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GENERALIZED FIBONACCI-LIKE SEQUENCE AND FIBONACCI SEQUENCE

BHUPENDRA SINGH GEHLOT

Department of Mathematics,
PMB Gujarati Science College, Indore (M.P.), India.

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ABSTRACT

In this paper, we study Generalized Fibonacci-Like Sequence M_n defined by the recurrence relation $M_n = M_{n-1} + M_{n-2}$, for all ($n\geq 2$) With $M_0 = 8$ and $M_1 = 8\sqrt{n}$, n being a fixed positive integer. we shall defined Binet 's formula and generating function of Generalized Fibonacci – Like sequence. Mainly, Induction method and Binet's formula will be used to establish properties of Generalized Fibonacci – Like sequence.

Mathematics subject classification: *11B39, 11B37, 11B99.*

Keywords: *Fibonacci Sequence, Lucas Sequence, Generalized Fibonacci – Like Sequence.*

1. INTRODUCTION

The generalization of Fibonacci and Lucas Sequence leads to several nice and interesting Sequence [3] [10]

The Sequence of Fibonacci number (F_n) is defined by

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2,\ F_0 = 0,\ F_1 = 1. \tag{1.1}$$

The Sequence of Lucas number $\{L_n\}$ is defined by

$$L_n = L_{n-1} + L_{n-2},\ n \geq 2,\ L_0 = 2,\ L_1 = 1 \tag{1.2}$$

The Binet's formula for Fibonacci Sequence is Given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\} \tag{1.3}$$

Where $\alpha = \frac{1+\sqrt{5}}{2} \approx$ Golden ratio ≈ 1.618

$\beta = \frac{1-\sqrt{5}}{2} \approx -0.618$

Similarly, the Binet's formula for Lucas Sequence is given by

$$L_n = \alpha^n + \beta^n = \left\{\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right\} \tag{1.4}$$

L_n denotes the n^{th} Lucas number of the Sequence. The first few number of this sequence are:

2, 1, 3, 4, 7, 11, 18, 29, 47, ...

In this paper, we present various properties of the Generalized Fibonacci – Like Sequence $\{M_n\}$ defined by

$$M_n = M_{n-1} + M_{n-2},\ \text{for all } n \geq 2 \tag{1.5}$$

With $M_0 = 8$ and $M_1 = 8\sqrt{n}$, n being a fixed positive integer.

The few terms of the Sequence $\{M_n\}$ are

$8,\ 8\sqrt{n}$ and $8+8\sqrt{n},\ 8+16\sqrt{n},\ 16+24\sqrt{n}$, and so on.

Corresponding Author: Bhupendra Singh Gehlot
Department of Mathematics, PMB Gujarati Science College, Indore (M.P.), India.

2. ADDITION OF TWO FIBONACCI SEQUENCES

Let us we consider Generalized Fibonacci number, with the recursion formula $M_{n+1} = M_n + M_{n-1}$, and an arbitrary initial numbers M_0 & M_1;
where

$M_0 = 8$, $M_1 = 8\sqrt{n}$

$M_2 = M_1 + M_0 = F_2 M_1 + F_1 M_0$
$= 1 * 8\sqrt{n} + 1 * 8$
$= 8\sqrt{n} + 8$
$= 8 + 8\sqrt{n}$

$M_3 = M_2 + M_1 = F_3 M_1 + F_2 M_0$
$= 2 * 8\sqrt{n} + 1 * 8$
$= 16\sqrt{n} + 8$
$= 8 + 16\sqrt{n}$

$M_4 = M_3 + M_2 = F_4 M_1 + F_3 M_0$
$= 3 * 8\sqrt{n} + 2 * 8$
$= 24\sqrt{n} + 16$
$= 16 + 24\sqrt{n}$

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$M_n = M_1 F_n + M_0 F_{n-1}$, $n \geq 2$
$M_n = 8\sqrt{n}\ F_n + 8 F_{n-1}$, $n \geq 2$

3. PRELIMINARY RESULTS OF GENERALIZED FIBONACCI – LIKE SEQUENCE

First we Introduce some basic results of Generalized Fibonacci – Like Sequence and Fibonacci Sequence.

The recurrence relation (1.1) has the characteristic Equation.

$$X^2 - X - 1 = 0 \text{ which has two roots } \alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2} \tag{3.1}$$

Now notice a few things about α and β

$$\alpha + \beta = 1;\ \alpha - \beta = \sqrt{5} \text{ and } \alpha\beta = -1$$

By substituting Binet's formula for F_n, F_{n-1} gives;

$$M_n = M_1 F_n + M_0 F_{n-1}$$

$$M_n = M_1 \left[\frac{\alpha^n - \beta^n}{\alpha-\beta}\right] + M_0 \left[\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha-\beta}\right]$$

$$= \frac{1}{\alpha-\beta}\left[M_1\alpha^n - M_1\beta^n + M_0\alpha^{n-1} - M_0\beta^{n-1}\right]$$

$$= \frac{1}{\alpha-\beta}\left[(M_1\alpha^n + M_0\alpha^{n-1}) - (M_1\beta^n + M_0\beta^{n-1})\right]$$

$$= \frac{1}{\alpha-\beta}\left[\left(M_1\alpha^n + \frac{M_0\alpha^n}{\alpha}\right) - \left(M_1\beta^n + \frac{M_0\beta^n}{\beta}\right)\right]$$

$$= \frac{1}{\alpha-\beta}\left[\alpha^n\left(M_1 + \frac{M_0}{\alpha}\right) - \beta^n\left(M_1 + \frac{M_0}{\beta}\right)\right]$$

$$\alpha\beta = -1 = -\beta = \frac{1}{\alpha}, -\alpha = \frac{1}{\beta}$$

$$= \frac{1}{\alpha-\beta}\left[\alpha^n(M_1 - M_0\beta) - \beta^n(M_1 - M_0\alpha)\right] \tag{3.2}$$

Equation (3.2) we called Binet's type formula for Generalized Fibonacci numbers.

The Generating function of $\{M_n\}$ is defined as

$$= \sum_{n=0}^{\infty} M_n X^n = \frac{8+8(\sqrt{n}-1)x}{1-x-x^2}$$

4. PROPERTIES OF GENERALIZED FIBONACCI – LIKE SEQUENCE

Sum of Generalized Fibonacci – Like terms:

Theorem 4.1: Sum of First n terms of the Generalized Fibonacci – Like Sequence $\{M_n\}$ is

$$M_1 + M_2 + M_3 + --- + M_n = \sum_{k=1}^{n} M_k = M_{n+2} - M_2$$
$$= M_{n+2} - \left(8 + 8\sqrt{n}\right)$$

Proof: In this the following relation holds:

$M_1 = M_3 - M_2$
$M_2 = M_4 - M_3$
$M_3 = M_5 - M_4$
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— — —
$M_{n-1} = M_{n+1} - M_n$
$M_n = M_{n+2} - M_{n+1}$

it follows the terms wise addition of all above equation that

$$M_1 + M_2 + M_3 + --- + M_n = M_{n+2} - M_2$$
$$= M_{n+2} - \left(8 + 8\sqrt{n}\right)$$

This identity becomes $M_1 + M_2 + M_3 + --- + M_{2n} = \sum_{k=1}^{2n} M_k$
$$= M_{2n+2} - \left(8 + 8\sqrt{n}\right)$$

Theorem 4.2: sum of the first n terms with odd indices is.

$$M_1 + M_3 + M_5 + --- + M_{2n-1} = \sum_{k=1}^{n} M_{2k-1}$$
$$= M_{2n} - M_0$$
$$= M_{2n} - 8$$

Proof: In this the following relation holds:

$M_1 = M_2 - M_0$
$M_3 = M_4 - M_2$
$M_5 = M_6 - M_4$
— — —
— — —
$M_{2n-1} = M_{2n} - M_{2n-2}$

Term wise addition of all above equations, gives

$$M_1 + M_3 + M_5 + --- + M_{2n-1} = M_{2n} - M_0$$
$$= M_{2n} - 8$$

Theorem 4.3: Sum of the first n terms with even indices is

$$M_2 + M_4 + M_6 + --- + M_{2n} = \sum_{k=1}^{n} M_{2k}$$
$$= M_{2n+1} - 8\sqrt{n}$$

Theorem 4.4: Sum of the square of first n terms of the Generalized Fibonacci – Like Sequence is

$$M_1^2 + M_2^2 + M_3^2 + --- + M_n^2 = \sum_{k=1}^{n} M_k^2$$
$$= M_n M_{n+1} - M_0 M_1$$

Proof: In this the following relation holds:

$M_1^2 = M_1 M_2 - M_0 M_1$
$M_2^2 = M_2 M_3 - M_1 M_2$
$M_3^2 = M_3 M_4 - M_2 M_3$
— — —
— — —
— — —
$M_n^2 = M_n M_{n+1} - M_{n-1} M_n$

It follows from term wise addition of all the above equation that

$$M_1^2 + M_2^2 + M_3^2 + --- + M_n^2 = M_n M_{n+1} - M_0 M_1 = M_n M_{n+1} - 64\sqrt{n}$$

5. CONNECTION FORMULAE

Theorem 5.1: Let n be a positive integer then

$$M_{n+1} + M_{n-1} = M_1 L_n + M_0 L_{n-1},\ n\geq 1$$

Proof: We shall prove this identities by induction on n.
For n = 1

$$M_2 + M_0 = 8\sqrt{n}\, L_1 + 8\, L_0$$
$$\left(8+8\sqrt{n}\right) + 8 = 8\sqrt{n} \times 1 + 8 \times 2$$
$$16 + 8\sqrt{n} = 8\sqrt{n} + 16$$

Which is true for n = 2

$$M_3 + M_1 = M_1L_2 + M_0L_1$$
$$\left(8+16\sqrt{n}\right) + 8\sqrt{n} = 8\sqrt{n} \times 3 + 8 \times 1$$
$$= 24\sqrt{n} + 8$$
$$8 + 24\sqrt{n} \text{ which is true .}$$

Suppose that identity holds for n = k – 2 and n = k – 1, Then

$$M_{k-1} + M_{k-3} = M_1L_{k-2} + M_0L_{k-3}$$
$$M_k + M_{k-2} = M_1L_{k-1} + M_0L_{k-2}$$

Adding equation and equation, we get

$$(M_{k-1} + M_k) + (M_{k-3} + M_{k-2}) = M_1[L_{k-1} + L_{k-2}] + M_0[L_{k-2} + L_{k-3}]$$
$$= (M_1L_k + M_0L_{k-1})$$

Which is our identity when n = k. Hence

$$M_{n+1} + M_{n-1} = M_1L_n + M_0L_{n-1}$$

Theorem 5.2: Let n be a positive integer then

$$M_{n+1} - M_{n-1} = M_1 F_n + M_0 F_{n-1},\ \text{for all } n \geq 1$$

Proof: we shall prove this identity by induction on n .
For n = 1

$$M_2 - M_0 = M_1F_1 + M_0F_0$$
$$\left(8+8\sqrt{n}\right) - 8 = 8\sqrt{n} \times 1 + 8 \times 0$$
$$8\sqrt{n} = 8\sqrt{n}$$

Which is true. for n = 2

$$M_3 - M_1 = M_1 F_2 + M_0 F_1$$
$$8 + 16\sqrt{n} - 8\sqrt{n} = 8\sqrt{n} \times 1 + 8 \times 1$$
$$8 + 8\sqrt{n} = 8\sqrt{n} + 8 \text{ Which is true.}$$

Suppose that identity holds for n = k – 2 and n = k – 1 , Then

$$M_{k-1} + M_{k-3} = M_1F_{k-2} + M_0F_{k-3}$$
$$M_k + M_{k-2} = M_1F_{k-1} + M_0F_{k-2}$$

Adding equation and equation, we get

$$(M_{k-1} + M_k) - (M_{k-3} + M_{k-2}) = M_1[F_{k-1} + F_{k-2}] + M_0[F_{k-2} + F_{k-3}]$$
$$M_{k+1} - M_{k-1} = M_1F_k + M_0F_{k-1} \textit{ which is our identitiy when } n = k.$$

Hence, $M_{n+1} - M_{n-1} = M_1F_n + M_0F_{n-1}$, for all n ≥ 1

Theorem 5.3: Let n be a positive integer then

$$\begin{vmatrix} M_n & F_n & 1 \\ M_{n+1} & F_{n+1} & 1 \\ M_{n+2} & F_{n+2} & 1 \end{vmatrix} = [F_n\, M_{n+1} - F_{n+1}\, M_n]$$

Proof: -Let $\Delta = \begin{vmatrix} M_n & F_n & 1 \\ M_{n+1} & F_{n+1} & 1 \\ M_{n+2} & F_{n+2} & 1 \end{vmatrix}$

Suppose $M_n = a,\ M_{n+1} = b,\ M_{n+2} = a + b$

$F_n = p,\ F_{n+1} = q,\ F_{n+2} = \text{p + q}$

Now $$\Delta = \begin{vmatrix} a & p & 1 \\ b & q & 1 \\ a+b & p+q & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$

$$\Delta = \begin{vmatrix} a-b & p-q & 0 \\ b & q & 1 \\ a+b & p+q & 1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$

$$\Delta = \begin{vmatrix} a-b & p-q & 0 \\ b-(a+b) & q-(p+q) & 0 \\ a+b & p+q & 1 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} a-b & p-q & 0 \\ -a & -p & 0 \\ a+b & p+q & 1 \end{vmatrix}$$

$$= [pb - aq]$$

$$= [F_n\, M_{n+1} - M_n\, F_{n+1}]$$

Theorem 5.4: Let n be a positive integer then

$$\begin{vmatrix} M_n & L_n & 1 \\ M_{n+1} & L_{n+1} & 1 \\ M_{n+2} & L_{n+2} & 1 \end{vmatrix} = [L_n\, M_{n+1} - L_{n+1}\, M_n]$$

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