_1) \\
 &= a \cdot a^{2n+2} \cdot a^{n+1} \cdot a^{n+4} \cdot a^{n+5} \dots a^{2n} \cdot 1 \cdot a^2 \cdot a^3 \dots a^{n-1} \cdot a^n \\
 &= 1 \cdot a \cdot a^2 \cdot a^3 \dots a^n \cdot a^{n+1} \cdot a^{n+4} \cdot a^{n+5} \cdot a^{n+6} \dots a^{2n} \cdot a^{2n+2} \\
 &= a^{(2n+2)(2n+3)/2} \\
 &= a^{2n^2+n-3} = a^{(2n+3)(n-1)} = 1_{\mathbb{A}}, \\
 \psi^*(f_0) &= \prod_{i=1}^n \psi(v_i) \cdot \psi(u) \cdot \prod_{i=1}^{n-1} \psi(v_i v_{i+1}) \cdot \psi(v_n v_1) \\
 &= a \cdot a^{2n+2} \cdot a^{n+1} \cdot a^{n+4} \cdot a^{n+5} \dots a^{2n} \cdot a^{2n+1} \cdot a^2 \cdot a^3 \dots a^{n-1} \cdot a^n \cdot a^{n+2} \cdot a^{n+3} \\
 &= a \cdot a^2 \cdot a^3 \dots a^{n-1} \cdot a^n \cdot a^{n+1} \cdot a^{n+2} \dots a^{2n} \cdot a^{2n+1} \cdot a^{2n+2} \\
 &= a^{(2n+2)(2n+3)/2} \\
 &= a^{(2n+3)(n+1)} = 1_{\mathbb{A}} \text{ and} \\
 \psi^*(f_2) &= \psi(v_1) \cdot \psi(v_2) \cdot \psi(u) \cdot \psi(uv_2) \cdot \psi(v_1 v_2) \cdot \psi(uv_1) \\
 &= a \cdot a^{2n+2} \cdot a^{2n+1} \cdot 1 \cdot a^{n+2} \cdot a^{n+3} \\
 &= a^{6n+9} = a^{(2n+3)3} = 1_{\mathbb{A}}.
 \end{aligned}$$

Thus  $\psi$  admits a face magic  $\mathbb{A}$ -labeling with face constant  $1_{\mathbb{A}}$ .

**Case ii:** Assume that vertex  $u$  is placed inside the cycle  $C_n$ . In this case, the resultant graph have the same faces as in case (i), but  $f_1$  is an outer face and  $f_0$  is an inner face. By the same labeling defined in case (i), the result follows.

A face magic  $\mathbb{A}$ -labeling of  $G$  obtained by duplicating an edge of  $C_7$  by a vertex is shown in Figure 3.



**Figure-3:** Face magic  $\mathbb{A}$ -labeling of  $G$  obtained by duplicating an edge of  $C_7$  by a vertex with face constant  $1_{\mathbb{A}}$ .

**Proposition 2.11:** The graph  $G$  obtained by duplication of any vertex in cycle  $C_n$  by an edge admits a face magic  $\mathbb{A}$ -labeling with face constant  $1_{\mathbb{A}}$ , for  $n \geq 3$ .

**Proof:** Let  $v_1, v_2, \dots, v_n$  be the vertices of the cycle  $C_n$ , for  $n \geq 3$ . By the graph isomorphism, duplicate the vertex  $v_1$  by an edge  $uv$ . Let the resultant graph be  $G$  whose vertex set is  $V(C_n) \cup \{u, v\}$ , edge set is  $E(C_n) \cup \{uv, uv_1, vv_1\}$  and face set is  $\{f_0, f_1, f_2\}$  where  $f_1$  is the cycle  $C_n$ ,  $f_2$  is the face bounded by the edges  $uv, uv_1, vv_1$  and  $f_0$  is the outer face bounded by all the edges of  $G$ .

Consider any cyclic group  $\langle a \rangle$  of order  $p + q = 2n + 5$ .

Define a labeling  $\psi: V(G) \cup E(G) \rightarrow \langle a \rangle$  as follows.

$$\begin{aligned} \psi(u) &= a^{n+1}, \\ \psi(v) &= a^{n+2} \\ \psi(v_i) &= \begin{cases} 1, & i = 1 \\ a^{i+1}, & 2 \leq i \leq n-1 \\ a^{n+3}, & i = n, \end{cases} \\ \psi(uv) &= a^{2n+4}, \\ \psi(uv_1) &= a, \\ \psi(vv_1) &= a^2, \\ \psi(v_i v_{i+1}) &= a^{n+3+i}, 1 \leq i \leq n-1 \text{ and} \\ \psi(v_n v_1) &= a^{2n+3}. \end{aligned}$$

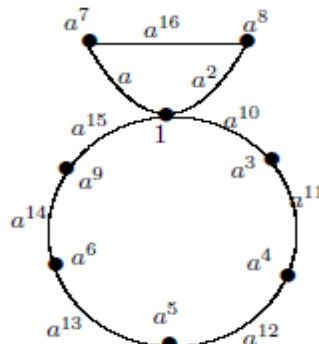
Then the induced face labeling is given by

$$\begin{aligned} \psi^*(f_1) &= \prod_{i=1}^n \psi(v_i) \cdot \prod_{i=1}^{n-1} \psi(v_i v_{i+1}) \cdot \psi(v_n v_1) \\ &= 1 \cdot a^3 \cdot a^4 \cdot \dots \cdot a^n \cdot a^{n+3} \cdot a^{n+4} \cdot a^{n+5} \cdot \dots \cdot a^{2n+2} \cdot a^{2n+3} \\ &= a^{[(2n+3)(2n+4)/2] - 1 - 2 - (n+1) - (n+2)} \\ &= a^{2n^2+5n} = a^{(2n+5)(n)} = 1_{\mathbb{A}} \text{ and} \\ \psi^*(f_2) &= \psi(u) \cdot \psi(v) \cdot \psi(v_1) \cdot \psi(uv_1) \cdot \psi(vv_1) \cdot \psi(uv) \\ &= a^{n+1} \cdot a^{n+2} \cdot 1 \cdot a \cdot a^2 \cdot a^{2n+4} \\ &= a^{4n+10} = a^{(2n+5)2} = 1_{\mathbb{A}}. \end{aligned}$$

Since  $G$  is outer planar and  $p + q = 2n + 5$  is odd, by proposition 2.3,  $\psi^*(f_0) = 1_{\mathbb{A}}$ .

Thus  $\psi$  admits a face magic  $\mathbb{A}$ -labeling with face constant  $1_{\mathbb{A}}$ .

A face magic  $\mathbb{A}$ -labeling of  $G$  obtained by duplicating a vertex of  $C_6$  by an edge is shown in Figure 4.



**Figure-4:** Face magic  $\mathbb{A}$ -labeling of  $G$  obtained by duplicating a vertex of  $C_6$  by an edge with face constant  $1_{\mathbb{A}}$ .

**Proposition 2.12:** *The graph  $G$  obtained by duplication of any vertex in path  $P_n$  by a vertex admits a face magic  $\mathbb{A}$ -labeling with face constant  $a^{(p+q)/2}$  if  $p+q$  is even or  $1_{\mathbb{A}}$  if  $p+q$  is odd, for  $n \geq 2$ .*

**Proof:** Let  $v_1, v_2, \dots, v_n$  be the vertices of the path  $P_n$ , for  $n \geq 2$ . Let the vertex  $v_i$  be duplicated by a vertex  $v_i'$ . Then the resultant graph is  $G$  which is outer planar with all its edges in the boundary of exterior face.

*Case i.*  $i = 1$  or  $n$

If any one of the end vertices is duplicated, then  $G$  is isomorphic to a tree graph having  $p + q = 2n + 1$  which is odd. By corollary 2.4,  $G$  is a Face magic graph over  $\mathbb{A}$  with face constant  $1_{\mathbb{A}}$ , for  $n \geq 1$ .

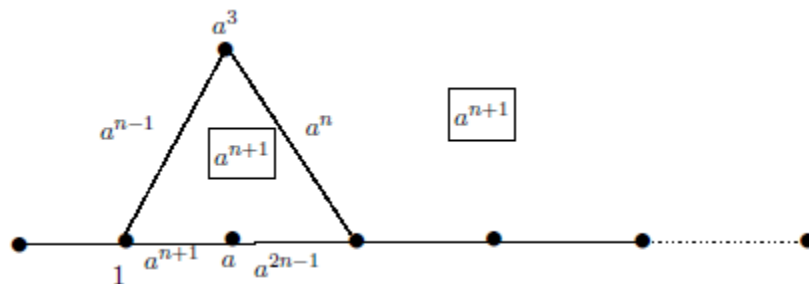
*Case ii.*  $2 \leq i \leq n-1$

Let the resultant graph  $G$  has vertex set  $V(G) = V(P_n) \cup \{v_i'\}$ , edge set  $E(P_n) \cup \{v_i'v_{i-1}, v_i'v_{i+1}\}$  and face set  $\{f_0, f_1\}$  where  $f_1$  is the face bounded by the edges  $v_{i-1}v_i, v_iv_{i+1}, v_i'v_{i-1}, v_i'v_{i+1}$  and  $f_0$  is the outer face bounded by all the edges of  $G$ .

Consider the cyclic group  $\langle a \rangle$  of order  $p + q = 2n + 2$ . Assign the labels  $1, a, a^2, a^3, a^{n-1}, a^n, a^{n+1}, a^{2n-1}$  to the vertices and edges of the face  $f_1$  and assign the remaining labels to the remaining vertices and edges of the graph  $G$ . Now, the weight of the face  $f_1 = a^{5n+5} = a^{2(2n+2)} \cdot a^{n+1} = a^{n+1}$ , since  $a^{2n+2} = 1_{\mathbb{A}}$ . Also the weight of the outer face  $f_0 = 1 \cdot a \cdot a^2 \cdot a^3 \dots a^{2n+1} = a^{n+1}$ .

Thus  $G$  obtained by duplication of any vertex in path  $P_n$  by a vertex is a face magic graph over  $\mathbb{A}$ .

A face magic  $\mathbb{A}$ -labeling of  $G$  obtained by duplicating a vertex of  $P_n$  by a vertex is shown in Figure 5.



**Figure-5:** Face magic A-labeling of  $G$  obtained by duplicating a vertex of  $P_n$  by a vertex.

**Proposition 2.13:** *The graph  $G$  obtained by duplication of any arbitrary edge in path  $P_n$  admits a face magic  $\mathbb{A}$ -labeling with face constant  $a^{(p+q)/2}$  if  $p+q$  is even or  $1_{\mathbb{A}}$  if  $p+q$  is odd, for  $n \geq 3$ .*

**Proof:** Let  $v_1, v_2, \dots, v_n$  be the vertices of the path  $P_n$ , for  $n \geq 3$ . Let the edge  $e = v_i v_{i+1}$  be duplicated by the edge  $e' = v'_i v'_{i+1}$ . Then the resultant graph is  $G$  which is outer planar with all its edges in the boundary of exterior face.

*Case i.  $i = 1$  or  $n$*

If any one of the edges  $v_1v_2$  or  $v_{n-1}v_n$  is duplicated, then  $G$  is isomorphic to a tree graph having  $p + q = 2n + 3$  which is odd. By corollary 2.4,  $G$  is a Face magic  $\mathbb{A}$ -graph with face constant  $1_{\mathbb{A}}$ , for  $n \geq 3$ .

**Case ii.**  $2 \leq i \leq n - 2$

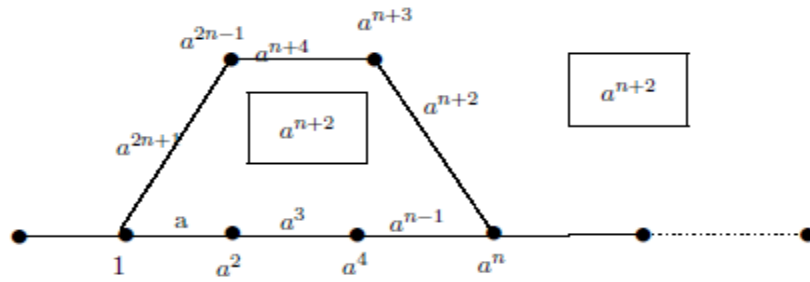
Let the resultant graph be  $G$  has vertex set  $V(P_n) \cup \{v'_i, v_{i+1}'\}$ , edge set  $E(P_n) \cup \{v'_i v_{i+1}', v'_i v_{i-1}, v_{i+1}' v_{i+2}\}$  and face set  $\{f_0, f_1\}$  where  $f_1$  is the face bounded by the edges  $\{v_{i-1} v_i, v_i v_{i+1}, v_{i+1} v_{i+2}, v_{i+2} v'_{i+1}, v'_i v'_{i+1}, v'_i v_{i-1}\}$  and  $f_0$  is the outer face bounded by all the edges of  $G$ .

Consider the cyclic group  $\langle a \rangle$  of order  $p + q = 2n + 4$ . Assign the labels  $1, a, a^2, a^3, a^4, a^{n-1}, a^n, a^{n+2}, a^{n+3}, a^{n+4}, a^{2n-1}, a^{2n+1}$  to the vertices and edges of the face  $f_1$  and assign the remaining labels to the remaining vertices and edges of the graph. Now, the weight of the face  $f_1 = a^{9n+18} = a^{4(2n+4)} \cdot a^{n+2} = a^{n+2}$ , since  $a^{2n+4} = 1_{\mathbb{A}}$ . Also, the weight of the outer face  $f_0 = 1 \cdot a \cdot a^2 \cdot a^3 \dots a^{2n+3} = a^{n+2}$ .

Thus  $G$  obtained by duplication of any edge in path  $P_n$  by an edge is a face magic graph over  $\mathbb{A}$ .

A face magic  $\mathbb{A}$ -labeling of  $G$  obtained by duplicating an edge of path  $P_n$  by an edge is shown in Figure 6.





**Figure-6:** Face magic  $\mathbb{A}$ -labeling of  $G$  obtained by duplicating an edge of path  $P_n$  by an edge.

**Proposition 2.14:** The graph  $G$  obtained by duplication of any edge by a vertex in path  $P_n$  admits a face magic  $\mathbb{A}$ -labeling with face constant  $a^{n+1}$ , for  $n \geq 2$ .

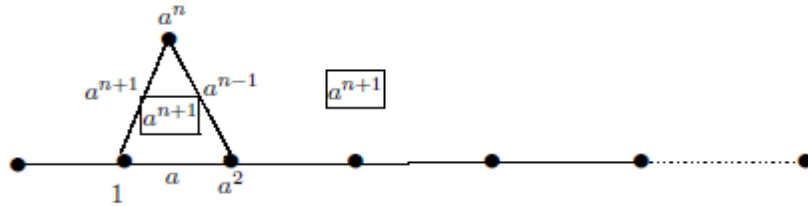
**Proof:** Let  $v_1, v_2, \dots, v_n$  be the vertices of the path  $P_n$ , for  $n \geq 2$ .

By duplicating any edge  $v_i v_{i+1}$  by a vertex  $u$ ,  $1 \leq i \leq n-1$ , the resultant graph  $G$  has vertex set  $V(P_n) \cup \{u\}$ , edge set  $E(P_n) \cup \{uv_i, uv_{i+1}\}$  and face set  $\{f_0, f_1\}$  where  $f_1$  is the face bounded by the edges  $v_i v_{i+1}, uv_{i+1}, uv_i$  and  $f_0$  is the outer face bounded by all the edges of  $G$ .

Consider the cyclic group  $\langle a \rangle$  of order  $p+q=2n+2$ . Assign the labels  $1, a, a^2, a^{n-1}, a^n, a^{n+1}$  to the vertices and edges of the face  $f_1$  and assign the remaining labels to the remaining vertices and edges of the graph. Now, the weight of the face  $f_1 = a^{3n+3} = a^{2n+2} \cdot a^{n+1} = a^{n+1}$ , since  $a^{2n+2} = 1_{\mathbb{A}}$ . Also the weight of the outer face  $f_0 = 1 \cdot a \cdot a^2 \cdot a^3 \dots a^{2n+1} = a^{n+1}$ .

Thus  $G$  obtained by duplication of any edge by a vertex in path  $P_n$  is a face magic graph over  $\mathbb{A}$ .

A face magic  $\mathbb{A}$ -labeling of  $G$  obtained by duplication of any edge by a vertex in path  $P_n$  is shown in Figure 7.



**Figure-7:** Face magic  $\mathbb{A}$ -labeling of  $G$  obtained by duplication of any edge by a vertex in path  $P_n$  with face constant  $a^{n+1}$ .

**Proposition 2.15:** The graph  $G$  obtained by duplication of any vertex by an edge in path  $P_n$  admits a face magic  $\mathbb{A}$ -labeling with face constant  $a^{n+2}$ .

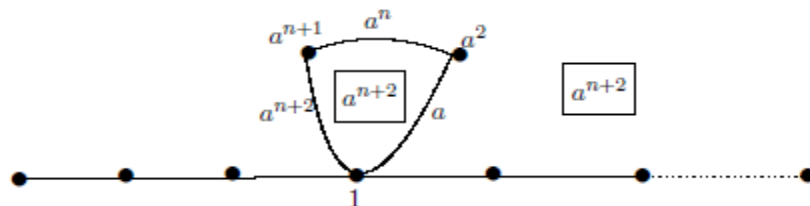
**Proof:** Let  $v_1, v_2, \dots, v_n$  be the vertices of the path  $P_n$ , for  $n \geq 1$ .

By duplicating any vertex  $v_i$  by an edge  $uv$ ,  $1 \leq i \leq n$ , the resultant graph  $G$  has vertex set  $V(P_n) \cup \{u, v\}$ , edge set  $E(P_n) \cup \{uv, uv_i, vv_i\}$  and face set  $\{f_0, f_1\}$  where  $f_1$  is the face bounded by the edges  $uv, uv_i, vv_i$  and  $f_0$  is the outer face bounded by all the edges of  $G$ .

Consider the cyclic group  $\langle a \rangle$  of order  $p+q=2n+4$ . Assign the labels  $1, a, a^2, a^n, a^{n+1}, a^{n+2}$  to the vertices and edges of the face  $f_1$  and assign the remaining labels to the remaining vertices and edges of the graph. Now, the weight of the face  $f_1 = a^{3n+6} = a^{2n+4} \cdot a^{n+2} = a^{n+2}$ , since  $a^{2n+4} = 1_{\mathbb{A}}$ . Also, the weight of the outer face  $f_0 = 1 \cdot a \cdot a^2 \cdot a^3 \dots a^{2n+3} = a^{n+2}$ .

Thus  $G$  obtained by duplication of any vertex by an edge in path  $P_n$  is a face magic graph over  $\mathbb{A}$ .

A face magic  $\mathbb{A}$ -labeling of  $G$  obtained by duplication of any vertex by an edge in path  $P_n$  is shown in Figure 8.



**Figure-8:** Face magic  $\mathbb{A}$ -labeling of  $G$  obtained by duplication of any vertex by an edge in path  $P_n$  with face constant  $a^{n+2}$ .

## REFERENCES

1. N. Hartsfield and G. Ringel, *Pearls in Graph Theory*, Academic Press, San Diego, 1990.
2. W. D. Wallis, *Magic Graphs*, Birhauser, Borton-Bacel-Berlin, 2001.
3. J. A. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, 5(2005), # DS6.
4. Dalibor Froncek, Group distance magic labeling of cartesian product of cycles, *Australasian journal of Combinatorics*, 55 (2013), 167–174.
5. D.Combe, A.M.Nelson, W.D.Palmer, Magic labelings of graphs over finite abelian groups, *Australasian journal of Combinatorics*, 29 (2004), 259–271.
6. Sylwia Cichacz, Note on Group Distance Magic Graphs  $G(C_4)$ , *Graphs and Combinatorics*, 30 (2014), 565–571.
7. A.Durai Baskar and S.Arockiaraj, Further Results on Super Geometric Mean Graphs, *International J.Math. Combin.*, 4 (2015), 104–128.
8. A.Durai Baskar, S.Arockiaraj and B.Rajendran, Geometric meanness of graphs obtained from paths, *Utilitas Mathematica*, 101 (2016), 45–68.
9. S.Arumugam, D. Froncek and N. Kamatchi, Distance Magic Graphs - A Survey, *J. Indones. Math. Soc. Special Ed.* (2011), 1–9.
10. A.Meena Kumari and S.Arockiaraj, On  $(1,1,0)$ -F-face magic mean labeling of some graphs, *Utilitas Mathematica*, 103 (2017), 139–159.
11. S.Arockiaraj and A.Meena Kumari, On F-face magic mean labeling of some planar graphs, *International Journal of Pure and Applied Mathematics*, 117, No.6 (2017), 1–8.
12. S.Arockiaraj and A.Meena Kumari, On F-face magic mean labeling of some duplicated graphs, *International Journal of Applied Mathematics*, 31, No.2 (2018), 231–240.

**Source of support: Nil, Conflict of interest: None Declared.**

**[Copy right © 2018. This is an Open Access article distributed under the terms of the International Journal of Mathematical Archive (IJMA), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.]**