

COMMON FIXED POINT THEOREM
BY USING COMPATIBLE MAPPINGS IN Menger SPACES

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ABSTRACT

We prove common fixed point theorem by using compatible mappings via in Menger spaces. some others in Menger as well as metric spaces. our result generalize many known results in Menger as well as metric spaces

Keywords: Fixed point, Fixed Point Theorem, Common Property, Finite Family, Compatible Mapping.

1. INTRODUCTION

There have been lots of generalizations of metric space. One such generalization is Menger space in which, used distribution functions instead of nonnegative real numbers as value of metric.

A Menger space is a space in which the concept of distance is considered to be a probabilistic, rather than deterministic. For detail discussion of Menger spaces and their applications we refer to Schweizer and Sklar [15]. The theory of Menger space is fundamental importance in probabilistic functional analysis.

A probabilistic metric space shortly PM-Space, is an ordered pair (X, F) consisting of a non empty set X and a mapping F from $X \times X$ to L , where L is the collection of all distribution functions (a distribution function F is non decreasing and left continuous mapping of reals in to $[0,1]$ with properties, $\inf F(x) = 0$ and $\sup F(x) = 1$).

The value of F at $(u, v) \in X \times X$ is represented by $F_{u,v}$. The function $F_{u,v}$ are assumed satisfy the following conditions;

- 1.1(a) $F_{u,v}(x) = 1$, for all $x > 0$, iff $u = v$;
- 1.1 (b) $F_{u,v}(0) = 0$, if $x = 0$;
- 1.1 (c) $F_{u,v}(x) = F_{v,u}(x)$;
- 1.1 (d) $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{u,w}(x + y) = 1$.

A mapping $t: [0,1] \times [0,1] \rightarrow [0,1]$ is a t-norm, if it satisfies the following conditions;

- 1.1 (e) $t(a, 1) = a$ for every $a \in [0,1]$;
- 1.1 (f) $t(0, 0) = 0$,
- 1.1 (g) $t(a, b) = t(b, a)$ for every $a, b \in [0,1]$;
- 1.1 (h) $t(c, d) \geq t(a, b)$ for $c \geq a$ and $d \geq b$
- 1.1 (i) $t(t(a, b), c) = t(a, t(b, c))$ where $a, b, c, d \in [0,1]$.

A Menger space is a triplet (X, F, t) , where (X, F) is a PM-Space, X is a non-empty set and a t -norm satisfying instead of 6.1(i) a stronger requirement.

- 1.1 (j) $F_{u,w}(x + y) \geq t(F_{u,v}(x), F_{v,w}(y))$ for all $x \geq 0, y \geq 0$.

For a given metric space (X, d) with usual metric d , one can put $F_{u,v}(x) = H(x - d(u, v))$ for all $x, y \in X$ and $t > 0$. where H is defined as;

$$H(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

and t-norm is defined as $t(a, b) = \min \{a, b\}$.

Our aim of this chapter, we establish a fixed point theorem in the setting of probabilistic metric space using weak compatibility.

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2. PRELIMINARIES

To proof of our result we need some known definitions which are follows;

Definition 2.1: A probabilistic metric space (PM- space) is an ordered pair (X, F) consisting of a non empty set X and a mapping F from $X \times X$ into the collections of all distribution $F \in R$. For $x, y \in X$ we denote the distribution function $F(x, y)$ by $F_{x,y}$ and $F_{x,y}(u)$ is the value of $F_{x,y}$ at u in R .

Definition 2.2: Self maps A and B of a Menger space $(X, F, *)$ are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if $Ax = Bx$ for some $x \in X$ then $ABx = BAx$.

Definition 2.3: Self maps A and B of a Menger space $(X, F, *)$ are said to be compatible if $F_{ABx_m, BAx_n}(t) \rightarrow 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n \rightarrow x, Bx_n \rightarrow x$ for some x in X as $n \rightarrow \infty$.

The concept of neighborhoods in Menger space was introduced by Schweizer and Skalar [91].

Definition 2.4: Let (X, F, t) be a Menger space. If $x \in X, \varepsilon > 0$ and $\lambda \in (0, 1)$, then (ε, λ) – neighborhood of x is called $U_x(\varepsilon, \lambda)$, is defined by

$$U_x(\varepsilon, \lambda) = \{y \in X: F_{x,y}(\varepsilon) > (1 - \lambda)\}.$$

an (ε, λ) – topology in X is the topology induced by the family

$$\{U_x(\varepsilon, \lambda): x \in X, \varepsilon > 0 \text{ and } \lambda \in (0, 1)\}$$

of neighborhood.

Remark 1: If t is continuous, then Menger space (X, F, t) is a Housdorff space in (ε, λ) – topology.

Let (X, F, t) be a complete Menger space and $A \subset X$. Then A is called a bounded set if

$$\liminf_{u \rightarrow \infty} \inf_{x,y \in A} F_{x,y}(u) = 1.$$

Definition 2.5: A sequence $\{x_n\}$ in (X, F, t) is said to be convergent to a point x in X if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $N = N(\varepsilon, \lambda)$ such that $x_n \in U_x(\varepsilon, \lambda)$ for all $n \geq N$ or equivalently $F(x_n, x; \varepsilon) > 1 - \lambda$ for all $n \geq N$.

Definition 2.6: A sequence $\{x_n\}$ in (X, F, t) is said to be Cauchy sequence if for every $\varepsilon > 0$ and $\lambda > 0, \exists$ an integer $N = N(\varepsilon, \lambda)$ such that $F(x_n, x_m, \varepsilon) > 1 - \lambda$ for all $n, m \geq N$.

Definition 2.7: A Menger space (X, F, t) with the continuous t –norm is said to be complete if every cauchy sequence in X converges to a point in X .

Definition 2.8: Let (X, F, t) be a Menger space, two mappings $f, g : X \rightarrow X$ are said to be weakly compatible if they commute at coincidence point.

Lemma 2.9: Let X be a set f, g OWC self maps of X . If f and g have a unique point of coincidence, $w = fx = gx$, then w is the unique common fixed point of f and g .

Lemma 2.10: Let $\{x_n\}$ be a sequence in a Menger space (X, F, t) , where t is continuous and $t(p, p) \geq p$ for all $p \in (0, 1)$ and $n \in N$

$$F(x_n, x_{n+1}, kp) \geq F(x_{n-1}, x_n, p), \text{ then } \{x_n\} \text{ is Cauchy sequence.}$$

Lemma 2.11: If (X, d) is a metric space, then the metric d induces a mapping $F: X \times X \rightarrow L$ defined by

$$F(p, q) = H(x - d(p, q)), \quad p, q \in R.$$

Further if $t: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is defined by $t(a, b) = \min\{a, b\}$, then (X, F, t) is a Menger space. It is complete if (X, d) is complete.

3. MAIN THEOREM

In this section we prove some common fixed point theorems by using seven compatible mappings in complete Menger spaces.

In fact we prove following results.

Theorem 3.1: Let A, B, S, T and P be self maps on a complete Menger space $(X, F, *)$ with $t * t \geq t$ for all $t \in [0, 1]$, satisfying;

3.1 (a) $P(X) \subseteq AB(X), P(X) \subseteq ST(X)$;

3.1 (b) there exists a constant $k \in (0, 1)$ such that

$$M_{P_x, P_y}(kt) \geq M_{AB_x, P_x}(t) * M_{P_x, S_T y}(t) * M_{AB_x, S_T y}(t) * \frac{M_{P_x, AB_x}(t) * M_{P_x, S_T y}(t)}{M_{S_T y, AB_x}(t)} * M_{P_x, P_y}(t) * M_{P_x, P_x}(t) *$$

$$M_{AB_x, P_y}(n - \alpha)t$$

for all $x, y \in X, \alpha \in (0, n)$ and $t > 0$,

3.1 (c) $PB = BP, PT = TP, AB = BA$ and $ST = TS$,

3.1 (d) A and B are continuous,

3.1 (e) the pair (P, AB) is compatible (if compatible then it is weak compatible). Then A, B, S, T and P have a common fixed point in X .

Proof: Since $P(X) \subseteq AB(X)$, for $x_0 \in X$, we can choose a point $x_0 \in X$ such that $Px_0 = ABx_1$. Since $P(X) \subseteq ST(X)$, for this point x_1 , we can choose a point $x_2 \in X$ such that $Px_1 = STx_2$. Thus by induction, we can define a sequence $y_n \in X$ as follows;

$$y_{2n} = Px_{2n} = ABx_{2n+1}$$

and

$$y_{2n+1} = Px_{2n+1} = STx_{2n+1}$$

for $n = 1, 2, \dots$ from 3.1 (b),

for all $t > 0$ and $\alpha = 2 - q$ with $q \in (0, 2)$, we have

$$M_{y_{2n+1}, y_{2n+2}}(kt) = M_{Px_{2n+1}, Px_{2n+2}}(kt) \geq M_{y_{2n+1}, y_{2n+1}}(t) * M_{y_{2n}, y_{2n+1}}(t) * M_{y_{2n+1}, y_{2n+2}}(t) * \frac{M_{y_{2n+1}, y_{2n}}(t) * M_{y_{2n+1}, y_{2n+1}}(t)}{M_{y_{2n+1}, y_{2n}}(t)} *$$

$$M_{y_{2n}, y_{2n+2}}(1 + q)t,$$

$$\begin{aligned} M_{y_{2n+1}, y_{2n+2}}(kt) &\geq M_{y_{2n}, y_{2n+1}}(t) * M_{y_{2n}, y_{2n+2}}(1 + q)t \\ &\geq M_{y_{2n}, y_{2n+1}}(t) * M_{y_{2n}, y_{2n+1}}(t) * M_{y_{2n+1}, y_{2n+2}}(qt) \\ &\geq M_{y_{2n}, y_{2n+1}}(t) * M_{y_{2n+1}, y_{2n+2}}(t) \end{aligned}$$

as $q \rightarrow 1$. Since $*$ is continuous and $M_{x,y}(\cdot)$ is continuous, letting $q \rightarrow 1$ in above equation

We get

$$M_{y_{2n+1}, y_{2n+2}}(kt) \geq M_{y_{2n}, y_{2n+1}}(t) * M_{y_{2n+1}, y_{2n+2}}(t) \dots \dots \quad 3.1 (i)$$

Similarly, we have

$$M_{y_{2n+2}, y_{2n+3}}(kt) \geq M_{y_{2n+1}, y_{2n+2}}(t) * M_{y_{2n+2}, y_{2n+2}}(t) \dots \dots \quad 3.1 (ii)$$

Thus from 2.1 (i) and 2.1 (ii), it follows that

$$M_{y_{n+1}, y_{n+2}}(kt) \geq M_{y_n, y_{n+1}}(t) * M_{y_{n+1}, y_{n+2}}(t)$$

for $n = 1, 2, \dots$ and then for positive integers n and p ,

$$M_{y_{n+1}, y_{n+2}}(kt) \geq M_{y_n, y_{n+1}}(t) * M_{y_{n+1}, y_{n+2}}\left(\frac{t}{k^p}\right).$$

Thus, since $M_{y_{n+1}, y_{n+1}}\left(\frac{t}{k^p}\right) \rightarrow 1$ as $p \rightarrow \infty$ we have

$$M_{y_{n+1}, y_{n+2}}(kt) \geq M_{y_n, y_{n+1}}(t).$$

y_n is Cauchy sequence in X and since X is complete, y_n converges to a point $z \in X$. Since Px_n, ABx_{2n+1} and STx_{2n+2} are subsequences of y_n , they also converge to the point z , since A, B are continuous and pair $\{P, AB\}$ is compatible and also weak compatible, we have

$$\lim_{n \rightarrow \infty} PABx_{2n+1} = ABz$$

$$\text{and } \lim_{n \rightarrow \infty} (AB)^2 x_{2n+1} = ABz,$$

From 2.1 (b) with $\alpha = 2$, we get

$$\begin{aligned} M_{PABx_{2n+1}, Px_{2n+2}}(kt) &\geq M_{(AB)^2 x_{2n+1}, STx_{2n+2}}(t) * M_{PABx_{2n+1}, STx_{2n+2}}(t) \\ &\quad M_{PABx_{2n+1}, Px_{2n+2}}(t) * M_{PABx_{2n+1}, PABx_{2n+1}}(t) * \\ &\quad * M_{(AB)^2 x_{2n+1}, STx_{2n+2}}(t) * \frac{M_{PABx_{2n+1}, (AB)^2 x_{2n+1}}(t) * M_{PABx_{2n+1}, STx_{2n+2}}(t)}{M_{STx_{2n+2}, (AB)^2 x_{2n+1}}(t)} \\ &\quad * M_{(AB)^2 x_{2n+1}, Px_{2n+2}}(t) \end{aligned}$$

which implies that

$$\begin{aligned} M_{ABz, z}(kt) &= \lim_{n \rightarrow \infty} M_{PABx_{2n+2}}(kt) \\ &\geq 1 * M_{ABz, z}(t) M_{ABz, z}(t) M_{ABz, z}(kt) * M_{ABz, z}(t) * \frac{1 * M_{ABz, z}(t)}{M_{z, ABz}(t)} * M_{ABz, z}(t) \end{aligned}$$

We have

$$ABz = z, \text{ since } M_{z,STz}(t) \geq M_{z,ABz}(t) = 1 \text{ for all } t > 0,$$

We get $STz = z$. Again by 3.1 (b) with $\alpha = 2$,

We have

$$M_{PABx_{2n+1}, Pz}(kt) \geq M_{(AB)^2x_{2n+1}, PABx_{2n+1}}(t) * M_{PABx_{2n+1}, STz}(t) * M_{PABx_{2n+1}, Pz}(t) * M_{PABx_{2n+1}, PABx_{2n+1}}(kt) \\ * M_{(AB)^2x_{2n+1}, STz}(t) * \frac{M_{PABx_{2n+1}, (AB)^2x_{2n+1}}(t) * M_{PABx_{2n+1}, STz}(t)}{M_{STz, (AB)^2x_{2n+1}}(t)} * M_{(AB)^2x_{2n+1}, Pz}(t)$$

which implies that

$$M_{ABz, Pz, Pz}(kt) = \lim_{n \rightarrow \infty} M_{PABx_{2n+1}, Pz}(kt) \\ \geq 1 * 1 * 1 * 1 * 1 * 1 * 1 * 1 M_{ABz, Pz}(t) \\ \geq M_{ABz, Pz}(t).$$

We have $ABz = Pz$. Now, we show that $Bz = z$. Infact, from 2.1(b) with $\alpha = n$, and 3.1 (c) we get,

$$M_{Bz, z}(kt) = M_{BPz, Pz}(kt) \\ = M_{PBz, Pz}(kt) \\ M_{PBz, Pz}(kt) \geq M_{PBz, STz}(t) * M_{ABBz, STz}(t) * M_{PBz, Pz}(t) * M_{PBz, PBz}(t) \\ * \frac{M_{PBz, ABBz}(t) * M_{PBz, z, z}(t)}{M_{z, PBz}(t)} * M_{PBz, z}(t) \\ = 1 * M_{Bz, z}(t) * M_{Bz, z}(t) * 1 * 1 * 1 * 1 * M_{Bz, z}(t) \\ = M_{Bz, z}(t).$$

which implies that $Bz = z$. Since $ABz = z$,

We have $Az = z$. Next, we show that $Tz = z$. Indeed from 6.3.1 (b) with $\alpha = 2$, and 2.1 (c) we get

$$M_{Tz, z}(kt) = M_{TPz, Pz}(kt) = M_{Pz, Pz}(kt) \\ \geq 1 * M_{z, Tz}(t) * M_{Pz, Pz}(t) * M_{Pz, Pz}(t) * M_{z, Tz}(t) * 1 * M_{z, Tz}(t) \\ \geq M_{Tz, z}(t).$$

which implies that $Tz = z$. Since $STz = z$, we have $Sz = STz = z$. Therefore, by combining the above results we obtain,

$$Az = Bz = Sz = Tz = Pz.$$

Therefore z is the common fixed point of A, B, S, T and P .

Finally, the uniqueness of the fixed point of A, B, S, T and P .

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