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CODING THEORY ON COMPANION MATRIX FOR THE FIBONACCI λ-NUMBERS

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ABSTRACT

K.Kuhapatnakul in [10], introduced companion matrix of the Fibonacci λ-numbers. In this paper showed the relation between companion matrices and Fibonacci matrices. We established the relations among the code matrix elements error detection and correction for this coding theory. Correction ability of this method is 93.33% for λ=1 and for λ=2 the correction ability is 99.8, In general correction ability of this method increases as λ increases.

Keywords: *companion matrices, Fibonacci matrices, Fibonacci λ-numbers.*

Mathematics Subject Classification 2010: *11B39, 94B35, 94B25, 11T71.*

1. INTRODUCTION

In the last decades the theory of Fibonacci numbers was complemented by the theory of the so-called Fibonacci Q-matrix [2]. This 2×2 square matrix is defined as $Q=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$.The k^{th} power of the Q-matrix can be defined as $Q^k=\begin{pmatrix}F_{k+1} & F_k\\ F_k & F_{k-1}\end{pmatrix}$ and det $Q^k = F_{k+1}F_{k-1} - {F_k}^2 = (-1)^k$ where $k = 0, \pm1, \pm2, \cdots$, and$F_k$ is k^{th} Fibonacci number and recurrence relation $F_{k+1}=F_k + F_{k-1}$ is called "Cassini formula" with terms the initial $F_1 = F_2 = 1$.

The Fibonacci λ-numbers can be defined as given integer λ=0, 1, 2, 3, 4….. And consider the following recurrence relation can be defined as $F_\lambda(k)= F_\lambda(k-1)+ F_\lambda(k-\lambda-1)$ with $k > \lambda+1$ with terms the initial $F_\lambda(1) = F_\lambda(2) = F_\lambda(\lambda) = F_\lambda(\lambda+1) = 1$.It generates an infinite number of recurrent sequences. In particular for $\lambda = 0$ then recurrence relation $F_0(k)= F_0(k-1)+ F_0(k-1)$ with n>1 with term the initial $F_0(1) = 1$.the recurrence relation generates the binary numbers:1,2,4,8,16,….2^{k-1}.., particular for $\lambda = 1$ then recurrence relation $F_1(k)= F_1(k-1)+ F_1(k-2)$ with n>2with term the initial $F_1(1) = F_1(2) = 1$. This recurrence relation "generates" the classical Fibonacci numbers: 0,1,1,2,3,5,8,13,21,34 ...

1.1. Some properties of Fibonacci λ-numbers:

1. $F_\lambda(0) = F_\lambda(-1) = F_\lambda(-2) = \cdots = F_\lambda(-\lambda+1) = F_\lambda(-\lambda-1) = F_\lambda(-\lambda-2) = \cdots = F_\lambda(-2\lambda+1) = 0$
2. $F_\lambda(-\lambda) = 1$ and $F_\lambda(-k) = (-1)^{-k+1}F_\lambda(k)$ with $\lambda > k$

Table-1: The "expanded" Fibonacci λ-numbers

k	5	4	3	2	1	0	-1	-2	-3	-4	-5
$F_1(k)$	5	3	2	1	1	0	1	-1	2	-3	5
$F_2(k)$	3	2	1	1	1	0	0	1	0	-1	1
$F_3(k)$	2	1	1	1	1	0	0	0	1	0	0
$F_4(k)$	1	1	1	1	1	0	0	0	0	1	0
$F_5(k)$	1	1	1	1	1	0	0	0	0	0	1

The following property for Fibonacci - λ numbers is proved in [3]

$$F_\lambda(1) + F_\lambda(2) + \cdots\cdots\cdots + F_\lambda(k) = F_p(k+\lambda+1) - 1 \quad (1)$$

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In case λ =0 in (1) reduces to the following well-known formula for the binary numbers

$$2^0 + 2^1 + 2^2 + \cdots\cdots\cdots + 2^{k-1} = 2^k - 1$$

$\mathrm{F_n}$ is known as classical Fibonacci numbers, Therefore the identity (1) becomes the following formula

$$F_1 + F_1 + \cdots\cdots\cdots + F_{\mathrm{k}} = F_{\mathrm{k+2}} - 1$$

is well known from the Fibonacci number theory[1]

Lemma: Let n≥ 1, $and\ \ k \geq 2\ be\ two\ integers\ then\ \ F_\lambda(n\lambda + k) - F_\lambda(n\lambda + 1) = \sum_{i=1}^{k-1} F_\lambda(n\lambda - \lambda + 1)$
In [10] it was proved

2. THE GENERALIZED FIBONACCI Q_λ –MATRICES

In [4], the generalized Fibonacci matrices can be defined as

$$Q_\lambda = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{2}$$

Where Det Q_λ=$(-1)^\lambda$, for given $\lambda = 0,1,2,3, \ldots..$ and $Q_\lambda -$ matrices of order $\lambda + 1$
$Q_\lambda -$ Matrices have following forms for $\lambda = 0,1,2,3, \ldots.$: Q_0=(1),

$$Q_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, Q_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\ Q_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

3. THE GENERALIZED FIBONACCI ${Q_\lambda}^{\boldsymbol{k}}$ –MATRICES

In [4], the generalized Fibonacci matrices can be defined as follows

$${Q_\lambda}^{\boldsymbol{k}} = \begin{pmatrix} F_\lambda(k+1) & F_\lambda(k) & \cdots & F_\lambda(k-\lambda+2) & F_\lambda(k-\lambda+1) \\ F_\lambda(k-\lambda+1) & F_\lambda(k-\lambda) & \cdots & F_\lambda(k-2\lambda+2) & F_\lambda(k-2\lambda+1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_\lambda(k-1) & F_\lambda(k-2) & \cdots & F_\lambda(k-\lambda) & F_\lambda(k-\lambda-1) \\ F_\lambda(k) & F_\lambda(k-1) & \cdots & F_\lambda(k-\lambda+1) & F_\lambda(k-\lambda) \end{pmatrix} \tag{4}$$

Where F_λ(k) is Fibonacci λ-number, k = 0,±1,±2,±3, ±4… and for given $\lambda = 0,1,2,3$ and

$$\text{Det } {Q_\lambda}^{\boldsymbol{k}} = (\text{Det } Q_\lambda)^k = (-1)^{\lambda\mathrm{k}} \quad \text{from the matrix theory [1, 8]} \tag{5}$$

Properties of generalized Fibonacci${Q_\lambda}^{\boldsymbol{k}} -$ matrices are ${Q_\lambda}^{\boldsymbol{k}} \times {Q_\lambda}^{\boldsymbol{l}} = {Q_\lambda}^{\boldsymbol{l}} \times {Q_\lambda}^{\boldsymbol{k}} = {Q_\lambda}^{\boldsymbol{k+l}}$ and ${Q_\lambda}^{\boldsymbol{k}} = {Q_\lambda}^{\boldsymbol{k-1}} + {Q_\lambda}^{\boldsymbol{k}-\lambda-1}$ for k = 0,±1,±2,±3, ±4… and for given $\lambda = 0,1,2,3 \ldots..$

4. THE GENERALIZED "CASSINI FORMULA"

For a given integers λ =0, 1, 2, 3 then the generalized "Cassini formula" [1] can be defined as

$${Q_\lambda}^{\boldsymbol{k}} = {Q_\lambda}^{\boldsymbol{k-1}} + {Q_\lambda}^{\boldsymbol{k}-\lambda-1} \tag{6}$$

For $\lambda = 1$ then "Cassini formula" can be represent as a recurrence relation

$$F_1(k) = F_1(k-1) + F_1(k-2) \text{ with n>2with term the initial } F_1(1) = F_1(2) = 1 \tag{7}$$

The Fibonacci ${Q_1}^k$ –matrix can be represent as

$${Q_1}^k = \begin{pmatrix} F_1(k+1) & F_1(k) \\ F_1(k) & F_1(k-1) \end{pmatrix} \text{ where } k = 0, \pm1, \pm2, \pm3, \pm4\ldots \tag{8}$$

$$\text{Det } {Q_1}^{\boldsymbol{k}} = (\text{Det } Q_1)^k = (-1)^{\mathrm{k}} \tag{9}$$

Table-2: Represents the "direct matrices${Q_1}^k$" and their "inverse matrices${Q_1}^{-k}$"

k	0	1	2	3	4
${Q_1}^k$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$
${Q_1}^{-k}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$	$\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$

5. THE COMPANION MATRIX FOR THE FIBONACCI $\lambda-$ numbers

In [10] it was introduced Fibonacci $\lambda-$ numbersdesingned as H_λ $-$matrix as companion matrix $\lambda=0,1,2,3\ \ldots.$

$$H_\lambda=\begin{pmatrix}1&1&1&1&\cdots&1&1\\1&0&0&0&\cdots&0&0\\1&1&0&0&\cdots&0&0\\\vdots&\vdots&\ddots&\ddots&\vdots&\vdots&\vdots\\1&1&1&1&\cdots&0&0\\1&1&1&1&\cdots&0&0\\1&1&1&1&\cdots&1&0\end{pmatrix} \tag{10}$$

Theorem 1: For $given\ integer\ \lambda=0,1,2,3\ \ldots$then Det ($H_\lambda$)= $(-1)^{\lambda^2}$ (11)

Proof: We have $H_\lambda=(Q_\lambda)^\lambda$
Since Det Q_λ=$(-1)^\lambda$
Det (H_λ)= (Det Q_λ $)^\lambda=(-1)^{\lambda^2}$

Theorem 2: For $given\ integer\ \lambda=0,1,2,3\ \ldots$we have

$$H_\lambda{}^{\boldsymbol{n}}=\begin{pmatrix}F_\lambda(n\lambda+1)&F_\lambda(n\lambda)&\cdots&F_\lambda(n\lambda-\lambda+2)&F_\lambda(n\lambda-\lambda+1)\\F_\lambda(n\lambda-\lambda+1)&F_\lambda(n\lambda-\lambda)&\cdots&F_\lambda(n\lambda-2\lambda+2)&F_\lambda(n\lambda-2\lambda+1)\\\vdots&\vdots&\vdots&\vdots&\vdots\\F_\lambda(n\lambda-1)&F_\lambda(n\lambda-2)&\cdots&F_\lambda(n\lambda-\lambda)&F_\lambda(n\lambda-\lambda-1)\\F_\lambda(n\lambda)&F_\lambda(n\lambda-1)&\cdots&F_\lambda(n\lambda-\lambda+1)&F_\lambda(n\lambda-\lambda)\end{pmatrix} \tag{12}$$

where $F_\lambda(n\lambda+k)=F_\lambda(n\lambda+1)+\sum_{i=1}^{k-1}F_\lambda(n\lambda-\lambda+1)$, n=$\pm1,\pm2,\pm3,\pm4$ ....
k = 2, 3, 4,…. and It was proved in [10]

Theorem 3: For $given\ integer\ \lambda=0,1,2,3\ \ldots.$and n=$\pm1,\pm2,\pm3,\pm4$ ....
then $H_\lambda{}^{\boldsymbol{n}}=Q_\lambda{}^{n\lambda}$ and Det $H_\lambda{}^{\boldsymbol{n}}=(-1)^{n\lambda^2}$ (13)

Case (i): if $\lambda=1$ then $H_1{}^{\boldsymbol{n}}=Q_1{}^{n}$=$\begin{pmatrix}F_1(n+1)&F_1(n)\\F_1(n)&F_1(n-1)\end{pmatrix}$=$\begin{pmatrix}1&1\\1&0\end{pmatrix}^n$

Det $H_1{}^{\boldsymbol{n}}$ =Det $Q_1{}^{\boldsymbol{n}}$ = (Det $Q_1)^n$ = $(-1)^n$ where n=$\pm2,\pm3,\pm4$ ....

Case (ii): if $\lambda=2$ then $H_2{}^{\boldsymbol{n}}=Q_2{}^{2n}=\begin{pmatrix}F_2(2n+1)&F_2(2n)&F_2(2n-1)\\F_2(2n-1)&F_2(2n-2)&F_2(2n-3)\\F_2(2n)&F_2(2n-1)&F_2(2n-2)\end{pmatrix}$

$$=\begin{pmatrix}1&1&1\\1&0&0\\1&1&0\end{pmatrix}^n$$

Det $H_2{}^{\boldsymbol{n}}$ =Det $Q_2{}^{\boldsymbol{2n}}$ = (Det $Q_2)^{2n}$ = 1 where n=$\pm2,\pm3,\pm4$ .... (14)

Continue this process we can generate matrices $H_\lambda{}^{\boldsymbol{n}}$ **fo**r λ =3, 4, 5, 6… and n=0, ±1, ±2, ±3, ±4 ....

6. PROPERTIES

1. $H_\lambda{}^{\boldsymbol{n}}H_\lambda{}^{\boldsymbol{m}}=H_\lambda{}^{\boldsymbol{m}}H_\lambda{}^{\boldsymbol{n}}=H_\lambda{}^{\boldsymbol{n+m}}$ (15)
   $since\ H_\lambda{}^{\boldsymbol{n}}=Q_\lambda{}^{n\lambda},H_\lambda{}^{\boldsymbol{m}}=Q_\lambda{}^{m\lambda}$
   $H_\lambda{}^{\boldsymbol{n}}H_\lambda{}^{\boldsymbol{m}}=Q_\lambda{}^{n\lambda}Q_\lambda{}^{m\lambda}=Q_\lambda{}^{(n+m)\lambda}$=$H_\lambda{}^{\boldsymbol{n+m}}$
2. $H_\lambda{}^{\boldsymbol{n}}=H_\lambda{}^{\boldsymbol{n-1}}+H_\lambda{}^{\boldsymbol{n-p-1}}$ (16)
   $H_\lambda{}^{\boldsymbol{n-1}}+H_\lambda{}^{\boldsymbol{n-p-1}}=Q_\lambda{}^{(n-1)\lambda}+Q_\lambda{}^{(n-\lambda-1)\lambda}=Q_\lambda{}^{n\lambda}$=$H_\lambda{}^{\boldsymbol{n}}$
3. Det ($H_\lambda{}^{\boldsymbol{n}}$)= $(Det\ H_\lambda)$ ^{n}=$(-1)^{n\lambda^2}$ (17)
   Det ($H_\lambda{}^{\boldsymbol{n}}$) = Det ($Q_\lambda{}^{\boldsymbol{n\lambda}}$) = $Det\ (Q_\lambda{}^{\boldsymbol{n}})^\lambda=(-1)^{n\lambda^2}$
   Det ($H_\lambda{}^{\boldsymbol{n}}$) = $(-1)^{n\lambda^2}=((-1)^{\lambda^2})^n=(Det\ H_\lambda)^n$

7. The Fibonaccicoding and decoding method

7.1. Determinant of the code matrix E:

The code matrix E is defined by the following formula E=M×($H_\lambda{}^{\boldsymbol{n}}$)
According to the matrix theory [5].

We have Det E=Det M× Det ($H_\lambda{}^{\boldsymbol{n}}$)= Det M× $(-1)^{n\lambda^2}$ (18)

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7.2. Relations among the code matrix elements for $\lambda = 1$

We can write the code matrix E and the initial message matrix M as following

$$E = \mathrm{M} \times ({H_1}^{n}) = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \begin{pmatrix} F_1(n+1) & F_1(n) \\ F_1(n) & F_1(n-1) \end{pmatrix} = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}$$

and $$\mathrm{M} = \mathrm{E} \times ({H_1}^{-n}) = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} \begin{pmatrix} -F_1(n-1) & F_1(n) \\ F_1(n) & -F_1(n+1) \end{pmatrix} = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$$

Since m_1, m_2, m_3, m_4 *are positive integers*

$$m_1 = -e_1F_1(n-1) + e_2F_1(n) > 0 \tag{19}$$
$$m_2 = e_1F_1(n) - e_2F_1(n+1) > 0 \tag{20}$$
$$m_3 = -e_3F_1(n-1) + e_4F_1(n) > 0 \tag{21}$$
$$m_4 = e_3F_1(n) - e_4F_1(n+1) > 0 \tag{22}$$

From (19) & (20), we get

$$\frac{F_1(n+1)}{F_1(n)} < \frac{e_1}{e_2} < \frac{F_1(n)}{F_1(n-1)} \tag{23}$$

From (21) & (22) ,we get

$$\frac{F_1(n+1}{F_1(n)} < \frac{e_3}{e_4} < \frac{F_1(n)}{F_1(n-1)} \tag{24}$$

From the inequalities (23) and (24), we obtain

$$\frac{e_1}{e_2} \approx \tau, \frac{e_3}{e_4} \approx \tau, \text{ Where } \tau = \frac{1+\sqrt{5}}{2} \tag{25}$$

For n=2k we obtain the similar relations given (25)

7.3. Relations among the code matrix elements for $\lambda = 2$

In this paper, we have developed relations among the code matrix elements for $\lambda = 2$

We can write the code matrix E and initial message M as following

$$E = \mathrm{M} \times ({H_2}^{n}) = \begin{pmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{pmatrix} \begin{pmatrix} F_2(2n+1) & F_2(2n) & F_2(2n-1) \\ F_2(2n-1) & F_2(2n-2) & F_2(2n-3) \\ F_2(2n) & F_2(2n-1) & F_2(2n-2) \end{pmatrix} = \begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix}$$

and $$\mathrm{M} = E \times ({H_2}^{-n}) = \begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix} \begin{pmatrix} F_2(2n+1) & F_2(2n) & F_2(2n-1) \\ F_2(2n-1) & F_2(2n-2) & F_2(2n-3) \\ F_2(2n) & F_2(2n-1) & F_2(2n-2) \end{pmatrix}^{-1} = \begin{pmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{pmatrix}$$

$$\mathrm{M} = \begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix} \times$$
$$\begin{pmatrix} \mathbf{F_2}^2(\mathbf{2n-2}) - \mathbf{F_2(2n-1)F_2(2n-3)} & \mathbf{F_2}^2(\mathbf{2n-1}) - \mathbf{F_2(2n)F_2(2n-2)} & \mathbf{F_2(2n)F_2(2n-3) - F_2(2n-1)F_2(2n-1)} \\ \mathbf{F_2(2n)F_2(2n-3) - F_2(2n-1)F_2(2n-1)} & \mathbf{F_2(2n+1)F_2(2n-2) - F_2(2n)F_2(2n-1)} & \mathbf{F_2}^2(\mathbf{2n-1}) - \mathbf{F_2(2n+1)F_2(2n-3)} \\ \mathbf{F_2}^2(\mathbf{2n-1}) - \mathbf{F_2(2n)F_2(2n-2)} & \mathbf{F_2}^2(\mathbf{2n}) - \mathbf{F_2(2n+1)F_2(2n-1)} & \mathbf{F_2(2n+1)F_2(2n-2) - F_2(2n)F_2(2n-1)} \end{pmatrix}$$

Since $m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8, m_9$ *are positive integers*

$$\mathrm{m}_1 = \mathrm{e}_1\left[\mathbf{F_2}^2(\mathbf{2n-2}) - \mathbf{F_2(2n-1)F_2(2n-3)}\right] + \mathrm{e}_2[\mathbf{F_2(2n)F_2(2n-3) - F_2(2n-1)F_2(2n-1)}] + \mathrm{e}_3\left[\mathbf{F_2}^2(\mathbf{2n-1}) - \mathbf{F_2(2n)F_2(2n-2)}\right] > 0 \tag{26}$$

$$\mathrm{m}_2 = \mathrm{e}_1\left[\mathbf{F_2}^2(\mathbf{2n-1}) - \mathbf{F_2(2n)F_2(2n-2)}\right] + \mathrm{e}_2[\mathbf{F_2(2n+1)F_2(2n-2) - F_2(2n)F_2(2n-1)}] + \mathrm{e}_3\left[\mathbf{F_2}^2(\mathbf{2n}) - \mathbf{F_2(2n+1)F_2(2n-1)}\right] > 0 \tag{27}$$

$$\mathrm{m}_3 = \mathrm{e}_1[\mathbf{F_2(2n)F_2(2n-3) - F_2(2n-1)F_2(2n-2)}] + \mathrm{e}_2\mathbf{F_2}^2(\mathbf{2n-1}) - \mathbf{F_2(2n+1)F_2(2n-3)}] + \mathrm{e}_3[\mathbf{F_2(2n+1)F_2(2n-2) - F_2(2n)F_2(2n-1)}] > 0 \tag{28}$$

$$\mathrm{m}_4 = \mathrm{e}_4\left[\mathbf{F_2}^2(\mathbf{2n-2}) - \mathbf{F_2(2n-1)F_2(2n-3)}\right] + \mathrm{e}_5[\mathbf{F_2(2n)F_2(2n-3) - F_2(2n-1)F_2(2n-1)}] + \mathrm{e}_6\left[\mathbf{F_2}^2(\mathbf{2n-1}) - \mathbf{F_2(2n)F_2(2n-2)}\right] > 0 \tag{29}$$

$$\mathrm{m}_5 = \mathrm{e}_4\left[\mathbf{F_2}^2(\mathbf{2n-1}) - \mathbf{F_2(2n)F_2(2n-2)}\right] + \mathrm{e}_5[[\mathbf{F_2(2n+1)F_2(2n-2) - F_2(2n)F_2(2n-1)}] + \mathrm{e}_6\left[\mathbf{F_2}^2(\mathbf{2n}) - \mathbf{F_2(2n+1)F_2(2n-1)}\right] > 0 \tag{30}$$

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$$m_6 = e_4[\mathbf{F_2(2n)F_2(2n-3) - F_2(2n-1)F_2(2n-2)}] + e_5\mathbf{F_2}^2\mathbf{(2n-1) - F_2(2n+1)F_2(2n-3)}] \\ +e_6[\mathbf{F_2(2n+1)F_2(2n-2) - F_2(2n)F_2(2n-1)}] > 0 \quad (31)$$

$$m_7 = e_7\left[\boldsymbol{F_2}^2\boldsymbol{(2n-2) - F_2(2n-1)F_2(2n-3)}\right] + e_8[\boldsymbol{F_2(2n)F_2(2n-3) - F_2(2n-1)F_2(2n-1)}] \\ + e_9\left[\boldsymbol{F_2}^2\boldsymbol{(2n-1) - F_2(2n)F_2(2n-2)}\right] > 0 \quad (32)$$

$$m_8 = e_7\left[\mathbf{F_2}^2\mathbf{(2n-1) - F_2(2n)F_2(2n-2)}\right] + e_8[\mathbf{F_2(2n+1)F_2(2n-2) - F_2(2n)F_2(2n-1)}] \\ + e_9\left[\mathbf{F_2}^2\mathbf{(2n) - F_2(2n+1)F_2(2n-1)}\right] > 0 \quad (33)$$

$$m_9 = e_7[F_2(2n)F_2(2n-3) - F_2(2n-1)F_2(2n-2)] + e_8\ [F_2{}^2(2n-1) - F_2(2n+1)F_2(2n-3)] \\ + e_9[F_2(2n+1)F_2(2n-2) - F_2(2n)F_2(2n-1)] > 0 \quad (34)$$

From (26)

$$\mathbf{e_1}\left[\mathbf{F_2}^2\mathbf{(2n-2)}\right] + \mathbf{e_2}[\mathbf{F_2(2n)F_2(2n-3)}] + \mathbf{e_3}\left[\mathbf{F_2}^2\mathbf{(2n-1)}\right] > \\ \mathbf{e_1}[\mathbf{F_2(2n-1)F_2(2n-3)}] + \mathbf{e_2}[\mathbf{F_2(2n-1)F_2(2n-1)}] + \mathbf{e_3}[\ \mathbf{F_2(2n)F_2(2n-2)}] \quad (35)$$

From (27)

$$\mathbf{e_1}\left[\mathbf{F_2}^2\mathbf{(2n-1)}\right] + \mathbf{e_2}[\mathbf{F_2(2n+1)F_2(2n-2)}] + \ \mathbf{e_3}\left[\mathbf{F_2}^2\mathbf{(2n)}\right] > \\ \mathbf{e_1F_2(2n)F_2(2n-2)}]+\mathbf{e_2F_2(2n)F_2(2n-1)}]+\mathbf{e_3}\ \mathbf{F_2(2n+1)F_2(2n-1)} \quad (36)$$

From (28)

$$\mathbf{e_1}[\mathbf{F_2(2n)F_2(2n-3)}] + \mathbf{e_2F_2}^2\mathbf{(2n-1)}] + \ \mathbf{e_3}[\mathbf{F_2(2n+1)F_2(2n-2)}]\ > \\ \mathbf{e_1F_2(2n-1)F_2(2n-1)} + \mathbf{e_2F_2(2n+1)F_2(2n-3)} + \mathbf{e_3F_2(2n)F_2(2n-1)} \quad (37)$$

Dividing both sides by $e_1[F_2(2n)F_2(2n-3)]$ *of* (35), $e_1F_2(2n)F_2(2n-2)$ *of* (36), $e_1F_2(2n-1)F_2(2n-2)$ *of* (37)

Therefore we get

$$\frac{\mathbf{e_3}}{\mathbf{e_1}}\left[\mathbf{F_2}^2\mathbf{(2n-1) - F_2(2n-2)F_2(2n)}\right] > \frac{\mathbf{e_2}}{\mathbf{e_1}}[\mathbf{F_2(2n-2)F_2(2n-1) - F_2(2n)F_2(2n-3)}] \\ +\left[\mathbf{F_2(2n-1)F_2(2n-3) - F_2}^2\mathbf{(2n-2)}\right] \quad (38)$$

$$\frac{\mathbf{e_3}}{\mathbf{e_1}}\left[\mathbf{F_2(2n-1)F_2(2n+1) - F_2}^2\mathbf{(2n)}\right] < \frac{\mathbf{e_2}}{\mathbf{e_1}}[\mathbf{F_2(2n-2)F_2(2n+1) - F_2(2n)F_2(2n-1)}] \\ +\left[\mathbf{F_2}^2\mathbf{(2n-1) - F_2(2n)F_2(2n-2)}\right] \quad (39)$$

$$\frac{\mathbf{e_3}}{\mathbf{e_1}}[\mathbf{F_2(2n+1)F_2(2n-2) - F_2(2n)F_2(2n-1)}] > \frac{\mathbf{e_2}}{\mathbf{e_1}}\left[\mathbf{F_2(2n+1)F_2(2n-3) - F_2}^2\mathbf{(2n-1)}\right] \\ +[\mathbf{F_2(2n+1)F_2(2n-2) - F_2(2n)F_2(2n-3)}] \quad (40)$$

Let A=$\boldsymbol{F_2}^2\boldsymbol{(2n-1) - F_2(2n-2)F_2(2n)}$**,**

Let **B** = $\boldsymbol{F_2(2n-1)F_2(2n+1) - F_2}^2\boldsymbol{(2n)}$,

Let $\boldsymbol{C = F_2(2n+1)F_2(2n-2) - F_2(2n)F_2(2n-1)}$

Case (1): If A > 0, B > 0, C > 0 then $\frac{e_3}{e_1} > U$ since from (38)

where $U=\frac{e_2}{e_1}\left[\frac{F_2(2n-2)F_2(2n-1)-F_2(2n)F_2(2n-3)}{F_2{}^2(2n-1)-F_2(2n-2)F_2(2n)}\right]+\left[\frac{F_2(2n-1)F_2(2n-3)-F_2{}^2(2n-2)}{F_2{}^2(2n-1)-F_2(2n-2)F_2(2n)}\right]$ (41)

Since from (39) $\frac{e_3}{e_1} < V$, Where $V=\frac{e_2}{e_1}\left[\frac{F_2(2n-2)F_2(2n+1)-F_2(2n)F_2(2n-1)}{F_2(2n-1)F_2(2n+1)-F_2{}^2(2n)}\right]+\left[\frac{F_2{}^2(2n-1)-F_2(2n)F_2(2n-2)}{F_2(2n-1)F_2(2n+1)-F_2{}^2(2n)}\right]$ (42)

From (41) and (42)

$$\frac{e_1}{e_2} > \frac{F_2(2n)}{F_2(2n-1)} \quad (43)$$

From (25) $\frac{e_3}{e_1} < W$,

Where $W = \frac{e_2}{e_1}\left[\frac{F_2(2n-3)F_2(2n+1)-F_2{}^2(2n-1)}{F_2(2n-2)F_2(2n+1)-F_2(2n)F_2(2n-1)}\right]+\left[\frac{F_2(2n-2)F_2(2n-2)-F_2(2n)F_2(2n-3)}{F_2(2n+1)F_2(2n-2)-F_2(2n)F_2(2n-1)}\right]$ (44)

From (40) and (42)

$$\frac{e_1}{e_2} < \frac{F_2(2n+1)}{F_2(2n)}$$

Therefore $\frac{F_2(2n)}{F_2(2n-1)} < \frac{e_1}{e_2} < \frac{F_2(2n+1)}{F_2(2n)}$ (45)

Similar fashion, we get $\frac{F_2(2n)}{F_2(2n-1)} < \frac{e_2}{e_3} < \frac{F_2(2n+1)}{F_2(2n)}, \frac{F_2(2n)}{F_2(2n-1)} < \frac{e_1}{e_3} < \frac{F_2(2n+1)}{F_2(2n)}$ (46)

Case (2): If A=0,B>0,C>0

$\frac{e_1}{e_2} > \frac{F_2(2n)}{F_2(2n-1)}$ Using A=0 (47)

From (37) and (38)

$\frac{e_1}{e_2} < \frac{F_2(2n+1)}{F_2(2n)}$ Using A=0 (48)

From (44) and (45)

Therefore $\frac{F_2(2n)}{F_2(2n-1)} < \frac{e_1}{e_2} < \frac{F_2(2n+1)}{F_2(2n)}$ (49)

Case (3): If A<0,B<0,C<0 then $\frac{e_3}{e_1} < U$,

where U=$\frac{e_2}{e_1}\left[\frac{F_2(2n-2)F_2(2n-1)-F_2(2n)F_2(2n-3)}{F_2{}^2(2n-1)-F_2(2n-2)F_2(2n)}\right]+\left[\frac{F_2(2n-1)F_2(2n-3)-F_2{}^2(2n-2)}{F_2{}^2(2n-1)-F_2(2n-2)F_2(2n)}\right]$ (50)

$\frac{e_3}{e_1} > V$, Where V=$\frac{e_2}{e_1}\left[\frac{F_2(2n-2)F_2(2n+1)-F_2(2n)F_2(2n-1)}{F_2(2n-1)F_2(2n+1)-F_2{}^2(2n)}\right]+\left[\frac{F_2{}^2(2n-1)-F_2(2n)F_2(2n-2)}{F_2(2n-1)F_2(2n+1)-F_2{}^2(2n)}\right]$ (51)

From (48) and (49)

We get $\frac{e_1}{e_2} < \frac{F_2(2n)}{F_2(2n-1)}$ (52)

From (38) $\frac{e_3}{e_1} < W$,

Where $W = \frac{e_2}{e_1}\left[\frac{F_2(2n-3)F_2(2n+1)-F_2{}^2(2n-1)}{F_2(2n-2)F_2(2n+1)-F_2(2n)F_2(2n-1)}\right]+\left[\frac{F_2(2n-2)F_2(2n-2)-F_2(2n)F_2(2n-3)}{F_2(2n+1)F_2(2n-2)-F_2(2n)F_2(2n-1)}\right]$ (53)

From (39) and (40)

$\frac{e_1}{e_2} > \frac{F_2(2n+1)}{F_2(2n)}$ (54)

Therefore $\frac{F_2(2n+1)}{F_2(2n)} < \frac{e_1}{e_2} < \frac{F_2(2n)}{F_2(2n-1)}$ (55)

Similar fashion, we get $\frac{F_2(2n+1)}{F_2(2n)} < \frac{e_2}{e_3} < \frac{F_2(2n)}{F_2(2n-1)}, \frac{F_2(2n+1)}{F_2(2n)} < \frac{e_1}{e_3} < \frac{F_2(2n)}{F_2(2n-1)}$ (56)

Similar fashion, it can be proved for remaining cases

$\frac{F_2(2n)}{F_2(2n-1)} \leqslant \frac{e_1}{e_2} \leqslant \frac{F_2(2n+1)}{F_2(2n)}$ (57)

$\frac{F_2(2n)}{F_2(2n-1)} \leqslant \frac{e_2}{e_3} \leqslant \frac{F_2(2n+1)}{F_2(2n)}$ (58)

$\frac{F_2(2n)}{F_2(2n-1)} \leqslant \frac{e_1}{e_3} \leqslant \frac{F_2(2n+1)}{F_2(2n)}$ (59)

Therefore, for large value of n

$\frac{e_1}{e_2} = \mu, \frac{e_2}{e_3} = \mu, \frac{e_1}{e_2} = \mu^2$, where $\mu = 1.465$ (60)

$\frac{e_1}{e_2} = \mu, \frac{e_2}{e_3} = \mu, \frac{e_1}{e_2} = \mu^2$ (61)

$\frac{e_1}{e_2} = \mu, \frac{e_2}{e_3} = \mu, \frac{e_1}{e_2} = \mu^2$ (62)

The generalized relation among the code elements where

E=$(e_{ij})_{(\lambda+1)\times(\lambda+1)}$,

$\frac{F_\lambda(2n)}{F_\lambda(2n-k)} \leqslant \frac{e_{ij}}{e_{ij+k}} \leqslant \frac{F_\lambda(2n+1)}{F_\lambda(2n-k+1)}$, for i, j=1,2,3,4………..($\lambda$ +1); k=1,2,3,….. λ (63)

8. ERROR DETECTION /CORRECTION

Because of the reasons arising in the channel, some errors may be occur in the code matrix E. So we try to determine and correct these errors using the properties of determinant in this process.

Case i: Let λ = 1 and the message matrix M be as follows

$$M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$$

The new method of the error detection is an application of the H_λ matrix. The basic idea of this method depends on calculating the determinants of M and E. Comparing the determinants obtained from the channel, the receiver can decide whether the code message E is true or not. Actually, we cannot determine which element of the code message is damaged. In order to find damaged element, we suppose different cases such as single error, two errors etc. Now, we consider the first case with a single error in the code matrix E. We can easily obtain that there are four places where single error. Since the elements of message matrix M are positive integers, we should find integer solutions of the equations from If there are not integer solutions of these equations, we find that our cases related to a single error is incorrect or an error can be occurred in the checking element "Det(M)". If Det(M) is incorrect, we use the relations given in (25) to check a correctness of the code matrix E.

Similarly, we can check the cases with double errors in the code matrix E. Let us consider the following case with double error in E

$$\text{(i)} \begin{pmatrix} x_1 & e_2 \\ e_3 & e_4 \end{pmatrix} \qquad (ii) \begin{pmatrix} e_1 & x_2 \\ e_3 & e_4 \end{pmatrix} \qquad (iii) \begin{pmatrix} e_1 & e_2 \\ x_3 & e_4 \end{pmatrix} \qquad (iv) \begin{pmatrix} e_1 & e_2 \\ e_3 & x_4 \end{pmatrix} \tag{64}$$

We can use following equations

$$x_1 e_4 - e_2 e_3 = (-1)^n Det\ M \tag{65}$$
$$e_1 e_4 - x_2 e_3 = (-1)^n Det\ M \tag{66}$$
$$e_1 e_4 - x_3 e_2 = (-1)^n Det\ M \tag{67}$$
$$e_1 x_4 - e_3 e_2 = (-1)^n Det\ M \tag{68}$$

By using above equations, we have to write following way

$$x_1 = \frac{(-1)^n Det\ M + e_2 e_3}{e_4} \tag{69}$$
$$x_2 = \frac{-(-1)^n Det\ M + e_1 e_4}{e_3} \tag{70}$$
$$x_3 = \frac{-(-1)^n Det\ M + e_1 e_4}{e_4} \tag{71}$$
$$x_4 = \frac{(-1)^n Det\ M + e_2 e_3}{e_1} \tag{72}$$

Since the elements of message matrix M are positive integers, we should find integer solutions of the equations from (68) to (71). If there are not integer solutions of these equations, we find that our cases related to a single error is incorrect or an error can be occurred in the checking element "Det(M)". If Det(M) is incorrect, we use the relations given in (25) to check a correctness of the code matrix E. Similarly, we can check the cases with double errors in the code matrix E.

Let us consider the following case with double error in E

$$\begin{pmatrix} x_1 & x_2 \\ e_3 & e_4 \end{pmatrix} \tag{73}$$

where x_1, x_2 are the damaged elements of E. Using the relation

$$Det\ \text{E}=(-1)^n Det\ M \tag{74}$$

We can write following equation for the matrix Det E=$x_1 e_4 - x_2 e_3$

$$x_1 e_4 - x_2 e_3 = (-1)^n Det\ M \tag{75}$$

Also, we know the following relation between $x_1\ and\ x_2$

$$\frac{x_1}{x_2} \approx \tau \tag{76}$$

It is clear that the equation (73) is a Diophantine equation. Because there are many solutions of Diophantine equations, we should choose the solutions x_1, x_2 satisfying the above checking relation (74). Using the similar approach, we can correct the triple errors in the code matrix E such that

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & e_4 \end{pmatrix} \tag{77}$$

Where x_1, x_2 and x_3 are damaged elements of E.

Consequently, our method depends on confirmation of various cases about damaged elements of E using the checking element Det(M) and checking relation x_1, x_2. Our correctness method permits us to correct 14 cases among the 15 cases, because our method is inadequate for the case with four errors. So we can say that correction ability of our method is $\frac{14}{15}$ = 0.9333≡93.33%

Case ii: Let **λ** = 2 and the message matrix M be as follows

$$M = \begin{pmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{pmatrix}$$

$$E = \mathrm{M} \times (H_2{}^{\boldsymbol{n}}) = \begin{pmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{pmatrix} \begin{pmatrix} F_2(2n+1) & F_2(2n) & F_2(2n-1) \\ F_2(2n-1) & F_2(2n-2) & F_2(2n-3) \\ F_2(2n) & F_2(2n-1) & F_2(2n-2) \end{pmatrix} = \begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix}$$

$$\mathrm{M} = E \times (H_2{}^{-\boldsymbol{n}}) = \begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix} \begin{pmatrix} F_2(2n+1) & F_2(2n) & F_2(2n-1) \\ F_2(2n-1) & F_2(2n-2) & F_2(2n-3) \\ F_2(2n) & F_2(2n-1) & F_2(2n-2) \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{pmatrix}$$

The code matrix E may contain single, double, . . ., nine fold error similar to [1]. Thus, there are$9_{C_1} + 9_{C_2} + 9_{C_3} + 9_{C_4} + 9_{C_5} + 9_{C_6} + 9_{C_7} + 9_{C_8} + 9_{C_9} = 2^{4^2} - 1 = 511$
codes of errors in the code matrix E. We can use hypotheses for single error, double error, . . ., nine fold error as [1]. The nine fold error of the code matrix is not correctable so the possibility to correct eight cases of the method is $\frac{510}{511}$ = 0:9980 ≡99:80%:

In general, for $\boldsymbol{\lambda}$ = t and n > $\boldsymbol{\lambda}$ + 1 = t + 1 the correct possibility of the method is

$$\frac{2^{(t+1)^2} - 2}{2^{(t+1)^2} - 1}$$

Therefore, for large value of $\boldsymbol{\lambda} = \mathbf{t}$, the correct possibility of the method is

$$\frac{2^{(t+1)^2} - 2}{2^{(t+1)^2} - 1} \approx 1 = 100\%$$

CONCLUSION

The Fibonacci coding/decoding method is the main application of the Fibonacci H_λ matrices. There lies a difference between the classical algebraic codes and companion matrix with Fibonacci coding method.

(1) The relation between companion matrices and Fibonacci matrices with Fibonacci $\lambda -$ numbers.
(2) The correct ability of the method for the case $\boldsymbol{\lambda}$ = 1 is 93.33% which exceeds essentially all well-known correcting codes.
(3) The correct ability of the method for the case $\boldsymbol{\lambda}$ = 2 is 99.80% which corrects up to eightfold of errors among nine folds.
(4) The correct ability of the method increases as $\boldsymbol{\lambda}$ increases.

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