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# ON THE LAPLACIAN ENERGY AND COLOR SIGNLESS LAPLACIAN ENERGY OF BOOK GRAPH $B_n$

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***ABSTRACT***

*A book graph $B_n$ is a graph $S_n \times P_2$ where $S_n$ is a star graph with $n$ edges. The energy $E(G)$ of a simple graph $G$ is the sum of the absolute values of the eigenvalues of the adjacency matrix of $G$. In this paper, the Energy, Laplacian Energy, and Color Signless Laplacian Energy of Book Graph $B_n$ were determined.*

***Keywords:*** *Book Graph, Laplacian Energy, Color Signless Laplacian Energy.*

## 1. INTRODUCTION

Let $G = (V, E)$ be a simple graph with $n = |V|$ vertices and $m = |E|$ edges. Let $A$ be the adjacency matrix of $G$, and let the eigenvalues of $A$ be

$$\lambda_1 \geq \cdots \geq \lambda_n .$$

In 1978, Gutman [3] defined the energy of a graph $G$, $E(G)$, as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

This concept originated from chemistry to estimate the total $\pi$-electron energy of a molecule. The conjugated hydrocarbons can be represented by a graph called molecular graph where carbon atoms are the vertices while carbon-carbon bonds are the edges and hydrogen atoms are ignored. The eigenvalues of the molecular graph are the energy level of the electron in the molecule. [4]

Recently, graph energy has become the focus of interest among mathematicians and different energies were observed such as Laplacian energy, color energy and color signless Laplacian energy.

In this paper, we calculate the Energy, Laplacian Energy, and Color Signless Laplacian Energy of a Book Graph $B_n$ which is a Cartesian product of a star graph $S_n$ and path graph $P_2$.

### 1.1 Laplacian Energy

Let $\mu_1 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$ be the eigenvalues of the Laplacian matrix $L = (L_{ij})$ of $G$ where

$$L_{ij} = \begin{cases} \delta_i, & \text{if } i = j \\ -1, & \text{if } i \neq j \text{ and the vertices } v_i, v_j \text{ are adjacent} \\ 0, & \text{if } i \neq j \text{ and the vertices } v_i, v_j \text{ are not adjacent} \end{cases}$$

where $\delta_i =$ degree of vertex $v_i$.

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The Laplacian energy of a graph $G$ is then defined as

$$LE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$$

This Laplacian energy of a graph $G$ has been recently defined by Gutman and Zhou. [5]

### 1.2 Color Signless Laplacian Energy

A coloring of graph $G$ is a coloring of its vertices such that no two adjacent vertices receive the same color. The minimum number of colors needed for coloring a graph $G$ is called chromatic number and denoted by $\chi(G)$.

Adiga, Sampathkumar, Sriraj, Shrikanth [1] have studied color energy which is the sum of the absolute values of the color eigenvalues of $G$.

The *color adjacency matrix* $A_c(G)= [a_{ij}]$ where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent with } c(v_i) \neq c(v_j) \\ -1, & \text{if } v_i \text{ and } v_j \text{ are non-adjacent with } c(v_i) \neq c(v_j) \\ 0. & \text{if } v_i = v_j \text{ or } v_i \text{ and } v_j \text{ are non-adjacent with } c(v_i) \neq c(v_j) \end{cases}$$

, $c(v_i)$ is the color of vertex $v_i$.

The *color energy* of a graph denoted by $E_c(G)$ is defined as

$$E_C(G) = \sum_{i=1}^{n} |\lambda_i|$$

where $\lambda_i$ of $A_c(G)$ are called the color eigenvalues of $G$.

Given the Signless Laplacian matrix of the graph $G$, of order $n$ and size $m$, is denoted by $LE^+(G) = D(G) + A(G)$, where $D(G)$ is the degree matrix and $A(G)$ is the adjacency matrix, the Signless Laplacian energy of the graph $G$ is

$$LE^+(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$$

where $\mu_i$ are the Signless Laplacian eigenvalues of the graph $G$.

Bhat and D'Souza [2] defined color Laplacian energy as

$$LE_c(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$$

where $\mu_i$ are eigenvalues of color Laplacian matrix $L_c(G) = D(G) - A_c(G)$.

Then, the color Signless Laplacian energy of $G$, denoted by $LE_c^+(G)$ is defined as

$$LE_c^+(G) = \sum_{i=1}^{n} \left|\mu_i{}^+ - \frac{2m}{n}\right|$$

where $\mu_i{}^+$are eigenvalues of color Signless Laplacian matrix $L_c^+(G) = D(G) + A_c(G)$.

### 1.3 Book Graph $B_n$

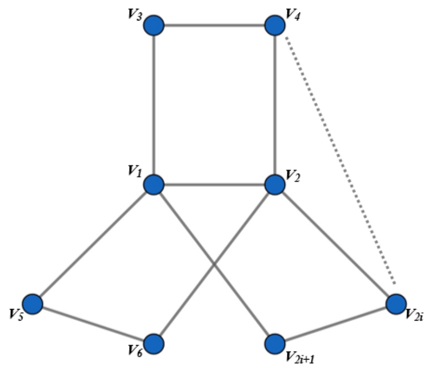


**Figure-1:** The book graph $B_n$

The book graph $B_n$ is a graph $S_n \times P_2$ where $S_n$ is a star graph with $n$ edges. The order of $B_n$ is $N = 2n + 2$, and its size is $M = 3n + 1$. The principle of labelling vertices of the book graph $B_n$ is that the center vertex of the first star graph is labelled by $V_1$ (which is adjacent to $V_2$) other vertices are labelled by $V_{2i+1}, 1 \leq i \leq n$, and center of the second star graph is labelled by $V_2$ (which is adjacent to $V_1$) and the other vertices are labelled by $V_{2i}, 2 \leq i \leq n + 1$.

### 1.4 Colored Book Graph $B_n$

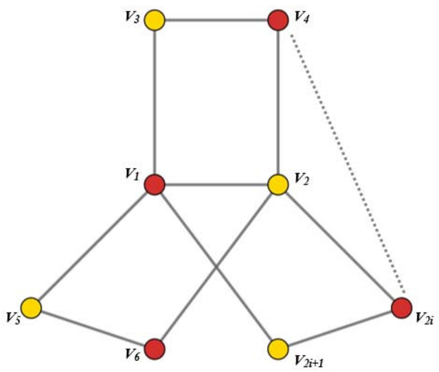


**Figure-2:** Colored book graph $B_n$

Colored book graph $B_n$, is a book graph with chromatic number $\chi(B_n) = 2$, with vertex color of red *(R)* and yellow *(Y)*. The principle of coloring, the vertices of the book graph $B_n$, is that $c(V_1) = R$ and $c(V_{2i+1}) = Y$, $1 \leq i \leq n$; and $c(V_2) = Y$ and $c(V_{2i}) = R, 2 \leq i \leq n + 1$.

## 2. MAIN RESULTS

This section presents the energy, Laplacian energy and Color Signless Laplacian energy of book graph $B_n$ and its proofs.

**Theorem 1:** Let $B_n$ be a book graph with order $N = 2n + 2$ and size $M = 3n + 1$. Then, the energy of $B_n$ is

$$\boldsymbol{E(B_n) = 2(2\sqrt{n} + n - 1)}$$

***Proof:*** The adjacency matrix of book graph $B_n$ is

$$A(B_n) = \begin{pmatrix} 0 & 1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 1 & 0 \\ 1 & 0 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & 0 & 0 & \ddots & \ddots & 0 & 0 & \vdots & \vdots \\ \vdots & \vdots & 0 & 0 & \ddots & \ddots & 0 & 0 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & 0 & 0 & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & 0 & \ddots & \ddots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of the matrix is given by

$$P(B_n, \lambda) = (\lambda + 1)^{n-1}(\lambda - 1)^{n-1}[\lambda - (\sqrt{n} - 1)][\lambda - (\sqrt{n} + 1)][\lambda - (-\sqrt{n} - 1)]$$
$$[\lambda - (\sqrt{n} + 1)]$$

Hence, the adjacency spectrum of book graph $B_{n,}$ is

$$E_{spec}(B_n) = \begin{Bmatrix} -1 & 1 & \sqrt{n} - 1 & \sqrt{n} + 1 & -\sqrt{n} - 1 & -\sqrt{n} + 1 \\ n - 1 & n - 1 & 1 & 1 & 1 & 1 \end{Bmatrix}$$

Therefore, the energy is given by

$$\begin{aligned} E(B_n) &= \sum_{i=1}^{N} |\lambda_i| \\ &= (n-1)|-1| + (n-1)|1| + |\sqrt{n} - 1| + |\sqrt{n} + 1| + |-\sqrt{n} - 1| + |-\sqrt{n} + 1| \\ &= 2(2\sqrt{n} + n - 1) \end{aligned}$$

**Theorem 2:** Let $B_n$ be a book graph with order $N = 2n + 2$ and size $M = 3n + 1$. Then, the Laplacian energy of $B_n$ is given by

$$LE(B_n) = 4n$$

***Proof:*** The Laplacian matrix of book graph $B_n$ is

$$L(B_n) = \begin{pmatrix} n+1 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & -1 & 0 \\ -1 & n+1 & 0 & -1 & \cdots & \cdots & \cdots & \cdots & 0 & -1 \\ -1 & 0 & 2 & -1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & 0 & 0 & \ddots & \ddots & 0 & 0 & \vdots & \vdots \\ \vdots & \vdots & 0 & 0 & \ddots & \ddots & 0 & 0 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & 0 & 0 & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & 0 & \ddots & \ddots & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 2 & -1 \\ 0 & -1 & 0 & 0 & \cdots & \cdots & 0 & 0 & -1 & 2 \end{pmatrix}$$

The characteristic polynomial of the Laplacian matrix is given by

$$P(B_n, \mu) = \mu(\mu - 1)^{n-1}(\mu - 2)(\mu - 3)^{n-1}[\mu - (n + 1)][\mu - (n + 3)]$$

Hence, the Laplacian spectrum of book graph $B_n$ is

$$LE_{spec}(B_n) = \begin{Bmatrix} 0 & 1 & 2 & 3 & n+1 & n+3 \\ 1 & n-1 & 1 & n-1 & 1 & 1 \end{Bmatrix}$$

Therefore, the Laplacian energy is given by

$$\begin{aligned} LE(B_n) &= \sum_{i=1}^{N} \left| \mu_i - \frac{2M}{N} \right| \\ &= \left| 0 - \frac{2(3n+1)}{2n+2} \right| + (n-1)\left| 1 - \frac{2(3n+1)}{2n+2} \right| + \left| 2 - \frac{2(3n+1)}{2n+2} \right| + \\ &\qquad (n-1)\left| 3 - \frac{2(3n+1)}{2n+2} \right| + \left| (n+1) - \frac{2(3n+1)}{2n+2} \right| + \left| (n+3) - \frac{2(3n+1)}{2n+2} \right| \\ &= 4n \end{aligned}$$

**Theorem 3:** Let $B_n$ be a book graph with order $N = 2n + 2$ and size $M = 3n + 1$. Then, the Color Signless Laplacian energy of $B_n$ if $n > 1$ is given by

$$\boldsymbol{LE_c^+(B_n) = 2(3n - 1)}$$

***Proof:*** The Color Signless Laplacian matrix of book graph $B_n$, is

$$LE_c^+(B_n) = \begin{pmatrix} n+1 & 1 & 1 & -1 & \cdots & \cdots & \cdots & \cdots & 1 & -1 \\ 1 & n+1 & -1 & 1 & \cdots & \cdots & \cdots & \cdots & -1 & 1 \\ 1 & -1 & 2 & 1 & -1 & 0 & \cdots & \cdots & -1 & 0 \\ -1 & 1 & 1 & 2 & 0 & -1 & \cdots & \cdots & 0 & -1 \\ \vdots & \vdots & -1 & 0 & \ddots & \ddots & -1 & 0 & \vdots & \vdots \\ \vdots & \vdots & 0 & -1 & \ddots & \ddots & 0 & -1 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & -1 & 0 & \ddots & \ddots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & -1 & \ddots & \ddots & 0 & -1 \\ 1 & -1 & -1 & 0 & \cdots & \cdots & -1 & 0 & 2 & 1 \\ -1 & 1 & 0 & -1 & \cdots & \cdots & 0 & -1 & 1 & 2 \end{pmatrix}$$

The characteristic polynomial of Color Signless Laplacian matrix is given by

$$P(B_n, \mu^+) = (\mu^+ - 2)^{n-1}(\mu^+ - 4)^{n-1}[\mu^+ - (n + 2)]^2(\mu^+ + n)[\mu^+ - (4 - n)]$$

Hence, the Color Signless Laplacian spectrum of book graph $B_n$, is

$$LE_{c\ spec}^{+}(B_n) = \begin{Bmatrix} 2 & 4 & n+2 & -n & 4-n \\ n-1 & n-1 & 2 & 1 & 1 \end{Bmatrix}$$

Therefore, the Color Signless Laplacian energy if $n > 1$ is given by

$$
\begin{aligned}
LE_c^+(B_n) &= \sum_{i=1}^{N} \left|\mu_i - \frac{2M}{N}\right| \\
&= (n-1)\left|2 - \frac{2(3n+1)}{2n+2}\right| + (n-1)\left|4 - \frac{2(3n+1)}{2n+2}\right| + 2\left|(n+2) - \frac{2(3n+1)}{2n+2}\right| + \left|-n - \frac{2(3n+1)}{2n+2}\right| \\
&\qquad + \left|(4-n) - \frac{2(3n+1)}{2n+2}\right| \\
&= 2(3n-1)
\end{aligned}
$$

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