International Journal of Mathematical Archive-9(12), 2018, 1-14
Available online through www.ijma.info ISSN 2229 – 5046

FIXED POINT THEOREMS FOR ψ-CONTRACTIONS AND φ-CONTRACTIONS IN QUASI-S-METRIC SPACES

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(Received On: 28-10-18; Revised & Accepted On: 06-12-18)

ABSTRACT

The notion of quasi-S-metric space is introduced based on the definition of S-metric space. We observe some properties (or consequences) of quasi-S-metric space. We also introduce the notion of ψ-contraction and φ-Contraction in quasi-S-metric space and prove two fixed point theorems followed by examples to illustrate the validity of our theorems.

Key Words: *S-metric space, quasi-S-metric space, ψ-contraction, φ-contraction.*

MSC: *54H25, 47H10.*

1. INTRODUCTION

The polish mathematician Stefen Banach [11] established the remarkable Banach Contraction Principle in 1922. The Banach Contraction Principle of fixed point is important as a source of existence and uniqueness theorems in different branches of Sciences.

Metric spaces are very important in mathematics and applied sciences. Many researchers put their effort and interest in the area of fixed point theory and established many generalizations of metric spaces in different ways [10].

In 1992, B.C.Dhage [2] introduced the notion of D-metric space. In 2007, S.Sedghi, N.Shobe and H.Zhou [15] introduced D*-metric spaces, which is a modification of D-metric space. Later on many researchers proved fixed point theorems in D*-metric spaces [7, 9].

In 2006, Z.Mustafa, B.I.Sims [21] introduced the new structure or generalization of metric space which is called a G-metric space. Many researchers extended their interest in this area [3, 4, 9, 20, 22, 23] and proved fixed point theorems.

In 2012, S.Sedghi *et al.* [16] introduced the notion of S-metric space which is a generalization of G-metric space [21] and D*-metric space [15] and proved fixed point theorems. Many researchers extended their interest in S-metric spaces and proved fixed point theorems [1, 5, 6, 8, 17, 18].

In this paper we introduce a new structure called quasi-S-metric space [QSMS] which is based on the definition of S-metric space, a generalization of S-metric space and also provided examples.

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We introduce the notion of ψ-contraction and φ-Contraction in qSMS and proved fixed point theorems for ψ-contractive and φ-Contractive self maps on a qSMS. Our results improved the results of K.P.R.Sastry, K.K.M.Sarma, P.Krishna Kumari and Sunitha Choudari [6].

2. PRELIMINARIES

In this section we present the necessary definitions and results which are used either tacitly or explicitly in the next section.

In 2006, Z.Mustafa and B. Sims [21] introduced the notion of G- metric spaces as a generalization of metric spaces.

2.1 Definition (Mustafa and B. Sims [21])
Let X be a non-empty set and $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following conditions, for all $x, y, z, a \in X$,
(2.1.1) $G(x, y, z) = 0$ if and only if $x = y = z$
(2.1.2) 0< $G(x, x, y)$, for all $x, y \in X$ with $x \neq y$
(2.1.3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $x \neq y$
(2.1.4) $G(x, y, z) = G\big(\pi(x, z, y)\big)$ where π is a permutation in $\{x, y, z\}$
(2.1.5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (Rectangle inequality).
G is called a G-metric on X and (X, G) is called a G-metric space.

In 2007, Sedghi *et al.* [15] introduced the notion of a D*-metric as follows.

2.2 Definition (S. Sedghi *et al.* [15])
Let X be a non-empty set. A generalized metric (or D^* metric) on X is a function $D^*: X^3 \to R^+$ that satisfies the following conditions: for each $x, y, z, a \in X$,
(2.2.1) $D^*(x, y, z) \geq 0$
(2.2.2) $D^*(x, y, z) = 0$ if and only if $x = y = z$
(2.2.3) $D^*(x, y, z) = D^*(\pi\{x, y, z\})$ (Symmetry) where π is a permutation function
(2.2.4) $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$
The pair (X, D^*) is called a Generalized metric space or D*-metric space.

The notion of partial metric is due to Matthews [12, 13]

2.3 Definition (Matthews [12, 13])
A partial metric on a non-empty set X is a function $p : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$,
(2.3.1) $p(x, x) = p(x, y) = p(y, y)$ if and only if $x = y$
(2.3.2) $p(x, x) \leq p(x, y)$ and $p(y, y) \leq p(x, y)$
(2.3.3) $p(x, y) = p(y, x)$
(2.3.4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$
The pair (X, p) is called a Partial metric space (PMS).

In 2012, Sedghi *et al.* introduced the notion of S-metric spaces as follows:

2.4 Definition (S. Sedghi, N. Shobe, A. Aliouche [16])
Let X be a non-empty set. An S-metric on X is a function $S: X^3 \to [0, \infty)$ that satisfies the following conditions: for all $x, y, z, a \in X$.
(2.4.1) $S(x, y, z) = 0$ if and only if $x = y$
(2.4.2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$
The function S is called an S- metric on X and the pair (X, S) is called an S-metric space.

The following are examples of S-metric spaces (Sedghi *et al.* [16])

2.5 Examples

2.5.1 Let $X = R^n$,
$S(x, y, z) = |y + z - 2x| + |y - z|$, then (X, S) is an S-metric space.

2.5.2 Let $X = R^n$,
$S(x, y, z) = |x - z| + |y - z|$, then (X, S) is an S-metric space.

2.5.3 Let (X, d) be a metric space.
Define $S(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in X$.
Then S is an S-metric space.

We observe the following in S-metric spaces (Sedghi *et al.* [16])

2.6 Observations
Let (X, S) be an S-metric space. Then

2.6.1 $S(x, x, y) = S(y, y, x)$

2.6.2 $S(x, x, z) \leq 2S(x, x, y) + S(z, z, y)$ and

2.6.3 $S(x, y, y) \leq S(x, x, y)$.

In 2016, Sedghi et al. introduced the notion of S_b-metric space as a generalization of S-metric space as follows

2.7 Definition (S. Sedghi *et al.* [14])
Let X be a non-empty set and $b \geq 1$ be a real number. Suppose that a mapping $S_b : X^3 \to [0, \infty)$ is a function satisfying the following properties:
(2.7.1) $0 < S_b\,(x, y, z)$for all $x, y, z \in X$ with $x \neq y \neq z$
(2.7.2) $S_b\,(x, y, z) = 0$ if and only if $x = y = z$
(2.7.3) $S_b\,(x, y, z) \leq b(S_b\,(x, x, a) + S_b\,(y, y, a) + S_b\,(z, z, a))$ for all $x, y, z, a \in X$.
Then the function S_b is called an S_b-metric on X and the pair (X, S_b) is called an S_b-metric space.

2.8 Remark: It should be noted that the class of S_b-metric spaces is effectively larger than that of S-metric spaces

3. MAIN RESULTS

In this section we introduce the notion of quasi-S-metric space (qSMS) based on the definition of S-metric space and provide examples. We also introduce the notion of ψ-contraction and φ-contraction on a quasi-S-metric space and prove fixed point theorems for ψ-contractions and φ-contractions.

3.1 Definition
A quasi-S-metric on a non-empty set X is a function $q : X \times X \times X \to [0, \infty)$ such that, for all $x, y, z \in X$,
(3.1.1) $q(x, x, x) = q(y, y, y) = q(z, z, z) = q(x, y, z)$ if and only if $x = y = z$
(3.1.2) $\max\{q(x, x, t), q(y, y, t), q(z, z, t)\} \leq q(x, y, z)$, for $t \in \{x, y, z\}$
(3.1.3) $q(x, y, z)$ is invariant of any permutation of x, y, z
(3.1.4) $q(x, y, t) \leq q(x, z, t) + q(z, y, t) - q(z, z, t)$ for all $, y, z, t \in X$.
The pair (X, q) is called a quasi-S-metric space.

3.2 Observation:

3.2.1. $q(x, y, y) = q(x, x, y)$

Proof: From (3.1.2)
$$\max\{q(x, x, t), q(y, y, t), q(z, z, t)\} \leq q(x, y, z), \text{ for } t \in \{x, y, z\}$$

By taking $t = y$, $z = y$, we have
$$\max\{q(x, x, y), q(y, y, y), q(y, y, y)\} \leq q(x, y, y)$$
$$\therefore q(x, x, y) \leq q(x, y, y) \tag{3.2.1.1}$$

By taking $t = x$, $z = x$, we have
$$\max\{q(x, x, x), q(y, y, x), q(x, x, x)\} \leq q(x, y, x)$$
$$\Rightarrow q(y, y, x) \leq q(x, y, x) = q(y, x, x)$$
$$\Rightarrow q(y, y, x) \leq q(y, x, x) \Rightarrow q(x, y, y) \therefore q(x, x, y) \tag{3.2.1.2}$$

From (3.2.1.1) and (3.2.1.2)
$$q(x, x, y) = q(x, y, y)$$

3.3 Definition: A sequence $\{x_n\}$ in the quasi-S-metric space (X, q) converges to a point $x \in X$ if $q(x_n, x_n, x) \to q(x, x, x)$. We write this as $x_n \to x$.

3.4 Definition: A sequence $\{x_n\}$ in the quasi-S-metric space (X, q) is called a Cauchy sequence if $\lim_{m,n\to\infty} q(x_n, x_n, x_m)$ exists and is finite.

3.5 Definition: A quasi-S-metric space is called complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ so that $q(x, x, x) = \lim_{m,n\to\infty} q(x_n, x_n, x_m)$

3.6 Now we observe the following:

3.6.1 If $x_n \to x$, then $q(x, x, x) = 0$

Proof: If $x_n \to x$, we have $q(x_n, x_n, x) \to 0$

From (3.1.2)

$$\max\{q(x,x,t), q(y,y,t), q(z,z,t)\} \le q(x,y,z), \text{ for } t \in \{x,y,z\}$$

Taking $t = x, y = z = x_n$

$$\max\{q(x,x,x), q(x_n,x_n,x), q(x_n,x_n,x)\} \le q(x,x_n,x_n) \to 0 \text{ as } n \to \infty$$
$$\Longrightarrow q(x,x,x) = 0$$

3.6.2 If $x_n \to x$, then $q(x_n, x_n, x_n) \to 0$ as $n \to \infty$ and $q(x, x_n, x_n) \to 0$ as $n \to \infty$

Proof: Since $x_n \to x$, we have $q(x_n, x_n, x) \to 0$

From (3.1.2)

$$\max\{q(x_n,x_n,t), q(x,x,t), q(x_n,x_n,t)\} \le q(x_n,x,x_n) \text{ for } t \in \{x, x_n\}$$

Taking $t = x_n$

$$\max\{q(x_n,x_n,x_n), q(x,x,x_n), q(x_n,x_n,x_n)\} \le q(x_n,x,x_n)$$
$$\therefore q(x_n,x_n,x_n) \to 0, q(x,x,x_n) \to 0$$

3.6.3 If $x_n \to x, x_n \to y$, then $q(x, y, x_n) \to 0$ as $n \to \infty$

Proof: Since $x_n \to x$, we have $q(x_n, x_n, x) \to 0$ and $x_n \to y$, we have $q(x_n, x_n, y) \to 0$

From (3.1.4), we have

$$q(x,y,x_n) \le q(x,x_n,x_n) + q(x_n,y,x_n) \to 0 \text{ as } n \to \infty$$

Hence $q(x, y, x_n) \to 0$ as $n \to \infty$

3.6.4 The limit is unique i.e. $x_n \to x, x_n \to y \Longrightarrow x = y$

If $x_n \to x, x_n \to y$, then $q(x_n, x_n, x) \to 0$ and $q(x_n, x_n, y) \to 0$

From (3.1.2)

$$\max\{q(x_n,x_n,t), q(x,x,t), q(x_n,x_n,t)\} \le q(x_n,x,x_n)$$
$$\Longrightarrow \max\{q(x_n,x_n,x_n), q(x,x,x_n), q(x_n,x_n,x_n)\} \le q(x_n,x,x_n)$$
$$\therefore q(x_n,x_n,x_n) \to 0, q(x,x,x_n) \to 0$$

Similarly we get $q(y, y, x_n) \to 0$

From (3.1.4)

$$q(x,y,x_n) \le q(x,x_n,x_n) + q(x_n,y,x_n) - q(x_n,x_n,x_n) \to 0 \text{ as } n \to \infty$$
$$\therefore q(x,y,x_n) \to 0 \text{ as } n \to \infty$$

From (3.1.2)

$$\max\{q(x,x,x), q(y,y,x), q(x_n,x_n,x)\} \le q(x,y,x_n) \to 0 \text{ as } n \to \infty$$
$$\therefore q(y,y,x) = 0$$

From (3.1.1)

$$0 = q(x,x,x) = q(y,y,y) = q(x,y,y) \Longrightarrow x = y$$

Therefore the limit is unique.

3.6.5 If $x_n \to x$, then $q(z,z,x_n) \to q(z,z,x)$

From (3.1.2), we have $q(z,z,x_n) \le q(z,x,x_n)$
$\le q(z,x_n,x_n) + q(x_n,x,x_n) - q(x_n,x_n,x_n)$

$$\therefore 0 \le q(z,x,x_n) - q(z,z,x_n) \le q(x_n,x,x_n) - q(x_n,x_n,x_n) \quad (3.6.5.1)$$
(since $q(z,x_n,x_n) = q(z,z,x_n) \to 0$ as $n \to \infty$)

Now $q(z,x_n,x) \le q(z,x,x) + q(x,x_n,x) - q(x,x,x)$ (by (3.1.4))
$\therefore \lim_{n\to\infty} \sup q(z,x_n,x) \le q(z,x,x) \le q(z,x_n,x)$ (by (3.1.2))
$\therefore \lim_{n\to\infty} \sup q(z,x_n,x) \le q(z,x,x) \le \lim_{n\to\infty} \inf q(z,x_n,x)$
$\therefore \lim_{n\to\infty} q(z,x_n,x) = q(z,x,x)$ or $q(z,z,x)$
$\therefore$ From (3.6.5.1)
$0 \le \lim\limits_{n\to\infty}\left(q(z,x,x_n) - q(z,z,x_n)\right) \le 0$
$0 \le q(z,z,x) - \lim\limits_{n\to\infty} q(z,z,x_n) \le 0$
$\therefore \lim_{n\to\infty} q(z,z,x_n) = q(z,z,x)$ or $q(z,z,x_n) \to q(z,z,x)$

Now we introduce a notation

3.7 Notation: Let $\Phi = \{\varphi\ \varphi: [0,\infty) \to [0,\infty)$, where φ is continuous, increasing, $\varphi(t) = 0$ if $t = 0$ and $\varphi(t) < t$ if $t > 0$ and $\sum \varphi^n(t) < \infty\}$.

3.8 Definition: Let (X,q) be a quasi-S-metric space, T be a self map on X and $\psi \in \Phi$.

Suppose $q(Tx,Ty,Tz) \le \psi(\max\{q(x,y,z), q(x,Tx,Tx), q(y,Ty,Ty), q(z,Tz,Tz)\})$ for all $x,y,z \in X$. Then T is called a ψ-contraction on X.

Now we prove the first main theorem for a ψ-contraction on a complete quasi-S-metric space.

3.9 Theorem: Let (X,q) be a complete quasi-S-metric space and $\psi \in \Phi$.
Suppose $T: X \to X$ is a ψ-contraction. That is
$$q(Tx,Ty,Tz) \le \psi\left(\max\{q(x,y,z), q(x,Tx,Tx), q(y,Ty,Ty), q(z,Tz,Tz)\}\right) \text{ for all } x,y,z \in X. \quad (3.9.1)$$
Then T has a unique fixed point.

Proof: Let $x_0 \in X$.

Write $x_{n+1} = Tx_n$ for $n = 0,1,2,\ldots$

If $x_n = x_{n+1}$ for some n, then x_n is a fixed point of T

Hence we may suppose that $x_n \ne x_{n+1}$ for $n = 0,1,2,\ldots$

Consider
$$\begin{aligned} q(x_{n+1},x_{n+2},x_{n+3}) &= q(Tx_n,Tx_{n+1},Tx_{n+2}) \\ &\le \psi\left(\max\left\{\begin{matrix} q(x_n,x_{n+1},x_{n+2}), q(x_n,Tx_n,Tx_n), \\ q(x_{n+1},Tx_{n+1},Tx_{n+1}), q(x_{n+2},Tx_{n+2},Tx_{n+2})\end{matrix}\right\}\right) \\ &= \psi\left(\max\left\{\begin{matrix} q(x_n,x_{n+1},x_{n+2}), q(x_n,x_{n+1},x_{n+1}), \\ q(x_{n+1},x_{n+2},x_{n+2}), q(x_{n+2},x_{n+3},x_{n+3})\end{matrix}\right\}\right) \\ &\le \psi\left(q(x_n,x_{n+1},x_{n+2})\right) \qquad (3.9.4) \\ &< q(x_n,x_{n+1},x_{n+2}) \end{aligned}$$

$\therefore \{q(x_n,x_{n+1},x_{n+2})\} \downarrow 0$

From (3.9.4) by induction, we get
$$q(x_{n+1},x_{n+2},x_{n+3}) \le \psi^n\left(q(x_0,x_1,x_2)\right) \quad (3.9.5)$$

By induction, we get
$$q(x_n,x_{n+1},x_{n+k}) \le \psi(t) + \psi^2(t) + \psi^3(t) + \cdots + \psi^{k-2}(t)$$

For any n, and $k = 3,4, \ldots$ where $t = q(x_n, x_{n+1}, x_{n+2})$

$$\therefore q(x_n, x_n, x_m) \leq q(x_n, x_{n+1}, x_m) \to 0 \text{ as } m, n \to \infty$$
$$= q(x_n, x_{n+1}, x_{n+k}) \text{ (m>n)}$$
$$\leq \psi(t) + \psi^2(t) + \psi^3(t) + \cdots + \psi^{k-2}(t) \text{ (where } t = q(x_n, x_{n+1}, x_{n+2}))$$
$$\leq \psi^n\big(q(x_0, x_1, x_2)\big) \text{ (by 3.9.5)}$$
$$\leq \textstyle\sum_{l=1}^{k-2} \psi^{n+l}\big(q(x_0, x_1, x_2)\big) \to 0 \text{ as } n \to \infty$$

$\therefore\ q(x_n, x_n, x_m) \to 0$ as $m, n \to \infty$

$\therefore \{x_n\}$ is Cauchy.

Now we show that T has a fixed point

Suppose $x_m \to l$ (since X is complete)

Consider

$$q(Tl, Tl, x_{m+1}) = q(Tl, Tl, Tx_m)$$
$$\leq \psi\left(\max\begin{Bmatrix} q(l, l, x_m), q(l, Tl, Tl), \\ q(l, Tl, Tl), q(x_m, Tx_m, Tx_m) \end{Bmatrix}\right)$$
$$= \psi\left(\max\begin{Bmatrix} q(l, l, x_m), q(l, Tl, Tl), \\ q(x_m, x_{m+1}, x_{m+1}) \end{Bmatrix}\right)$$

Letting $m \to \infty$, we get

$$q(Tl, Tl, l) \leq \psi\left(\max\begin{Bmatrix} q(l, l, l), q(l, Tl, Tl), \\ q(l, Tl, Tl), q(l, l, l) \end{Bmatrix}\right) \text{ (by (3.6.5))}$$
$$\leq \psi(\max\{q(l, l, l), q(l, Tl, Tl)\})$$
$$= \psi\big(q(l, Tl, Tl)\big) \text{ (by (3.6.1))}$$
$$\leq \psi\big(q(Tl, Tl, l)\big)$$

$\therefore q(Tl, Tl, l) \leq \psi\big(q(Tl, Tl, l)\big)$

$\therefore q(Tl, Tl, l) = 0$

$\therefore Tl = l$

$\therefore l$ is a fixed point of T

Now we show that fixed point of T is unique

Suppose x and y are two fixed points of T

So $Tx = x, Ty = y$

From (3.9.1)

$$q(x, x, y) = q(Tx, Tx, Ty)$$
$$\leq \psi\left(\max\begin{Bmatrix} q(x, x, y), q(x, Tx, Tx) \\ q(x, Tx, Tx), q(y, Ty, Ty) \end{Bmatrix}\right)$$
$$= \psi\left(\max\begin{Bmatrix} q(x, x, y), q(x, x, x) \\ q(x, x, x), q(y, y, y) \end{Bmatrix}\right)$$
$$= \psi(\max\{q(x, x, y), q(x, x, x), q(y, y, y)\})$$
$$\leq \psi\big(q(x, x, y)\big)$$

(From (3.1.2), we have $q(x, x, x) \leq\ q(x, x, y), q(y, y, y) \leq\ q(y, y, x)$)

$\therefore q(x, x, y) \leq \psi\big(q(x, x, y)\big)$

$\therefore\ q(x, x, y) = 0$

$\therefore x = y$

Thus fixed point of T is unique.

3.10 Corollary: Let (X, q) be a complete quasi-S-metric space. Let $T: X \to X$ and $0 \leq \lambda < 1$.

Suppose $q(Tx, Ty, Tz) \leq \lambda\, q(x, Ty, Tz)$ for all $x, y, z \in X$.

Then T has a unique fixed point.

Proof: Define $\psi(t) = \lambda t$ for $t \geq 0$. Then $\psi \in \Phi$ and from (3.9.1), we have

$$q(Tx, Ty, Tz) \leq \lambda q(x, y, z) = \psi\big(q(x, y, z)\big)$$
$$\leq \psi\left(\max\begin{pmatrix} q(x, y, z), q(x, Tx, Tx) \\ q(y, Ty, Ty), q(z, Tz, Tz) \end{pmatrix}\right)$$

Now the result follows from theorem 3.9.

3.11 Definition: Let (X,q) be a quasi-S-metric space, T be a self map on X and $\varphi \in \Phi$.

Suppose $$q(Tx,Ty,Tz) \le \varphi\left(max\left(\begin{matrix} max\begin{Bmatrix} q(x,y,z), q(x,Tx,Tx), \\ q(y,Ty,Ty), q(z,Tz,Tz) \end{Bmatrix}, \\ \frac{1}{2} max\begin{Bmatrix} q(x,Ty,Ty), q(x,Tz,Tz), q(y,Tx,Tx), \\ q(y,Tz,Tz), q(z,Tx,Tx), q(z,Ty,Ty) \end{Bmatrix} \end{matrix} \right)\right) \forall\, x,y,z \in X.$$

Then T is called a φ-contraction on X.

Now we prove our second main theorem, which involves the members of the class Φ.

3.12 Theorem: Let (X,q) be a Complete quasi-S-metric space, $\varphi \in \Phi$ and let $T : X \to X$ be such that

$$q(Tx,Ty,Tz) \le \varphi\left(max\left(\begin{matrix} max\begin{Bmatrix} q(x,y,z), q(x,Tx,Tx), \\ q(y,Ty,Ty), q(z,Tz,Tz) \end{Bmatrix}, \\ \frac{1}{2} max\begin{Bmatrix} q(x,Ty,Ty), q(x,Tz,Tz), q(y,Tx,Tx), \\ q(y,Tz,Tz), q(z,Tx,Tx), q(z,Ty,Ty) \end{Bmatrix} \end{matrix} \right)\right) \forall\, x,y,z \in X.$$

Then T has a unique fixed point.

Proof: Let $x_0 \in X$, and define inductively $x_{n+1} = Tx_n$,$n = 0,1,2,\ldots$

If $x_{n+1} = x_n$ for some n, then x_n is a fixed point of T.

Hence we may suppose that $x_{n+1} \ne x_n$ for $n = 0,1,2,\ldots$

Consider

$$\begin{aligned} q(x_n,x_{n+1},x_{n+2}) &= q(Tx_{n-1},Tx_n,Tx_{n+1}) \\ &\le \varphi\left(\max\left(\begin{matrix} max\begin{Bmatrix} q(x_{n-1},x_n,x_{n+1}), q(x_{n-1},Tx_{n-1},Tx_{n-1}), \\ q(x_n,Tx_n,Tx_n), q(x_{n+1},Tx_{n+1},Tx_{n+1}) \end{Bmatrix}, \\ \frac{1}{2} max\begin{Bmatrix} q(x_{n-1},Tx_n,Tx_n), q(x_{n-1},Tx_{n+1},Tx_{n+1}), \\ q(x_n,Tx_{n-1},Tx_{n-1}), q(x_n,Tx_{n+1},Tx_{n+1}) \\ , q(x_{n+1},Tx_{n-1},Tx_{n-1}), q(x_{n+1},Tx_n,Tx_n) \end{Bmatrix} \end{matrix} \right)\right) \\ &\le \varphi\left(\max\left(\begin{matrix} max\begin{Bmatrix} q(x_{n-1},x_n,x_{n+1}), q(x_{n-1},x_n,x_n), \\ q(x_n,x_{n+1},x_{n+1}), q(x_{n+1},x_{n+2},x_{n+2}) \end{Bmatrix}, \\ \frac{1}{2} max\begin{Bmatrix} q(x_{n-1},x_{n+1},x_{n+1}), q(x_{n-1},x_{n+2},x_{n+2}), \\ q(x_n,x_n,x_n), q(x_n,x_{n+2},x_{n+2}) \\ , q(x_{n+1},x_n,x_n), q(x_{n+1},x_{n+1},x_{n+1}) \end{Bmatrix} \end{matrix} \right)\right) \end{aligned}$$

From (3.1.2) we have

$$\begin{aligned} &q(x_{n-1},x_n,x_n) \le q(x_{n-1},x_n,x_{n+1}) \\ &q(x_n,x_{n+1},x_{n+1}) \le q(x_n,x_{n+1},x_{n+2}) \\ &q(x_{n+1},x_{n+2},x_{n+2}) \le q(x_{n+1},x_{n+2},x_n) \\ &q(x_{n-1},x_{n+1},x_{n+1}) \le q(x_{n-1},x_n,x_{n+1}) \\ &q(x_{n-1},x_{n+2},x_{n+2}) \le q(x_{n-1},x_{n+1},x_{n+2}) \\ &q(x_n,x_n,x_n) \le q(x_n,x_{n+1},x_{n+2}) \\ &q(x_n,x_{n+2},x_{n+2}) \le q(x_n,x_{n+1},x_{n+2}) \\ &q(x_{n+1},x_n,x_n) \le q(x_{n+1},x_{n-1},x_n) \\ &q(x_{n+1},x_{n+1},x_{n+1}) \le q(x_{n+1},x_n,x_{n+2}) \end{aligned}$$

$$\begin{aligned} \therefore q(x_n,x_{n+1},x_{n+2}) &\le \varphi\left(\max\begin{Bmatrix} max\begin{Bmatrix} q(x_{n-1},x_n,x_{n+1}), q(x_{n-1},x_n,x_{n+1}), \\ q(x_n,x_{n+1},x_{n+2}), q(x_{n+1},x_n,x_{n+2}) \end{Bmatrix}, \\ \frac{1}{2} max\begin{Bmatrix} q(x_{n-1},x_n,x_{n+1}), q(x_{n-1},x_{n+1},x_{n+2}), \\ q(x_n,x_{n+1},x_{n+2}), q(x_n,x_{n+1},x_{n+2}), \\ q(x_{n+1},x_{n-1},x_n), q(x_{n+1},x_n,x_{n+2}) \end{Bmatrix} \end{Bmatrix}\right) \\ &= \varphi\left(\max\begin{Bmatrix} \max\{q(x_{n-1},x_n,x_{n+1}), q(x_n,x_{n+1},x_{n+2})\}, \\ \frac{1}{2} q(x_{n-1},x_{n+1},x_{n+2}) \end{Bmatrix}\right) \\ &= \varphi\left(\max\begin{Bmatrix} max\{q(x_{n-1},x_n,x_{n+1}), q(x_n,x_{n+1},x_{n+2})\}, \\ \frac{1}{2}[q(x_{n-1},x_n,x_{n+1}) + q(x_n,x_{n+2},x_{n+1})] \end{Bmatrix}\right) \text{ (by (3.1.4))} \end{aligned}$$

$$\therefore q(x_n,x_{n+1},x_{n+2}) \le \varphi(\max\{q(x_{n-1},x_n,x_{n+1}), q(x_n,x_{n+1},x_{n+2})\})$$

If $q(x_n, x_{n+1}, x_{n+2})$ is maximum, we get

$$q(x_n, x_{n+1}, x_{n+2}) \leq \varphi\big(q(x_n, x_{n+1}, x_{n+2})\big)$$

Then $q(x_n, x_{n+1}, x_{n+2}) = 0$

Hence $x_{n+1} = x_n$ which is a contradiction, by (3.1.1)

Therefore $q(x_{n-1}, x_n, x_{n+1})$ is maximum

$$\text{i.e., } q(x_n, x_{n+1}, x_{n+2}) \leq \varphi\big(q(x_{n-1}, x_n, x_{n+1})\big) < q(x_{n-1}, x_n, x_{n+1}) \tag{3.12.1}$$

Hence $\{q(x_{n-1}, x_n, x_{n+1})\}$ is a strictly decreasing sequence

Suppose $\{\, q(x_{n-1}, x_n, x_{n+1})\} \downarrow \alpha$

On letting $n \to \infty$, we get

$$\lim_{n\to\infty} q(x_{n-1}, x_n, x_{n+1}) = \alpha \leq \varphi(\alpha)$$

Hence $\alpha = 0$

Therefore $q(x_{n-1}, x_n, x_{n+1}) \to 0$ as $n \to \infty$

Now we show that $\{x_n\}$ is Cauchy

Consider

$$\begin{aligned} q(x_{n+1}, x_{n+1}, x_{n+k}) &\leq q(x_{n+1}, x_{n+2}, x_{n+k}) \quad \text{(by 3.1.2)} \\ &= q(x_{n+1}, x_{n+k}, x_{n+2}) \\ &\leq q(x_{n+1}, x_{n+3}, x_{n+2}) + q(x_{n+3}, x_{n+k}, x_{n+2}) - q(x_{n+3}, x_{n+3}, x_{n+2}) \\ &\leq q(x_{n+1}, x_{n+2}, x_{n+3}) + q(x_{n+2}, x_{n+k}, x_{n+3}) \\ &= q(x_{n+1}, x_{n+2}, x_{n+3}) + q(x_{n+2}, x_{n+4}, x_{n+3}) + q(x_{n+4}, x_{n+k}, x_{n+3}) \\ &\leq q(x_{n+1}, x_{n+2}, x_{n+3}) + q(x_{n+2}, x_{n+3}, x_{n+4}) + q(x_{n+3}, x_{n+k}, x_{n+4}) \end{aligned}$$

Therefore by induction we get

$$\begin{aligned} q(x_{n+1}, x_{n+1}, x_{n+k}) &\leq q(x_{n+1}, x_{n+2}, x_{n+3}) + q(x_{n+2}, x_{n+3}, x_{n+4}) + \cdots + q\big(x_{n+(k-2)}, x_{n+(k-1)}, x_{n+k}\big) \\ &\leq \varphi\big(q(x_n, x_{n+1}, x_{n+2})\big) + \varphi\big(q(x_{n+1}, x_{n+2}, x_{n+3})\big) + \cdots + \varphi\big(q\big(x_{n+(k-3)}, x_{n+(k-2)}, x_{n+(k-1)}\big)\big) \end{aligned}$$

(by 3.9.2)

Now

$$\begin{aligned} \varphi\big(q(x_{n+1}, x_{n+2}, x_{n+3})\big) &\leq \varphi\Big(\varphi\big(q(x_n, x_{n+1}, x_{n+2})\big)\Big) \\ &= \varphi^2\big(q(x_n, x_{n+1}, x_{n+2})\big) \end{aligned}$$

$$\begin{aligned} \varphi\big(q(x_{n+2}, x_{n+3}, x_{n+4})\big) &\leq \varphi\Big(\varphi\big(q(x_{n+1}, x_{n+2}, x_{n+3})\big)\Big) \\ &= \varphi\Big(\varphi\Big(\varphi\big(q(x_n, x_{n+1}, x_{n+2})\big)\Big)\Big) \\ &= \varphi^3\big(q(x_n, x_{n+1}, x_{n+2})\big) \end{aligned}$$

Therefore by induction

$$\begin{aligned} \therefore \varphi\Big(q\big(x_{n+(k-3)}, x_{n+(k-2)}, x_{n+(k-1)}\big)\Big) &= \varphi\Big(q\big(x_{n+(k-4)}, x_{n+(k-3)}, x_{n+(k-2)}\big)\Big) \\ &\leq \varphi\Big(\varphi^{k-3}\big(q(x_n, x_{n+1}, x_{n+2})\big)\Big) \\ &= \varphi^{k-2}\big(q(x_n, x_{n+1}, x_{n+2})\big) \end{aligned}$$

$$\therefore q(x_{n+1}, x_{n+1}, x_{n+k}) \leq \varphi(t) + \varphi^2(t) + \cdots + \varphi^{k-2}(t) \to 0 \text{ as } k \to \infty, \text{ where } t = q(x_n, x_{n+1}, x_{n+2})$$

(Since $\varphi \in \Phi$)

$$\therefore\ q(x_n, x_n, x_m) \to 0 \text{ as } m, n \to \infty$$

Therefore $\{x_n\}$ is Cauchy.

Let $\{x_n\} \to p$ as $n \to \infty$

Now we show that p is a fixed point of T

Consider

$$q(Tp,Tp,x_{m+1}) = q(Tp,Tp,Tx_m)$$
$$\leq \varphi\left(\max\left\{\begin{array}{l} max\left\{\begin{array}{c} q(p,p,x_m), q(p,Tp,Tp), \\ q(p,Tp,Tp), q(x_m,Tx_m,Tx_m)\end{array}\right\}, \\ \frac{1}{2}max\left\{\begin{array}{c} q(p,Tp,Tp), q(p,Tx_m,Tx_m), \\ q(p,Tp,Tp), q(p,Tx_m,Tx_m), \\ q(x_m,Tp,Tp), q(x_m,Tp,Tp)\end{array}\right\}\end{array}\right\}\right)$$

On letting $m \to \infty$

$$q(Tp,Tp,p) \leq \varphi\left(\max\left(\begin{array}{l} max\left\{\begin{array}{c} q(p,p,p), q(p,Tp,Tp), \\ q(p,Tp,Tp), q(p,Tp,Tp)\end{array}\right\}, \\ \frac{1}{2}max\left\{\begin{array}{c} q(p,Tp,Tp), q(p,Tp,Tp), \\ q(p,Tp,Tp), q(p,Tp,Tp), \\ q(p,Tp,Tp), q(p,Tp,Tp)\end{array}\right\}\end{array}\right)\right)$$
$$= \varphi\left(\max\left(\begin{array}{c} max\{q(p,p,p), q(p,Tp,Tp)\}, \\ \frac{1}{2}q(p,Tp,Tp)\end{array}\right)\right)$$
$$\leq \varphi\big(q(p,Tp,Tp)\big) = \varphi\,(q(Tp,Tp,p)) \text{ by } (3.2.1)$$

$\therefore\ q(Tp,Tp,p) \leq \varphi\big(q(Tp,Tp,p)\big) < q(Tp,Tp,p)$, a contradiction if $Tp = p$

$\therefore Tp = p$

Thus p is a fixed point of T.

Now we show that fixed point of T is unique

Let p and r be fixed points of T then $Tp = p$ and $Tr = r$

Consider

$$q(p,p,r) = q(Tp,Tp,Tr)$$
$$\leq \varphi\left(\max\left(\begin{array}{l} max\left\{\begin{array}{c} q(p,p,r), q(p,Tp,Tp), \\ q(p,Tp,Tp), q(r,Tr,Tr)\end{array}\right\}, \\ \frac{1}{2}max\left\{\begin{array}{c} q(p,Tp,Tp), q(p,Tr,Tr), \\ q(p,Tp,Tp), q(p,Tr,Tr), \\ q(r,Tp,Tp), q(r,Tp,Tp)\end{array}\right\}\end{array}\right)\right)$$
$$= \varphi\left(\max\left(\begin{array}{l} max\left\{\begin{array}{c} q(p,p,r), q(p,p,p), \\ q(p,p,p), q(r,r,r)\end{array}\right\}, \\ \frac{1}{2}max\left\{\begin{array}{c} q(p,p,p), q(p,r,r), \\ q(p,p,p), q(p,r,r), \\ q(r,p,p), q(r,p,p)\end{array}\right\}\end{array}\right)\right)$$
$$= \varphi\left(\max\left(\begin{array}{l} max\{q(p,p,r), q(p,p,p), q(r,r,r)\}, \\ \frac{1}{2}max\{q(p,p,p), q(p,r,r), q(r,p,p)\}\end{array}\right)\right)$$

From (3.1.2) , we have $q(p,p,p) \leq q(p,p,r)$, $\ q(r,r,r) \leq q(r,r,p)$

$\therefore q(p,p,r) \leq \varphi\big(q(p,p,r)\big)$

$\therefore q(p,p,r) = 0$

$\therefore p = r$

We conclude the paper with an example of a quasi-S-metric space.

4. EXAMPLES

4.2 Example: Let $X = \{[a,b] | a,b \in R^+, a \leq b\}$ and define $q([a,b],[c,d],[e,f]) = \max\{b,d,f\} - \min\{a,c,e\}$. Then $q: X \times X \times X \to R^+$ is a quasi-S-metric and (X,q) is a complete qSMS.

Define $T: X \to X$ by $Tx = \frac{x}{2}$ i.e. $[a,b] \in X \Longrightarrow T([a,b]) = \left[\frac{a}{2}, \frac{b}{2}\right]$ and

Define ψ:$[0,\infty) \to [0,\infty)$ by $\psi(t) = \frac{t}{2}$ then $\psi \in \Phi$.

Show that T is a ψ-contraction.
i.e. $q(Tx,Ty,Tz) \le \psi(\max\{q(x,y,z), q(x,Tx,Tx), q(y,Ty,Ty), q(z,Tz,Tz)\})$ (4.2.1)

Solution: Let $x = [a,b]$, $y = [c,d]$, $z = [e,f]$

So that $Tx = \left[\frac{a}{2},\frac{b}{2}\right]$, $Ty = \left[\frac{c}{2},\frac{d}{2}\right]$, $Tz = \left[\frac{e}{2},\frac{f}{2}\right]$
L.H.S (4.2.1)

$$q(Tx,Ty,Tz) = q\left(\left[\frac{a}{2},\frac{b}{2}\right],\left[\frac{c}{2},\frac{d}{2}\right],\left[\frac{e}{2},\frac{f}{2}\right]\right)$$
$$= \max\left\{\frac{b}{2},\frac{d}{2},\frac{f}{2}\right\} - \min\left\{\frac{a}{2},\frac{c}{2},\frac{e}{2}\right\} \quad (4.2.2)$$

R.H.S (4.2.1)
$\psi(\max\{q(x,y,z), q(x,Tx,Tx), q(y,Ty,Ty), q(z,Tz,Tz)\})$

$$= \psi\left(\max\left\{\begin{matrix} q([a,b],[c,d],[e,f]), q\left([a,b],\left[\frac{a}{2},\frac{b}{2}\right],\left[\frac{a}{2},\frac{b}{2}\right]\right), \\ q\left([c,d],\left[\frac{c}{2},\frac{d}{2}\right],\left[\frac{c}{2},\frac{d}{2}\right]\right), q\left([e,f],\left[\frac{e}{2},\frac{f}{2}\right],\left[\frac{e}{2},\frac{f}{2}\right]\right)\end{matrix}\right\}\right)$$
$$= \psi\left(\max\left\{\begin{matrix} \max\{b,d,f\} - \min\{a,c,e\}, \max\left\{b,\frac{b}{2},\frac{b}{2}\right\} - \min\left\{a,\frac{a}{2},\frac{a}{2}\right\}, \\ \max\left\{d,\frac{d}{2},\frac{d}{2}\right\} - \min\left\{c,\frac{c}{2},\frac{c}{2}\right\}, \max\left\{f,\frac{f}{2},\frac{f}{2}\right\} - \min\left\{e,\frac{e}{2},\frac{e}{2}\right\}\end{matrix}\right\}\right)$$
$$= \psi\left(\max\left\{\max\{b,d,f\} - \min\{a,c,e\}, \left(b-\frac{a}{2}\right), \left(d-\frac{c}{2}\right), \left(f-\frac{e}{2}\right)\right\}\right)$$
$$= \max\left\{\max\left\{\frac{b}{2},\frac{d}{2},\frac{f}{2}\right\} - \min\left\{\frac{a}{2},\frac{c}{2},\frac{e}{2}\right\}, \left(\frac{b}{2}-\frac{a}{4}\right), \left(\frac{d}{2}-\frac{c}{4}\right), \left(\frac{f}{2}-\frac{e}{4}\right)\right\} \quad (4.2.3)$$

So (4.2.2) ≤ (4.2.3)

Therefore L.H.S≤R.H.S

By Theorem (3.9), T has a unique fixed point.

Clearly $x = [a,b]$ = [0,0] is a fixed point of T and is unique.

4.3 Example: Let (X,q) be a complete quasi-S-metric space, $\varphi \in \Phi$ and let $T: X \to X$ be such that

$$q(Tx,Ty,Tz) \le \varphi\left(max\left(\begin{matrix} max\left\{\begin{matrix} q(x,y,z), q(x,Tx,Tx), \\ q(y,Ty,Ty), q(z,Tz,Tz)\end{matrix}\right\}, \\ \frac{1}{2}max\left\{\begin{matrix} q(x,Ty,Ty), q(x,Tz,Tz), q(y,Tx,Tx), \\ q(y,Tz,Tz), q(z,Tx,Tx), q(z,Ty,Ty)\end{matrix}\right\}\end{matrix}\right)\right) \forall\, x,y,z \in X. \quad (4.3.1)$$

Define $Tx = \frac{x}{2}$, $\varphi(t)=\frac{t}{2}$
Then T has a unique fixed point in X.

Solution: Let $x = [a,b]$, $y = [c,d]$, $z = [e,f]$ and

Define $q(x,y,z) = q([a,b],[c,d],[e,f]) = \max\{b,d,f\} - \min\{a,c,e\}$.

From (4.3.1) L.H.S= $q(Tx,Ty,Tz) = q\left(\left[\frac{a}{2},\frac{b}{2}\right],\left[\frac{c}{2},\frac{d}{2}\right],\left[\frac{e}{2},\frac{f}{2}\right]\right)$
$= \max\left\{\frac{b}{2},\frac{d}{2},\frac{f}{2}\right\} - \min\left\{\frac{a}{2},\frac{c}{2},\frac{e}{2}\right\}$

From (4.3.1)

$$\text{R.H.S}= \varphi\left(max\left(\begin{matrix} max\left\{\begin{matrix} q([a,b],[c,d],[e,f]), q\left([a,b],\left[\frac{a}{2},\frac{b}{2}\right],\left[\frac{a}{2},\frac{b}{2}\right]\right), \\ q\left([c,d],\left[\frac{c}{2},\frac{d}{2}\right],\left[\frac{c}{2},\frac{d}{2}\right]\right), q\left([e,f],\left[\frac{e}{2},\frac{f}{2}\right],\left[\frac{e}{2},\frac{f}{2}\right]\right)\end{matrix}\right\}, \\ \frac{1}{2}max\left\{\begin{matrix} q\left([a,b],\left[\frac{c}{2},\frac{d}{2}\right],\left[\frac{c}{2},\frac{d}{2}\right]\right), q\left([a,b],\left[\frac{e}{2},\frac{f}{2}\right],\left[\frac{e}{2},\frac{f}{2}\right]\right), \\ q\left([c,d],\left[\frac{a}{2},\frac{b}{2}\right],\left[\frac{a}{2},\frac{b}{2}\right]\right), q\left([c,d],\left[\frac{e}{2},\frac{f}{2}\right],\left[\frac{e}{2},\frac{f}{2}\right]\right) \\ q\left([e,f],\left[\frac{a}{2},\frac{b}{2}\right],\left[\frac{a}{2},\frac{b}{2}\right]\right), q\left([e,f],\left[\frac{c}{2},\frac{d}{2}\right],\left[\frac{c}{2},\frac{d}{2}\right]\right)\end{matrix}\right\}\end{matrix}\right)\right)$$

$$= \varphi\left(max \begin{pmatrix} \max\begin{Bmatrix} \max\{b,d,f\} - \min\{a,c,e\}, \max\left\{a,\frac{a}{2},\frac{a}{2}\right\} - \min\left\{b,\frac{b}{2},\frac{b}{2}\right\}, \\ \max\left\{d,\frac{d}{2},\frac{d}{2}\right\} - \min\left\{c,\frac{c}{2},\frac{c}{2}\right\}, \max\left\{f,\frac{f}{2},\frac{f}{2}\right\} - \min\left\{e,\frac{e}{2},\frac{e}{2}\right\} \end{Bmatrix}, \\ \frac{1}{2}\max\begin{Bmatrix} \max\left\{b,\frac{d}{2},\frac{d}{2}\right\} - \min\left\{a,\frac{c}{2},\frac{c}{2}\right\}, \max\left\{b,\frac{f}{2},\frac{f}{2}\right\} - \min\left\{a,\frac{e}{2},\frac{e}{2}\right\}, \\ \max\left\{d,\frac{b}{2},\frac{b}{2}\right\} - \min\left\{c,\frac{a}{2},\frac{a}{2}\right\}, \max\left\{d,\frac{f}{2},\frac{f}{2}\right\} - \min\left\{c,\frac{e}{2},\frac{e}{2}\right\}, \\ \max\left\{f,\frac{b}{2},\frac{b}{2}\right\} - \min\left\{e,\frac{a}{2},\frac{a}{2}\right\}, \max\left\{f,\frac{d}{2},\frac{d}{2}\right\} - \min\left\{e,\frac{c}{2},\frac{c}{2}\right\} \end{Bmatrix} \end{pmatrix} \right)$$

$$= \varphi\left(max \begin{pmatrix} \max\left\{\max\{b,d,f\} - \min\{a,c,e\}, \left(b-\frac{a}{2}\right), \left(d-\frac{c}{2}\right), \left(f-\frac{e}{2}\right)\right\}, \\ \frac{1}{2}\max\begin{Bmatrix} \max\left\{b,\frac{d}{2}\right\} - \min\left\{a,\frac{c}{2}\right\}, \max\left\{b,\frac{f}{2}\right\} - \min\left\{a,\frac{e}{2}\right\}, \\ \max\left\{d,\frac{b}{2}\right\} - \min\left\{c,\frac{a}{2}\right\}, \max\left\{d,\frac{f}{2}\right\} - \min\left\{c,\frac{e}{2}\right\}, \\ \max\left\{f,\frac{b}{2}\right\} - \min\left\{e,\frac{a}{2}\right\}, \max\left\{f,\frac{d}{2}\right\} - \min\left\{e,\frac{c}{2}\right\} \end{Bmatrix} \end{pmatrix} \right)$$

Therefore

L.H.S = $\max\left\{\frac{b}{2},\frac{d}{2},\frac{f}{2}\right\} - \min\left\{\frac{a}{2},\frac{c}{2},\frac{e}{2}\right\}$

$= \varphi(\max\{b,d,f\} - \min\{a,c,e\})$ $\left(\text{since } \varphi(t) = \frac{t}{2}\right)$

$$\leq \varphi\left(\max\begin{Bmatrix} \max\left\{\max\{b,d,f\} - \min\{a,c,e\}, \left(b-\frac{a}{2}\right), \left(d-\frac{c}{2}\right), \left(f-\frac{e}{2}\right)\right\}, \\ \frac{1}{2}\max\begin{Bmatrix} \max\left\{b,\frac{d}{2}\right\} - \min\left\{a,\frac{c}{2}\right\}, \max\left\{b,\frac{f}{2}\right\} - \min\left\{a,\frac{e}{2}\right\}, \\ \max\left\{d,\frac{b}{2}\right\} - \min\left\{c,\frac{a}{2}\right\}, \max\left\{d,\frac{f}{2}\right\} - \min\left\{c,\frac{e}{2}\right\}, \\ \max\left\{f,\frac{b}{2}\right\} - \min\left\{e,\frac{a}{2}\right\}, \max\left\{f,\frac{d}{2}\right\} - \min\left\{e,\frac{c}{2}\right\} \end{Bmatrix} \end{Bmatrix} \right) \text{(since } \varphi \text{ is increasing)}$$

Therefore L.H.S ≤ R.H.S

By Theorem (3.11), T has a unique fixed point.

Clearly [0, 0] is a unique fixed point.

4.4 Example: Let $X = \{0,1,2,\dots\}$ and $q: X \times X \times X \to [0,\infty)$ is a quasi-S-metric defined by $q(x,y,z) = \max\{x,y,z\}$ for all $x,y,z \in X$ and (X,q) is a complete quasi-S-metric space. Define $T : X \to X$ by $Tx = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ 0 & \text{otherwise} \end{cases}$

Define $\varphi: [0,\infty) \to [0,\infty)$ by $\varphi(t) = \frac{t}{2}$ then $\varphi \in \Phi$**.** Now we show that T is a φ-contraction.

i.e., $q(Tx,Ty,Tz) \leq \varphi\left(max \begin{pmatrix} max\begin{Bmatrix} q(x,y,z), q(x,Tx,Tx), \\ q(y,Ty,Ty), q(z,Tz,Tz) \end{Bmatrix}, \\ \frac{1}{2}max\begin{Bmatrix} q(x,Ty,Ty), q(x,Tz,Tz), q(y,Tx,Tx), \\ q(y,Tz,Tz), q(z,Tx,Tx), q(z,Ty,Ty) \end{Bmatrix} \end{pmatrix} \right)$

$$\leq \left(max \begin{pmatrix} max\{q(x,y,z)\}, \\ \frac{1}{2}max\begin{Bmatrix} q(x,y,z), q(x,y,z), q(x,y,z), \\ q(x,y,z), q(x,y,z), q(x,y,z) \end{Bmatrix} \end{pmatrix} \right)$$

$$= \varphi\big(q(x,y,z)\big)$$

$$= \frac{1}{2}\, q(x,y,z)$$

$\therefore q(Tx,Ty,Tz) \leq \frac{1}{2}\, q(x,y,z) = \frac{1}{2}\max(x,y,z) = \varphi\big(q(x,y,z)\big)$

$\therefore\ q(Tx,Ty,Tz) \leq \varphi\big(q(x,y,z)\big)$

Therefore by theorem (3.11), T has a fixed point.

Clearly 0 is the fixed point which is unique.

4.1 Example: Let $X = \{[a,b]\ a,b \in R, a \leq b\}$ and define $q([a,b],[c,d],[e,f]) = \max\{b,d,f\} - \min\{a,c,e\}$. Then $q: X \times X \times X \to R^+$ is a quasi-S-metric space.

Sol: **(4.1.1)** Let $X = [a,b], Y = [c,d], Z = [e,f]$

Let us suppose that $q(x,x,x) = q(y,y,y) = q(z,z,z) = q(x,y,z)$

$\Rightarrow q([a,b],[a,b],[a,b]) = q([c,d],[c,d],[c,d]) = q([e,f],[e,f],[e,f]) = q([a,b],[c,d],[e,f])$

$\Rightarrow \max\{b,b,b\} - \min\{a,a,a\} = \max\{d,d,d\} - \min\{c,c,c\} = \max\{f,f,f\} - \min\{e,e,e\}$

$= \max\{b,d,f\} - \min\{a,c,e\}$

$\Rightarrow b - a = d - c = f - e = b - a$ (case 1)

$b - a = d - c = f - e = d - c$ (case2)

$b - a = d - c = f - e = f - e$ (case3)

In either case [a, b]=[c, d]=[e, f] ⟹ x=y=z

Conversely if x=y=z

$\Rightarrow [a,b] = [c,d] = [e,f]$

Then

$$q([a,b],[a,b],[a,b]) = q([c,d],[c,d],[c,d]) = q([e,f],[e,f],[e,f]) = q([a,b],[c,d],[e,f])$$

$\therefore\ q(x,x,x) = q(y,y,y) = q(z,z,z) = q(x,y,z)$ iff x=y=z

4.1.2 Now we show that $\max\{q(x,x,t), q(y,y,t), q(z,z,t)\} \le q(x,y,z)$ for $t \in \{x,y,z\}$

When $t = x$

$$\begin{aligned}\max\{q(x,x,t), q(y,y,t), q(z,z,t)\} &= q(x,x,t) \text{ (say)}\\ &= q([a,b],[a,b],[a,b])\\ &= \max\{b,b,b\} - \min\{a,a,a\}\\ &= b - a\end{aligned}$$

When $t = y$, then

$$\begin{aligned}q(x,x,y) &= q([a,b],[a,b],[c,d])\\ &= \max\{b,b,d\} - \min\{a,a,c\}\\ &= \max\{b,d\} - \min\{a,c\}\\ &\le \max\{b,d,f\} - \min\{a,c,e\}\\ &= q(x,y,z)\end{aligned}$$

When $t = z$, then

$$\begin{aligned}q(x,x,z) &= q([a,b],[a,b],[e,f])\\ &= \max\{b,b,f\} - \min\{a,a,e\}\\ &= \max\{b,f\} - \min\{a,e\}\\ &\le \max\{b,d,f\} - \min\{a,c,e\}\\ &= q(x,y,z)\end{aligned}$$

$\therefore q(x,x,t) \le q(x,y,z)$ for $t = x,y,z$

Similarly $q(y,y,t) \le q(x,y,z)$ for $t = x,y,z$

$q(z,z,t) \le q(x,y,z)$ for $t = x,y,z$

$\therefore \max\{q(x,x,t), q(y,y,t), q(z,z,t)\} \le q(x,y,z)$ for $t \in \{x,y,z\}$

Hence Proved

4.1.3 We have $q(x,y,z) = q([a,b],[c,d],[e,f])$

$$\begin{aligned}&= \max\{b,d,f\} - \min\{a,c,e\}\\ q(y,z,x) &= q([c,d],[e,f],[a,b])\\ &= \max\{d,f,b\} - \min\{c,e,a\}\\ &= \max\{b,d,f\} - \min\{a,c,e\}\end{aligned}$$

Similarly $q(z,x,y) = \max\{b,d,f\} - \min\{a,c,e\}$

$\therefore q(x,y,z)$ is invariant of any permutation of x,y,z

4.1.4 claim: $q(x,y,t) \le q(x,z,t) + q(z,y,t) - q(z,z,t)\ \forall\, x,y,z,t \in X$

L.H.S= $q(x,y,t) = q([a,b],[c,d],[p,q])$ where $t = \{p,q\}$

$= \max\{b,d,q\} - \min\{a,c,p\}$

R.H.S

$q(x,z,t) + q(z,y,t) - q(z,z,t)$

$= q([a,b],[e,f],[p,q]) + q([e,f],[c,d],[p,q]) - q([e,f],[e,f],[p,q])$

$= \max\{b,f,q\} - \min\{a,e,p\} + \max\{f,d,q\} - \min\{e,c,p\} - [\max\{f,f,q\} - \min\{e,e,p\}]$

But $\max\{b,d,q\} \leq \max\{b,f,q\} + \max\{f,d,q\} - \max\{f,f,q\}$
i.e.,$\delta \leq \alpha + \beta - \gamma$, where $\delta = \max\{b,d,q\}$, $\alpha = \max\{b,f,q\}$,
$\beta = \max\{f,d,q\}$ and $\gamma = \max\{f,f,q\}$

Now $b \leq \alpha \leq \alpha + \beta - \gamma$
$d \leq \beta \leq \alpha + \beta - \gamma$
$q \leq \gamma \leq \alpha + \beta - \gamma$
$\therefore \max\{b,d,q\} \leq \alpha + \beta - \gamma$
$\therefore \delta \leq \alpha + \beta - \gamma$

Now $\min\{a,c,p\} \geq \min\{a,e,p\} + \min\{e,c,p\} - \min\{e,e,p\}$
$\delta' \geq \alpha' + \beta' - \gamma'$, where $\delta' = \min\{a,c,p\}$, $\alpha' = \min\{a,e,p\}$,
$\beta' = \min\{e,c,p\}$ and $\gamma' = \min\{e,e,p\}$
$a \geq \alpha' \geq \alpha' + \beta' - \gamma'$
$c \geq \beta' \geq \alpha' + \beta' - \gamma'$
$p \geq \alpha' \geq \alpha' + \beta' - \gamma'$
$\therefore \delta' \geq \alpha' + \beta' - \gamma'$
i.e., $-\min\{a,c,p\} \leq \alpha' + \beta' - \gamma'$

$$\begin{aligned}
\therefore \max\{b,d,q\} - \min\{a,c,p\} &= \delta - \delta' \\
&\leq \alpha + \beta - \gamma - (\alpha' + \beta' - \gamma') \\
&\leq (\alpha - \alpha') + (\beta - \beta') - (\gamma - \gamma') \\
&= (\max\{b,f,q\} - \min\{a,e,p\}) + (\max\{f,d,q\} - \min\{e,c,p\}) - (\max\{f,f,q\} - \min\{e,e,p\}) \\
&= q([a,b],[e,f],[p,q]) + q([e,f],[c,d],[p,q]) - q([e,f],[e,f],[p,q]) \\
&= q(x,z,t) + q(z,y,t) - q(z,z,t)
\end{aligned}$$
$\therefore q(x,y,t) \leq q(x,z,t) + q(z,y,t) - q(z,z,t)\ \forall\, x,y,z \in X$

Hence (X,q) is a quasi-S-metric space.

The following examples show that every quasi-S-metric space (X,q) gives raise to a family of partial metrics on X.

4.5 Example: Suppose $a \in X$ and (X,q) is a quasi-S-metric on X. Define $q_a\colon X \times X \to R^+$ such that $q_a(x,y) = q(x,y,a)$. Then q_a is a partial metric on X

4.5.1 Let $q_a(x,x) = q_a(x,y) = q_a(y,y)$
$\Rightarrow q(x,x,a) = q(x,y,a) = q(y,y,a)$
$\Rightarrow x = y = a$ (since q is a quasi S-metric)

Conversely if $x = y = a$, then $q(x,x,a) = q(x,y,a) = q(y,y,a)$
Then $q_a(x,x) = q_a(x,y) = q_a(y,y)$. Hence we proved $q_a(x,x) = q_a(x,y) = q_a(y,y)$ iff x = y

4.5.2 $q_a(x,x) = q(x,x,a)$

But from (3.1.4), we have

$$\begin{aligned}
q(x,x,a) &\leq q(x,z,a) + q(z,x,a) - q(z,z,a) \\
&\leq q(x,z,a) + q(z,x,a) \\
&\leq q(x,y,a) + q(y,x,a) \\
&= 2q(x,y,a) \\
&< q(x,y,a)
\end{aligned}$$
$\therefore q(x,x,a) \leq q(x,y,a) = q_a(x,y)$
$\therefore q_a(x,x) \leq q_a(x,y)$

Similarly $q_a(y,y) \leq q_a(x,y)$

4.5.3 $q_a(x,y) = q(x,y,a) = q(y,x,a) = q_a(y,x)$

4.5.4 $q_a(x,y) = q(x,y,a) \leq q(x,z,a) + q(z,y,a) - q(z,z,a)\ \forall\, x,y,z,a$
$= q_a(x,z) + q_a(z,y) - q_a(z,z)$
i.e., $q_a(x,y) \leq q_a(x,z) + q_a(z,y) - q_a(z,z)$
$\therefore$ The pair (X,q_a) is a partial metric space.

4.6 Example: Let $X = \{0,1,2,\ldots\}$ and $q\colon X \times X \times X \to [0,\infty)$ is a defined by $q(x,y,z) = \max\{x^2,y^2,z^2\}$ then (X,q) is a quasi-S-metric space.

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Source of support: Nil, Conflict of interest: None Declared.