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WEAKLY CONVEX DOUBLY CONNECTED DOMINATION IN GRAPHS UNDER SOME BINARY OPERATIONS

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ABSTRACT

Let G be a connected simple graph. A weakly convex dominating set S of G is a weakly convex doubly connected dominating set if S is a doubly connected dominating set of G. The weakly convex doubly connected domination number of G, denoted by $\gamma_{ccc}^{w}(G)$, is the smallest cardinality of a convex doubly connected dominating set S of G. In this paper, we characterized the weakly convex doubly connected dominating sets of the composition and Cartesian product of graphs.

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Keywords: *dominating set, doubly connected dominating set, convex dominating set, convex doubly connected dominating set*

1. INTRODUCTION

Let G be a connected simple graph. A subset S of $V(G)$ is a dominating set of G if for every $v \in (V(G)\backslash S)$, there exists $x \in S$ such that $xv \in E(G)$. The domination number $\gamma(G)$ of G is the smallest cardinality of a dominating set of G. A graph G is connected if there is at least one path that connects every two vertices $x, y \in V(G)$, otherwise, G is disconnected. A component of a graph is a maximal connected subgraph. Clearly, if a graph has only one component, then it is connected, otherwise it is disconnected. A dominating set $S \subseteq V(G)$ is called a connected dominating set of G if the subgraph $\langle S \rangle$ induced by S is connected. The connected domination number of G, denoted by $\gamma_c(G)$, is the smallest cardinality of a connected dominating set of G. A connected dominating set of cardinality $\gamma_c(G)$ is called a γ_c-set of G. A set $S \subseteq V(G)$ is a doubly connected dominating set if it is dominating and both $\langle S \rangle$ and $\langle V(G) \setminus S \rangle$ are connected. The doubly connected domination number of G, denoted by $\gamma_{cc}(G)$, is the smallest cardinality of a doubly connected dominating set S of G. A doubly connected dominating set of cardinality $\gamma_{cc}(G)$ is called a γ_{cc}-set of G. Studies on doubly connected domination in graphs are found in [1, 2, 3, 4, 5].

For any two vertices u and v in a connected graph, the distance $d_G(u, v)$ between u and v is the length of a shortest path in G. A u-v path of length $d_G(u, v)$ is also referred to as u-v geodesic. A subset C of $V(G)$ is called a convex set of G if for every two vertices $u, v \in C$, the vertex-set of every u-v geodesic is contained in C. A subset C of $V(G)$ is called a weakly convex set of G if for every two vertices $u, v \in C$, there exists a u-v geodesic whose vertices bolong to C. Convexity in graphs was studied in [6,7,8,9]. Some variants of convex domination in graphs are found in [10, 11, 12, 13, 14, 15, 16, 17]

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A dominating set of G which is weakly convex is called a weakly convex dominating set. The weakly convex domination number of G, denoted by $\gamma_{wcon}(G)$, is the smallest cardinality of a weakly convex dominating set of G. A dominating set S which is also convex is called a convex dominating set of G. The convex domination number $\gamma_{con}(G)$ of G is the smallest cardinality of a convex dominating set of G. A convex dominating set of cardinality $\gamma_{con}(G)$ is called a γ_{con}-set of G. A weakly convex dominating set S of G is a weakly convex doubly connected dominating set if S is a doubly connected dominating set of G. The \emph{weakly convex doubly connected domination number of G, denoted by $\gamma_{ccc}^{w}(G)$, is the smallest cardinality of a weakly convex doubly connected dominating set S of G. A weakly convex doubly connected dominating set of cardinality $\gamma_{ccc}^{w}(G)$ is called a γ_{ccc}^{w}-set of G. For general concepts we refer the reader to [19].

2. RESULTS

The following remarks are immediate from the definitions.

Remark 2.1: *Let G be a connected graph. If $C \subseteq V(G)$ is convex dominating set, then C is a weakly convex dominating set of G.*

Remark 2.2: Let *G be a nontrivial connected graph of order n. Then*

(i) $\gamma(G) \leq \gamma_{wcon}(G) \leq \gamma_{ccc}^{w}(G) \leq \gamma_{ccc}(G)$, *and*
(ii) $1 \leq \gamma_{ccc}^{w}(G) \leq n$.

The composition of two graphs G and H is the graph $G[H]$ with vertex-set $V(G[H]) = V(G) \times V(H)$ and edge-set $E(G[H])$ satisfying the following conditions: $(x,u)(y,v) \in E(G[H])$ if and only if either $xy \in E(G)$ or $x = y$ and $uv \in E(H)$.

A subset C of $V(G[H]) = V(G) \times V(H)$ can be written as $C = \bigcup_{x\in S}(\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for every $x \in S$. We shall be using this form to denote any subset C of $V(G[H])$.

The following results are needed for the characterization of the weakly convex doubly connected dominating sets of the composition to two of graphs.

Theorem 2.3[7]: *Let G be connected graph of order $m \geq 2$ and H any graph. A subset $C = \bigcup_{x\in S}(\{x\} \times T_x)$ is a weakly convex dominating set of $G[H]$ if and only if S is a weakly convex dominating set of G, where T_x is a dominating set of H with $diam_H(\langle T_x\rangle) \leq 2$ if $|S| = 1$.*

Remark 2.4: *Let G and H be non-complete connected graphs. If S is a weakly convex dominating set of G with $|S| \geq 2$, then a subset $C = \bigcup_{x\in S}(\{x\} \times T_x)$ is a weakly convex dominating set of $G[H]$.*

The following result is the characterization of the weakly convex doubly connected dominating sets of the composition to two of graphs.

Theorem 2.5: *Let G and H be non-complete connected graphs. A subset $C = \bigcup_{x\in S}(\{x\} \times T_x)$ is a weakly convex doubly connected dominating set of $G[H]$ if and only if S is a weakly convex dominating set of G and T_x is a weakly convex set of H and one of the following holds:*

i) $S = \{x\}$ *and* T_x *is a dominating set of H with* $diam_H(\langle T_x\rangle) \leq 2$, *where* $T_x \neq V(H)$ *whenever* $\langle V(G) \setminus S\rangle$ *is not connected.*
ii) $S = V(G) \setminus \{z\}$.
iii) $S = S_1 \cup \{z\} = V(G)$ *and* $\langle V(H) \setminus T_z\rangle$ *is connected.*
iv) $T_x \neq V(H)$ *for all* $x \in S$ *whenever* $S \neq V(G) \setminus \{z\}$.

Proof: Suppose that a subset $C = \bigcup_{x\in S}(\{x\} \times T_x)$ is a weakly convex doubly connected dominating set of $G[H]$. Then C is a weakly convex dominating set of $G[H]$. Thus S is a weakly convex dominating set of G by Theorem 2.3. Suppose that T_x is not a weakly convex set of H. Let $S = \{x\}$ and $T_x = \{a,b\}$ such that $ab \notin E(H)$\$. Then $C = \{(x,a),(x,b)\}$ and $(x,a)(x,b) \notin E(G[H])$ contrary to our assumption that C is a weakly convex doubly connected dominating set of $G[H]$. Thus, T_x must be a weakly convex set of H. Further, $S = \{x\}$ and T_x is a dominating set of H with $diam_H(\langle T_x\rangle) \leq 2$ holds by Theorem 2.3. Suppose that $\langle V(G) \setminus S\rangle$ is not connected. If $T_x = V(H)$, then

$$V(G[H]) \setminus C = \big(V(G) \times V(H)\big) \setminus \big(S \times V(H)\big) = (V(G) \setminus S) \times V(H).$$

Since $\langle V(G) \setminus S\rangle$ is not connected, it follows that $\langle V(G[H]) \setminus C\rangle$ is not connected contrary to our assumption that C is doubly connected dominating set of $G[H]$. Thus, $T_x \neq V(H)$. This proves statement i).

Next, suppose that $|S| \neq 1$. If $S = V(G) \setminus \{z\}$ then we are done with statement ii). Suppose that $S \neq V(G) \setminus \{z\}$. Consider the following cases:

Case-1: Suppose that $S = V(G)$. Then consider $S = S_1 \cup \{z\} = V(G)$. Let $|T_z| = 1$. Since H is non-complete connected graph, let b be a fixed element of $V(H) \setminus T_z$ such that $bu \notin E(H)$ for all $u \in V(H) \setminus T_z$. Then $(z,b)(z,u) \notin E(G[H])$ for all $(z,b),(z,u) \in V(G[H]) \setminus C$. This implies that C is not a doubly connected dominating set of $G[H]$ contrary to our assumption. Thus, $bu \in E(H)$ for all $u \in V(H) \setminus T_z$ and hence $\langle V(H) \setminus T_z \rangle$ is connected. Similarly if $|T_z| \geq 2$, then $\langle V(H) \setminus T_z \rangle$ is connected. This prove statement iii).

Case-2: Suppose that $S \subset V(G) \setminus \{z\}$. Let $S = V(G) \setminus \{(z,w)\}$ such that $wz \notin E(G)$. If $T_x = V(H)$ for all $x \in S$, then $V(G[H]) \setminus C = \cup_{v \in \{z,w\}, u \in V(H)} \{(v,u)\backslash\}$. Since $wz \notin E(G)$, it follows that $(w,u)(z,u) \notin E(G[H])$ for all $(w,u),(z,u) \in V(G[H]) \setminus C$. Thus, C is not a doubly connected dominating set of $G[H]$ contrary to our assumption. Hence $T_x \neq V(H)$. This proves statement iv).

For the converse, suppose that S is a weakly convex dominating set of G and T_x is a weakly convex set of H and one of the following statements $i), ii), iii), or\ iv$) holds.

First, suppose that statement i) holds. Then $C = \cup_{x \in S}(\{x\} \times T_x)$ is a weakly convex dominating set of $G[H]$ by Theorem 2.3. Clearly, if $\langle V(G) \setminus S \rangle$is connected, then C is a weakly convex doubly connected dominating set of $G[H]$. Suppose that $\langle V(G) \setminus S \rangle$ is not connected. Let $T_x \neq V(H)$. If $T_x = \{a\}$ then $C = \{(x,a)\}$. Since H is non-complete connected graph, let $b,c \in V(H) \setminus T_x$. If $bc \in E(H)$, then $(x,b)(z,c) \in E(G[H])$ for all $z \in N_G(x)$, that is, $\langle V(G[H]) \setminus C \rangle$ is connected. Suppose that $bc \notin E(H)$\$. Since H is connected, there exists a path $[b = u_1, u_2, \ldots, u_s = c]$ such that $[(x,b),(z,u_2),\ldots,(z,c))]$ is a path in $\langle V(G[H]) \setminus C \rangle$ for all $z \in N_G(x)$, that is, $\langle V(G[H]) \setminus C \rangle$ is connected. Thus, C is a doubly connected dominating set of $G[H]$. Similarly, if $|T_x| \geq 2$, then C is a doubly connected dominating set of $G[H]$.

Next, suppose that ii) holds. If $T_x = V(H)$ for each $x \in S$, then T_x is a dominating set of H. Since S is a weakly convex dominating set of G, it follows that $C = \cup_{x \in S}(\{x\} \times T_x)$ is a weakly convex dominating set of $G[H]$ by Theorem 2.3. Since G is a non-complete connected graphs, $|V(G)| \geq 3$ and $|S| \neq 1$. Since H is non-complete connected graph, let $a,b \in V(H)$. If $ab \in E(H)$, then $(z,a)(z,b) \in E(G[H])$ for all $(z,a),(z,b) \in V(G[H]) \setminus C$ and for all $z \in V(G) \setminus S$. Thus, $\langle V(G[H]) \setminus C \rangle$ is connected, and hence C is a doubly connected dominating set of $G[H]$. Suppose that $ab \notin E(H)$. Since H is connected, there exists a path $[a = u_1, u_2, \ldots, u_s = b]$ such that for all $z \in V(G) \setminus S, [(z,a),(z,u_2),\ldots,(z,b))]$ is a path in $\langle V(G[H]) \setminus C \rangle$, that is, $\langle V(G[H]) \setminus C \rangle$ is connected. Thus, C is a doubly connected dominating set of $G[H]$. Thus, $\langle V(G[H]) \setminus C \rangle$ is connected. Hence C is a doubly connected dominating set of $G[H]$. Now, suppose that $T_x \neq V(H)$ for some $x \in S$. Consider T_x is a dominating set of H for each $x \in S$. Then $C = \cup_{x \in S}(\{x\} \times T_x)$ is a weakly convex dominating set of $G[H]$ by Theorem 2.3. This further implies that C is connected dominating set in $G[H]$. Since S is a dominating set in G, there exists $x \in S$ such that $xz \in E(G)$ for all $z \in V(G) \setminus S$. Let $a \in V(H) \setminus T_x$ for each $x \in S$. Then $(x,a)(z,u) \in E(G[H])$ for all $u \in V(H)$ and $(x,a)(y,a) \in E(G[H])$ for all $y \in N_G(x)\ (y \neq z)$. Thus, $\langle V(G[H]) \setminus C \rangle$ is connected. Hence C is a doubly connected dominating set of $G[H]$. Consider T_x is not a dominating set of H for each $x \in S$. Since S is a weakly convex dominating set of G and $|S| \geq 2$, then C is a weakly convex dominating set of $G[H]$ by Remark 2.4. Let $a \in V(H) \setminus T_x$. By following similar arguments used earlier, C is a doubly connected dominating set of $G[H]$.

Suppose that statement iii) holds. Consider the following cases.

Case-1: Suppose that $T_x = V(H) for\ all\ x \in S_1$.
Since S is a weakly convex dominating set of G and T_x is a dominating set of H, it follows that C is a weakly convex dominating set of $G[H]$ by Theorem 2.3. If $T_z = V(H)$, then

$$\begin{aligned}
C &= \cup_{x \in S}(\{x\} \times T_x) \\
&= \cup_{x \in (S_1 \cup \{z\})}(\{x\} \times T_x) \\
&= \cup_{x \in S_1}(\{x\} \times T_x) \cup (\{z\} \times T_z) \\
&= \cup_{x \in S_1}\big(\{x\} \times V(H)\big) \cup (\{z\} \times V(H)) \\
&= \cup_{x \in S}(\{x\} \times V(H) \\
&= S \times V(H) = V(G) \times V(H) = V(G[H]).
\end{aligned}$$

This implies that $V(G[H]) \setminus C = \emptyset$ and hence $\langle V(G[H]) \setminus C \rangle$ is connected.

If $T_z \neq V(H)$, then let $\{a\} \subseteq T_z \subset V(H)$. Consider $T_z = \{a\}$. Since H is a non-complete connected graph, $|V(H)| \geq 3$. Let $b,c \in V(H) \setminus T_z$. If $bc \in E(H)$, then $(z,b)(z,c) \in E(G[H])$ for all $(z,b),(z,c) \in V(G[H]) \setminus C$ and hence $\langle V(G[H]) \setminus C \rangle$ is connected. Suppose that $bc \notin E(H)$. Since $\langle V(H) \setminus T_z \rangle$ is connected, there exists a path $[b = v_1, v_2, \ldots, v_r = c]$ in $\langle V(H) \setminus T_z \rangle$ such that $[(z,b),(z,v_2),\ldots,(z,c)]$ is also a path in $\langle V(G[H]) \setminus C \rangle$. Thus, $\langle V(G[H]) \setminus C \rangle$ is connected. Similarly, if $\{a\} \subset T_z$, then $\langle V(G[H]) \setminus C \rangle$ is connected. Consider $T_z = V(H) \setminus \{a\}$. Then $V(G[H]) \setminus C = \{(z,a)\}$ and hence $\langle V(G[H]) \setminus C \rangle$ is connected. Thus, C is a doubly connected dominating set of $G[H]$.

Case-2: Suppose that $T_x \neq V(H)$ for all $x \in S$.
Since G is a non-complete connected graph and $S = V(G)$, it follows that $|S| \geq 3$. Since S is a weakly convex dominating set of G with $|S| \geq 3$, $C = \cup_{x\in S}(\{x\} \times T_x)$ is a weakly convex dominating set of $G[H]$ by Remark 2.4. This implies that $\langle C\rangle$ is connected. Let $a \in V(H) \setminus T_x$ and $x, y \in S$. If $xy \in E(G)$, then $(x,a)(y,a) \in E(G[H])$ for all $(x,a),(y,a) \in V(G[H]) \setminus C$. This implies that $\langle V(G[H]) \setminus C\rangle$ is connected. Suppose that $xy \notin E(G)$. Since G is connected, there exist a path $[x = x_1, x_2, \ldots, x_r = y]$ in G such that $[(x,a),(x_2,a),\ldots,(y,a)]$ is also a path in $\langle V(G[H]) \setminus C\rangle$. Thus, $\langle V(G[H]) \setminus C\rangle$ is connected, that is, C is a doubly connected dominating set of $G[H]$.

Finally, suppose that statement $iv)$ holds. If $|S| = 1$, then C is a weakly convex doubly connected dominating set of $G[H]$ by statement i). Suppose that $|S| \geq 2$. If $S = V(G)$, then C is a weakly convex doubly connected dominating set of $G[H]$ by statement iii). If $S = V(G) \setminus \{z\}$, then C is a weakly convex doubly connected dominating set of $G[H]$ by statement ii). Since S is a weakly convex dominating set of G with $|S| \geq 2$, $C = \cup_{x\in S}(\{x\} \times T_x)$ is a weakly convex dominating set of $G[H]$ by Remark 2.4. Let $w \in (V(G) \setminus \{z\}) \setminus S$ and $a \in V(H) \setminus T_x$.

Case-1: If $wz \in E(G)$, then $(w,a)(z,a) \in E(G[H])$ for all $(w,a),(z,a) \in V(G[H]) \setminus C$, that is, $\langle V(G[H]) \setminus C\rangle$ is connected.

Case-2: Suppose that $wz \notin E(G)$. Since G is connected, there exists a path $[w = x_1, x_2, \ldots, x_r = z]$ in G such that $[(w,a),(x_2,a),\ldots,(z,a)]$ is a path in $\langle V(G[H]) \setminus C\rangle$. Thus, $\langle V(G[H]) \setminus C\rangle$ is connected, that is, C is a doubly connected dominating set of $G[H]$.

Accordingly, C is a weakly convex doubly connected dominating set of $G[H]$. ∎

As a consequence of Theorem 2.5, we obtain the following result.

Corollary 2.6: *Let G and H be non-complete connected graphs. Then*
$$\gamma_{ccc}^{w}(G[H]) = \begin{cases} 1 & if\ \gamma(G) = 1\ and\ \gamma(H) = 1 \\ k & if\ \gamma_{wcon}(G) = k\ where\ k \geq 2 \end{cases}$$

Proof: Suppose that $\gamma(G) = 1$ and $\gamma(H) = 1$. Let $S = \{x\}$ be a γ-set in G and $T_x = \{a\}$ be a γ-set in H. Then S is a weakly convex dominating set of G and $T_x \neq V(H)$ is a weakly convex dominating set of H with $diam_H(\langle T_x\rangle) < 2$. Thus $C = \cup_{x\in S}[\{x\} \times T_x] = \{(x,a)\}$ is a weakly convex doubly connected dominating set of $G[H]$ by Theorem 2.5. Hence, $\gamma_{ccc}^{w}(G[H]) = |C| = 1$.

Suppose that $\gamma_{wcon}(G) = k$ where $k \geq 2$. Let $S = \{x_1, x_2, \ldots, x_k\}$ be a γ_{wcon}-set in G. Since S is a weakly convex dominating set of G with $|S| \geq 2$, a subset $C = \cup_{x\in S}(\{x\} \times T_x)$ is a weakly convex dominating set of $G[H]$ by Remark 2.4. Let $T_x = \{a\}$ for all $x \in S$. Then $C = \{(x_1,a),(x_2,a),\ldots,(x_k,a)\}$, that is, $|C| = k$. Let $x, y \in S$ and let $b \in V(H) \setminus T_v$ for all $v \in S$. If $xy \in E(G)$, then by similar arguments used to prove Theorem 2.5, C is a weakly convex doubly connected dominating set of $G[H]$. Similarly, if $xy \notin E(G)$, then C is a weakly convex doubly connected dominating set of $G[H]$. Thus, $\gamma_{ccc}^{w}(G[H]) \leq |C| = k$. Since $k = \gamma_{wcon}(G[H]) \leq \gamma_{ccc}^{w}(G[H])$ by Remark 2.2, it follows that $\gamma_{ccc}^{w}(G[H]) = k$. ∎

The Cartesian product of two graphs G and H is the graph $G \boxdot H$ with vertex-set $V(G \boxdot H) = V(G) \times V(H)$ and edge-set $E(G \boxdot H)$ satisfying the following conditions: $(x,a)(y,b) \in E(G \boxdot H)$ if and only if either $xy \in E(G)$ and $a = b$ or $x = y$ and $ab \in E(H)$.

The next result is needed for the characterization of the weakly convex doubly connected dominating sets of the Cartesian product of two of graphs.

Lemma 2.7: *Let G and H be non-tivial connected graphs. Then $C = \cup_{x\in S}[\{x\} \times T_x]$ is a weakly convex dominating set of $G \boxdot H$ if S is a weakly convex dominating set of G and $T_x = V(H)$ for all $x \in S$, or $S = V(G)$ and $T_x = V(H)$ is a weakly convex dominating set of H for all $x \in S$.*

Proof: Suppose that $C = \cup_{x\in S}[\{x\} \times T_x]$ is not a weakly convex dominating set of $G \boxdot H$. Let $(x,a) \in C$. If there exists $(y,a) \in C$ whose vertices in any (x,a)-(y,a) geodesic are not all in C, then for each $x \in S$, there exists $y \in S$ whose vertices in any x-y geodesic are not all in S, that is, S is not a weakly convex dominating set of G. If there exists $(x,b) \in C$ whose vertices in any (x,a)-(x,b) geodesic are not all in C, then for each $a \in T_x$, there exists $b \in T_x$ (for all $x \in S$) whose vertices in any a-b geodesic are not all in T_x, that is, T_x is not a weakly convex dominating set of H for all $x \in S$. ∎

The following result is the characterization of the weakly convex doubly connected dominating sets of the Cartesian product of two of graphs.

Theorem 2.8: *Let G and H be non-trivial connected graphs. Then $C = \cup_{x\in S}[\{x\} \times T_x]$ is a weakly convex doubly connected dominating set of $G \boxdot H$ if and only if S is a weakly convex dominating set of G and H is a weakly convex dominating set of H and one of the following statements holds:*

i) $S \neq V(G)$ *and* $T_x = V(H)$ *for all* $x \in S$ *where* $\langle V(G) \setminus S\rangle$ *is connected.*

ii) $S = V(G)$ *and* $T_x \neq V(H)$ *for all* $x \in S$ *where* $\langle V(H) \setminus T_x\rangle$ *is connected.*

iii) $S = S_1 \cup S_2$ *where* $S_1 = \{x \in V(G): T_x = V(H)\}$, $S_2 = \{x \in V(G): T_x \neq V(H)\}$, $\langle S_1\rangle$ *is connected,* $\langle S_2\rangle$ *is connected, and* $\langle V(H) \setminus T_z\rangle$*is connected for all* $z \in S_2$

(iv) $T_x = T_{x'} \cup T_{x''}$ *where* $T_{x'} = \{a \in V(H): S = V(G)\}$, $T_{x''} = \{a \in V(H): S \neq V(G)\}$, $\langle T_{x'}\rangle$ *is connected,* $\langle T_{x''}\rangle$ *is connected, and* $\langle V(G) \setminus S'\rangle$ *is connected where* $S' = \{x \in V(G): a \in T_{x''}\}$.

Proof: Suppose that $C = \cup_{x\in S}[\{x\} \times T_x]$ is a weakly convex doubly connected dominating set of $G \boxdot H$. Suppose that S is not a weakly convex dominating set of G. Let $x \in S$. If S is not a dominating set of G, then there exists $y \in V(G) \setminus S$ such that $xy \notin E(G)$. Let $a \in T_x$ for all $x \in S$. Then there exists $(y,a) \in V(G \boxdot H) \setminus C$ such that $(x,a)(y,a) \notin E(G \boxdot)$ for all $(x,a) \in C$. Hence C is not a dominating set of $G \boxdot H$ contrary to our assumption. If S is not a weakly convex set in G, then $|S| \geq 2$. Let $x,y \in S$ such that $xy \notin E(G)$. For each $v \in S$, let $a \in T_v$. If $|S| = 2$, then $(x,a)(y,a) \notin E(G \boxdot H)$ for all $(x,a),(y,a) \in C$. If $S \geq 3$, then there exists $z \in V(G) \setminus S$ such that for every x-y geodesic in $\langle S\rangle$, $z \in I_G[x,y]$. Thus, for every (x,a)-(y,a) geodesic in $\langle C\rangle$, $(z,a) \in I_{G\boxdot H}[(x,a),(y,a)]$ where $(z,a) \in V(G \boxdot H) \setminus C$. This is contrary to our assumption that C is a weakly convex dominating set of $G \boxdot H$. Thus, S must be a weakly convex dominating set of G. Similarly, for each $x \in S$, T_x is a weakly convex dominating set of H. Now, consider first that $S \neq V(G)$. Let $z \in V(G) \setminus S$. If $T_x \neq V(H)$ for all $x \in S$, then let $a \in V(H) \setminus T_x$ for all $x \in S$. Then $(x,a)(z,a) \in E(G \boxdot H)$ for all $x \in N_G(z)$ and $(z,a)(z,b) \in E(G \boxdot H)$ for all $b \in N_H(a)$. Since $(x,a),(z,b) \notin C$, it follows that (z,a) is not dominated by any element of C\$. This is contrary to our assumption that C is a dominating set of $G \boxdot H$. Thus, $T_x = V(H)$ for all $x \in S$. Suppose that $\langle V(G) \setminus S\rangle$ is not connected. Then there exists $w \in V(G) \setminus S$ such that no path z-w exists in $\langle V(G) \setminus S\rangle$. Let $a \in V(H)$. Then no path (z,a)-(w,a) exists in $\langle V(G \boxdot H) \setminus C\rangle$. This implies that $\langle V(G \boxdot H) \setminus C\rangle$ is not connected contrary to our assumption that C is a doubly connected dominating set of $G \boxdot H$. Thus, $\langle V(G) \setminus S\rangle$ must be connected. This proves statement $i)$. Similarly, if $V(G) = S$, then statement $ii)$ holds.

Next, suppose that $S = S_1 \cup S_2$ where $S_1 = \{x \in V(G): T_x = V(H)\}$, $S_2 = \{x \in V(G): T_x \neq V(H)\}$. Suppose that $|V(G)| = 2$. If $|V(H)| = 2$, then $|S_1| = 1$ and $|S_2| = 1$, Hence $\langle S_1\rangle$ is connected and $\langle S_2\rangle$ is connected. Clearly $\langle V(H) \setminus T_z\rangle$ is connected for all $z \in S_2$. Similarly, if $|V(H)| \geq 3$, then $\langle S_1\rangle$ is connected, $\langle S_2\rangle$ is connected. Suppose that $\langle V(H) \setminus T_z\rangle$ is not connected for some $z \in S_2$. Then there exists $a,b \in T_z$ such that an a-b geodesic is not a path in $\langle T_z\rangle$ for all for some $z \in S$. Thus, there exists $(z,a),(z,b) \in C$ such that a (z,a)-(z,b) geodesic is not a path in $\langle C\rangle$. This contradict to our assumption that C is a weakly convex set of $G \boxdot H$. Thus, $\langle V(H) \setminus T_z\rangle$ must be connected for all $z \in S_2$. Similarly, if $|V(H)| = 2$ and $|V(G)| \geq 3$, then $\langle S_1\rangle$ is connected, $\langle S_2\rangle$ is connected, and $\langle V(H) \setminus T_z\rangle$ is connected for all $z \in S_2$. Suppose that $|V(G)| \geq 3$ and $|V(H)| \geq 3$. Let $x,y \in S$. If $\langle S_1\rangle$ is not connected, then every x-y geodesic is not a path in $\langle S_1\rangle$. Thus, every (x,a)-(y,a) geodesic for all $a \in T_x$ for all $x \in S$ is not a path in $\langle C\rangle$. This contradict to our assumption that C is a weakly convex set of $G \boxdot H$. Thus, $\langle S_1\rangle$ must be connected. Similarly, $\langle S_2\rangle$ is connected. Further, suppose that $\langle V(H) \setminus T_z\rangle$ is not connected for some $z \in S_2$. Let $a,b \in V(H) \setminus T_z$. Then every a-b geodesic is not a path in $\langle V(H) \setminus T_z\rangle$. Thus, every (a,x)-(b,x) geodesic is not a path in $V(G \boxdot H) \setminus C$. This contradict to our assumption that C is a weakly convex set of $G \boxdot H$. Thus, $\langle V(H) \setminus T_z\rangle$ must be connected for all $z \in S_2$. This proves statement $iii)$. Similarly, statement $iv)$ holds.

For the converse, suppose that S is a weakly convex dominating set of G and H is a weakly convex dominating set of H and one of the statements $i), ii), iii)$, or $iv)$ holds. Then $C = \cup_{x\in S}[\{x\} \times T_x]$ is a weakly convex dominating set of $G \boxdot H$ by Lemma 2.7. Suppose first that statement $i)$ holds. Let $z \in V(G) \setminus S$. Consider $|V(G) \setminus S| = 1$. Since H is connected, there exists an a-b path in H such that (z,a)-(z,b) is a path in $\langle V(G \boxdot H) \setminus C\rangle$. This implies that C is a doubly connected dominating set of $G \boxdot H$. Hence C is a weakly convex doubly connected dominating set of $G \boxdot H$. Consider that $|V(G) \setminus S| \geq 2$. Since $\langle V(G) \setminus S\rangle$ is connected, there exists a z-w path in $\langle V(G) \setminus S\rangle$ such that (z,a)-(w,a) is a path in $\langle V(G \boxdot H) \setminus C\rangle$. This implies that C is a doubly connected dominating set of $G \boxdot H$. Hence C is a weakly convex doubly connected dominating set of $G \boxdot H$. Similarly, if $ii)$ holds, then C is a weakly convex doubly connected dominating set of $G \boxdot H$.

Next, suppose that $iii)$ holds. Let $a \in V(H) \setminus T_z$ for all $z \in S_2$. Consider that $|S_2| = 1$. Then $(z,a) \in V(G \boxdot H)\setminus C$. If $|V(H) \setminus T_z| = 1$, then $V(G \boxdot H) \setminus C = \{(z,a)\}$, that is, $\langle V(G \boxdot H) \setminus C\rangle$ is connected and hence C is weakly convex doubly connected dominating set of $G \boxdot H$. Suppose that $|V(H) \setminus T_z| \geq 2$. Then there exists $b \in V(H) \setminus T_z$ such that a-b is a path in $\langle V(H) \setminus T_z\rangle$ for all $a \in V(G) \setminus T_z$. Thus, for each $(z,a) \in V(G \boxdot H) \setminus C$, there exists $(z,b) \in V(G \boxdot H) \setminus C$ such that (z,a)-(z,b) is a path in $\langle V(G \boxdot H) \setminus C\rangle$. This implies that $\langle V(G \boxdot H) \setminus C\rangle$ is connected and hence C is a weakly convex doubly connected dominating set of $G \boxdot H$. Similarly, if statement $iv)$ holds, then C is a weakly convex doubly connected dominating set of $G \boxdot H$. ■

The next result is the consequence of Theorem 2.8.

Corollary 2.9: *Let G and H be non-trivial connected graphs. Then*

$$\gamma_{ccc}^{w}(G \boxdot H) = (max\{|V(G)|, |V(H)|\})(min\{|V(G)|, |V(H)|\} - 1)$$

if S is a weakly convex dominating set of G and T_x is a weakly convex dominating set of H for all $x \in S$ and one of the following statements holds:

i) $S = V(G) \setminus \{z\}$ and $T_x = V(H)$ for all $x \in S$ and $|V(G)| \leq |V(H)|$.
ii) $S = V(G)$ and $T_x = V(H) \setminus \{a\}$ for all $x \in S$ and $|V(G)| \geq |V(H)|$.

Proof: Suppose that S is a weakly convex dominating set of G and T_x is a weakly convex dominating set of H for all $x \in S$ and one of the statements i) or ii) holds. Then $C = \cup_{x\in S}[\{x\} \times T_x]$ is a weakly convex doubly connected dominating set of $G \boxdot H$ by Theorem 2.8. Further, $C = S \times V(H)$ or $C = V(G) \times T_x$ for all $x \in S$.
Let $|C| = min\{|S \times V(H)|, |V(G) \times T_x|\}$ for all $x \in S$.

$$\gamma_{ccc}^{w}(G \boxdot H) \leq |C| = min\{|S \times V(H)|, |V(G) \times T_x|\} = min\{|S||V(H)|, |V(G)||T_x|\}.$$

If i) holds, then

$$\begin{aligned} |C| &= |S \times V(H)| = |S||V(H)| \\ &= (min\{|S|, |V(H)|\})(max\{|V(G)|, |V(H)|\}) \\ &= (min\{|V(G)| - 1, |V(H)|\})(max\{|V(G)|, |V(H)|\}) \\ &= (min\{|V(G)|, |V(H)|\} - 1)(max\{|V(G)|, |V(H)|\}). \end{aligned}$$

If ii) holds, then

$$\begin{aligned} |C| &= |V(G) \times T_x| = |V(G)||T_x| \\ &= (max\{|V(G)|, |V(H)|\})(min\{|V(G)|, |T_x|\}) \\ &= (max\{|V(G)|, |V(H)|\})(min\{|V(G)|, |V(H)| - 1\}) \\ &= (max\{|V(G)|, |V(H)|\})(min\{|V(G)|, |V(H)|\} - 1). \end{aligned}$$

Thus, $\gamma_{ccc}^{w}(G \boxdot H) \leq (max\{|V(G)|, |V(H)|\})(min\backslash\{|V(G)|, |V(H)|\} - 1)$. Type equation here.

Since C is also a weakly convex dominating set of $G \boxdot H$, it follows that $\gamma_{wcon}(G \boxdot H) \leq |C|$. Let $(x, a) \in C$ and $C' = C \setminus \{(x, a)\}$. Then $(x, a)(z, a) \in E(G \boxdot H)$ for all $z \in N_G(x)$ and $(z, a)(z, b) \in E(G \boxdot H)$ for all $b \in N_H(a)$. If $z \in V(G) \setminus S$, then $(z, a) \in V(G \boxdot H) \setminus C'$ is not dominated by any element of C' since $(x, a), (z, b) \notin C'$. This implies that C' is not a weakly convex dominating set of $G \boxdot H$ and hence C is a minimum weakly convex dominating set of $G \boxdot H$. Thus, $|C| = \gamma_{wcon}(G \boxdot H) \leq \gamma_{ccc}^{w}(G \boxdot H)$ by Remark 2.2.
Therefore $\gamma_{ccc}^{w}(G \boxdot H) = |C| = (max\{|V(G)|, |V(H)|\})(min\{|V(G)|, |V(H)|\} - 1)$. ■

REFERENCES

1. Arriola, B. and Canoy Jr., S.R. *Doubly Connected Domination in the Corona and Lexicographic Product of Graphs*. Applied Mathematical Sciences, Vol. 8, 2014, no. 31, 1521-1533.
2. Cyman, J., Lemanska, M., and Raczek, J. *On the Doubly Connected Domination Number of a Graph*, Cent. Eur. J. Math, 4(2006), 34-45.
3. Enriquez, E.L. *Convex Doubly Connected Domination in Graphs Under Some Binary Operations*
4. Aunzo Jr.,R.T. and Enriquez, E.L. Convex *Doubly Connected Domination in Graphs*. Applied Mathematical Sciences, Vol. 9, 2015, no. 135, 6723-6734. http://dx.doi.org/10.12988/ams.2015.58540
5. Estrada, G.M, Loquias, C.M., Enriquez, E.L., Hinoguin, R.N. *Weakly Convex Doubly Connected Domination in the Join and Corona of Graphs*
6. Harary, F. and Nieminen, J.,*Convexity in graphs*. J. Differ Geom. 16(1981), 185-190.
7. Chartrand, G. and Zhang, P. *Convex sets in graphs*. Congressus Numerantium, 136 (1999), 19-32.
8. Lemanska, M. *Weakly convex and convex domination numbers*. Opuscula Mathematica, 24 (2004), 181-188.
9. S.R. Canoy Jr., S.R. and Garces, I.J.L. *Convex Sets Under Some Graphs Operations*, Graphs Comb. 18(2002), 787-793.
10. Enriquez, E.L. and Canoy Jr., S.R., *Secure Convex Domination in a Graph*. International Journal of Mathematical Analysis, Vol. 9, 2015, no. 7, 317-325.
11. Enriquez, E.L. and Canoy Jr., S.R. *On a Variant of Convex Domination in a Graph*
12. Loquias, C. M. and Enriquez, E.L. *On Secure Convex and Restrained Convex Domination in Graphs*. International Journal of Applied Engineering, Vol. 11, 2016, no. 7, 4707-4710.
13. Enriquez, E.L. and Canoy Jr., S.R. Restrained *Convex Dominating Set in the Corona and Products of Graphs*
14. Enriquez, E.L. *Secure Restrained Convex Domination in Graphs*. International Journal of Mathematical Archive, Vol. 8, 2017, no. 7, 1-5.
15. Enriquez, E.L. *Secure Convex Dominating Sets in Corona of Graphs*
16. Enriquez, E.L. and Canoy Jr., S.R., *Secure Convex Dominating Sets in Products of Graphs*
17. Enriquez, E.L. On Some Operations of Secure Restrained Convex Domination in Graphs

18. Leonida, R.E. *Weakly Convex Domination and Weakly Connected Independent Domination in Graphs*, Ph.D. Thesis, MSU-Iligan Institute of Technology, 2013
19. Chartrand, G. and Zhang, P. A *First Course in Graph Theory*, Dover Publication, Inc., New York, 2012.

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