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THE SIGN SYMMETRIC P0-MATRIX COMPLETIONPROBLEM

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ABSTRACT

In this paper, we study completions for sign symmetric P_0-matrices. We obtained that digraphs that include all loops and have sign symmetric P_0-completion are null graphs and complete digraphs.

Key Words: *Matrix completion, digraphs, partial matrix, sign symmetric P_0-matrix.*

1. INTRODUCTION

In this section we discuss related work leading to this research, full definitions of terms are given in the next section.

Choi *et al* [1] studied "The P_0-Matrix Completion Problem", in 2002. In 2003, research was conducted on the nonnegative P_0-matrix completion problem in [2] and in the same years another was done on the weakly sign symmetric P_0-matrix completion problem in [3]. Figure 1.1 shows completion problems for various classes of P_0-matrices.

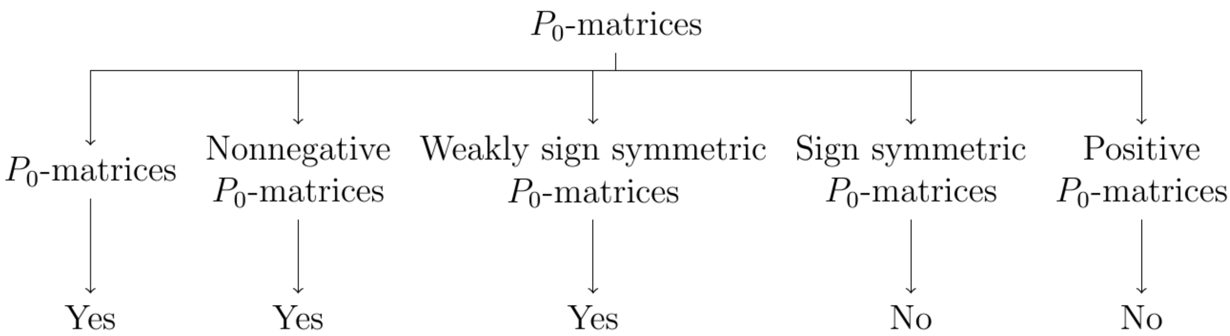

Figure 1.1: Completion problems for various classes of P_0-matrices.

In Figure 1.1, "Yes" means it has been researched and complete classifications of digraphs up to order 4 was given, and "No" means there has been little attention given on the class. In this research we are interested in the class of sign symmetric P_0-matrices.

2. PRELIMINARIES

Definition 2.1: A **P_0-matrix A** is a matrix in which every principal minor of the matrix A is nonnegative [1].

Definition 2.2: A **positive (nonnegative) P_0-matrix** is a P_0-matrix in which all entries are positive (nonnegative).

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Definition 2.3: A P_0-matrix A is called a **weakly sign symmetric P_0 -matrix (resp. sign symmetric P_0 -matrix)** if $a_{ij}a_{ji} \geq 0$ (resp. either $a_{ij}a_{ji} > 0$ or $a_{ij} = 0 = a_{ji}$) for all i and j .

Example 2.4: Consider the following matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Matrix A, B and C are P_0-matrices since every principal minor for each matrix is nonnegative.

Next, let us consider the sign conditions on the entries: matrix B is a nonnegative P_0-matrix, matrix B and C are weakly sign symmetric P_0-matrices product of twin entries is nonnegative and finally, observe that matrix B is both sign symmetric P_0-matrix and positive P_0-matrix.

Definition 2.5: A $P_{0,1}$**-matrix** is a P_0-matrix whose diagonal entries are positive, A **positive** $P_{0,1}$**-matrix** is a $P_{0,1}$ -matrix in which all entries are positive [5].

Proposition 2.6 A matrix is a positive $P_{0,1}$ -matrix if and only if it is positive P_0-matrix

Proof: From Definition 2.5, a positive $P_{0,1}$ -matrix is a $P_{0,1}$ -matrix in which all entries are positive meaning the condition that all diagonal entries are positive has been stated and therefore it is also correct to say a positive $P_{0,1}$-matrix is a P_0 -matrix in which all entries are positive or simply is a positive P_0 -matrix.

Conversely, a positive P_0-matrix is a P_0-matrix in which all entries are positive hence all diagonal entries are positive, therefore it also correct to say it a positive $P_{0,1}$ -matrix.

Definition 2.7: A **partial matrix** is a matrix in which some entries are specified while others are free to be chosen (from a certain set). A partial matrix is a **partial sign symmetric P_0-matrix** if determinants of all fully specified principal sub-matrices are nonnegative and either $a_{ij}a_{ji} > 0$ or $a_{ij} = 0 = a_{ji}$ for all i and j where a_{ij} and a_{ji} are specified [1, 2].

Graph theoretic approach will be used in completing these partial matrices, and some definitions are given as follows.

Definition 2.8: A graph $G = (V_G, E_G)$ is a finite non-empty set of positive integers V_G, whose members are called vertices and a set of E_G (unordered) pairs $\{u, v\}$ of vertices called edges of G. A graph whose edge-set is empty is a null graph [4].

Definition 2.9: A digraph $D = (V_D, E_D)$ is a graph G with ordered pairs (u, v) of vertices and arc where u the initial vertex is and v is the terminal vertex. The order of a digraph D denoted n is the number of vertices of D. A digraph is complete digraph if it includes all possible arcs between its vertices [4]. If an arc exist between vertices i and j then we say position (i, j) of the partial matrix is specified.

Remark 2.10: Whenever all the diagonal entries are unspecified then complete the partial matrix by assigning sufficiently large values, problems arise when some diagonal entries are unspecified while others are specified. If we consider a partial sign symmetric P_0-matrix $A = \begin{bmatrix} d_1 & 1 \\ 1 & 0 \end{bmatrix}$, it is impossible to complete since $\det(A) = -1$ for any values of d_i .

From Remark 2.10, we consider digraphs that have all loops specified and therefore partial matrices considered have all diagonal entries specified.

Definition 2.11: A **completion** of a partial matrix is a specific choice of values for the unspecified entries [5]. A zero **completion** is a completion of partial matrix in which all unspecified entries are assigned zeros. A digraph has **sign symmetric P_0-matrix completion** if every partial sign symmetric P_0-matrix that specifies the digraph can be completed to a sign symmetric P_0-matrix.

3. MAIN RESULTS

Theorem 3.1: A null graph of order n has a sign symmetric P_0-completion.

Proof: Consider a partial sign symmetric P_0-matrix A specifying a null digraph of order n. Perform zero completion to A, by assigning all off-diagonal positions to zeros that is $x_{ij} = 0$ for all $i \neq j$. Since all diagonal positions are nonnegative that is $d_i \geq 0$ for $i = 1,\dots,n$ then the completed matrix is a diagonal matrix which is also sign symmetric P_0-matrix A_c due to the fact that the determinant of every principal sub-matrix is nonnegative that is $\det A_c(\alpha) \geq 0$ for $\alpha \subseteq \{1,\dots,n\}$. Therefore, a null graph of order n has a sign symmetric P_0-completion.

Example 3.2 verifies the theorem above.

Example 3.2

Consider a partial sign symmetric P_0-matrix $A = \begin{bmatrix} d_1 & x_{12} & x_{13} \\ x_{21} & d_2 & x_{23} \\ x_{31} & x_{32} & d_3 \end{bmatrix}$ specifying a null digraph of order 3.

Setting all the unspecified entry to zero that is $x_{12} = x_{21} = x_{13} = x_{31} = x_{23} = x_{32} = 0$ gives:

A completed matrix $A_c = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$ of partial matrix A, which can be confirmed by the following calculations

$$\det A_c(i) = d_i \geq 0 \text{ for } i = 1,2,3., \det A_c(1,2) = d_1 d_2 \geq 0,$$

$$\det A_c(1,3) = d_1 d_3 \geq 0, \ \det A_c(2,3) = d_2 d_3 \geq 0, \ \det A_c = d_1 d_2 d_3 \geq 0.$$

Since all the principal minors are nonnegative, then completed matrix A_c is a sign symmetric P_0-matrix.

Therefore, a null digraph of order 3 has sign symmetric P_0-completion.

The next theorem gives other set of digraphs that have sign symmetric P_0-completion.

Theorem 3.3: If a digraph D is not a null digraph and has a sign symmetric P_0-completion then it is a complete digraph.

Proof: Let a $n \times n$ matrix A be a partial sign symmetric P_0-matrix with all diagonal entries equal to zero that is $d_i = 0$ for $i = 1,\dots,n$. Since digraph D is not a null digraph then at least one entry is specified, assume the $(i,j)-$ position is specified $a_{ij} > 0$ and $(j,i)-$ position is unspecified x_{ji} then it is impossible to assign values to x_{ji} such that $c_{ij} x_{ji} > 0$ because $\det A_c(i,j) = -c_{ij} x_{ji} < 0$ for any $x_{ji} > 0$. This implies that if $(i,j)-$ position is specified then $(j,i)-$ position must be specified hence digraph D specified by $n \times n$ matrix A is a complete digraph.

The following counter example is used for more understanding of Theorem 3.3

Example 3.4

Consider a partial sign symmetric P_0-matrix $A = \begin{bmatrix} 0 & 1 \\ x & 0 \end{bmatrix}$, but A does not have sign symmetric P_0-completion since it is impossible to assign a value to x such that $x > 0$ and $\det A \geq 0$.

CONCLUSION AND RECOMMENDATIONS

In this paper we have shown that digraphs that include all loops and have sign symmetric P_0-completion are null graphs and complete digraphs. It is recommended that similar research be done on the positive P_0-matrix completion.

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