MORE ON $\delta g\beta$-IRRESOLUTE FUNCTIONS IN TOPOLOGICAL SPACES AND RELATED GROUPS

SANJAY TAHILIANI*
PGT/Lecturer, Maths, N. K. Bagrodia P.S, Sector 9, Rohini, New Delhi-85, India.

(Received On: 10-12-18; Revised & Accepted On: 01-02-19)

ABSTRACT

A function $f: X \rightarrow Y$ is said to be $\delta g\beta$-irresolute if the inverse image of every $\delta g\beta$-closed set in $Y$ is $\delta g\beta$-closed set in $X$. Some properties of these functions were obtained and relations with group theory has been studied.

Key words: $g\beta$-irresolute, $\delta g\beta$-irresolute, homeomorphism group.

AMS Subject classification: 54C08.

1. INTRODUCTION

Throughout the present paper, $X$ and $Y$ denote topological spaces. Let $A$ be a subset of $X$. We denote the interior and closure of $A$ by $\text{Int}(A)$ and $\text{Cl}(A)$ respectively.

A subset $A$ of a topological space $X$ is said to be $\beta$-open [1] or semi-preopen [3] (if $A \subseteq \text{Cl}($Int$(\text{Cl}(A)))$. The complement of $\beta$-open set is $\beta$-closed. The intersection of all $\beta$-closed sets containing $A$ is called $\beta$-closure [2] of $A$ and is denoted by $\beta\text{Cl}(A)$. Further $A$ is said to be regular open if $A = \text{Int}(\text{Cl}(A))$ and it is said to be regular closed if $A = \text{Cl}(\text{Int}(A))$. It is said to be $\pi$-open [10] if it is finite union of regular open sets and $\delta$-open [9] if for each $x \in A$, there exists a regular open set $V$ such that $x \in V \subseteq A$. Every $\pi$-open set is $\delta$-open. Also $\delta$-closure [9] of $A$, denoted by $\delta\text{Cl}(A)$ is defined to be the set of all $x \in X$ such that $A \cap \text{Int}(\text{Cl}(U)) \neq \emptyset$ for every open neighbourhood $U$ of $x$. If $A = \delta\text{Cl}(A)$, then $A$ is called $\delta$-closed. The complement of $\delta$-closed set is $\delta$-open. Also, $A$ is said to be generalized semi-preclosed [6] (briefly, gsp-closed) or $g\beta$-closed (resp. $\pi g\beta$-closed [8], $\delta g\beta$-closed [5]) if $\beta\text{Cl}(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is open (resp $\pi$-open, $\delta$-open) in $X$.

2. PRELIMINARIES

Definition 2.1: A function $f: X \rightarrow Y$ is said to be $\pi g\beta$-irresolute [8] (resp. $\delta g\beta$-irresolute [5]) if the inverse image of every $\pi g\beta$-closed (resp. $\delta g\beta$-closed) set in $Y$ is $\pi g\beta$-closed (resp. $\delta g\beta$-closed) set in $X$.

Remark 2.1: Every $\pi g\beta$-irresolute function is $\delta g\beta$-irresolute but not conversely as can be seen from the following example which is example 4.6 of [7]:

Example 2.1: Let $(X,\tau)$ be the Moore plane (also known as Niemytzki plane). Set $S = \{(x, y): x$ is irrational and $y=0\}$. Let $A = \{(x, y): y<2\} - S$ and let $B = \{(x, y): x^2+(y-4)^2=1\}$. Let $\sigma$ be the topology on the upper half plane generated by $A$ and $B$. Now consider the identity function $f: (X,\tau) \rightarrow (X,\sigma)$. Note that in $(X,\tau)$, $B$ is regular open and $A$ is union of regular open sets, that is, $A$ can be represented as the union of all open balls of radius 1 tangent to the $x$-axis at the rational along with corresponding rational. Note that every such set is regular open. Thus $f$ is $\delta g\beta$-irresolute. But $A$ is regular open in $(X,\sigma)$ and there is no way $A$ can be represented as finite union of regular sets of $(X,\tau)$. Thus $f$ is not $\pi g\beta$-irresolute function.

Theorem 2.1: Every homeomorphism is $\delta g\beta$-irresolute

Proof: It is obvious from the fact that every homeomorphism is $\pi g\beta$-irresolute function ([8], Theorem 2.3 (iv)) and Remark 2.1.

Corresponding Author: Sanjay Tahiliani*
P.G.T./Lecturer, Maths, N.K. Bagrodia, Sector 9, Rohini, New Delhi-85, India.
Definition 2.2: A function f: (X,τ)→(Y,σ) is called δβc-homeomorphism if f is a δβ- irresolute and f⁻¹ is δβc-
irresolute.
For a topological space (X,τ), we introduce the following:
h(τ,τ)= \{f | f: (X,τ)→(Y,τ) is a homeomorphism, δβcch(X,τ)= \{f | f: (X,τ)→(X,τ) is a δβc-homeomorphism\}.

Theorem 2.2: For a topological space (X,τ), h(τ,τ)⊆ δβcch(X,τ).

Proof: Let fε h(τ,τ). Then by Theorem 2.1 and Definition 2.2, it is shown that f and f⁻¹ are δβc-homeomorphism, that is, fε δβcch(X,τ).

Theorem 2.3: The collection δβcch(X,τ) forms a group under the composition of functions.

Proof: A binary operation nₓ : δβcch(X,τ) × δβcch(X,τ) → δβcch(X,τ) is well defined by nₓ(a,b) = boa, where boa: X→X is a composite function of the functions a and b such that (boa)(x) = b(a(x)) for every point x∈X. Indeed by ([5], Theorem 4.12 (iii)), it is shown that for every δβc-homeomorphisms a and b, the composition boa is also δβc-homeomorphism. Namely, for every pair (a,b)ε δβcch(X,τ), nₓ(a,b) = boa∈ δβcch(X,τ). Then it is claimed that the binary operation nₓ : δβcch((X,τ) × δβcch(X,τ) → δβcch(X,τ) satisfies the axiom of group, namely, putting a.b = nₓ(a,b), the following properties hold in δβcch(X,τ):
(1) (a.b).c = (a.(b.c)) holds for every a, b, c ∈ δβcch(X,τ).
(2) For all a ∈ δβcch(X,τ), there exists an element e ∈ δβcch(X,τ) such that a.e = a = e.a hold in δβcch(X,τ).
(3) For each element a ∈ δβcch(X,τ), there exists an element a⁻¹ ∈ δβcch(X,τ) such that a.a⁻¹ = e = a⁻¹.a hold in δβcch(X,τ).
Indeed, the proof of (1) is obvious. The proof of (2) is obtained by taking e= 1ₓ, where 1ₓ is the identity function on X and using the fact that identity function is always δβc-irresolute. Proof of (3) is obtained by taking a⁻¹ = a⁻¹ for each a ∈ δβcch(X,τ) and Definition 2.2, where a⁻¹ is inverse of a. Therefore by definition of groups, the pair (δβcch(X,τ), nₓ) forms a group under the compositions of functions.

Theorem 2.4: The homeomorphism group h(X,τ) is a subgroup of the group δβcch(X,τ).

Proof: It is obvious that 1ₓ : (X,τ)→(X,τ) is a homeomorphism and so h(X,τ)≠ ∅. It follows by Theorem 2.2 that h(X,τ)⊆ δβcch(X,τ). Let a,b ε h(X,τ). Then we have nₓ(a,b⁻¹) = b⁻¹oa∈ ch(X,τ), where nₓ : δβcch(X,τ)×δβcch(X,τ) → δβcch(X,τ) is a binary operation. (Theorem 2.3). Therefore, the group h(X,τ) is a subgroup of δβcch(X,τ).

Theorem 2.5: If (X,τ) and (Y,σ) are homeomorphic, then δβcch(X,τ) ≅ δβcch(Y,σ).

Proof: It follows from the assumption that there exist a homeomorphism say f: (X,τ)→(Y,σ). We define a function f*: δβcch(X,τ)→δβcch(Y,σ) by f*(a) = foa⁻¹ for every aε δβcch(X,τ). By Theorem 2.2, the bijections foa⁻¹ and (foa)⁻¹ are δβc-irresolutes and so f* is well defined. The induced function f* is a homomorphism. Indeed f*(nₓ(a,b)) = fob⁻¹ofa⁻¹ = (f*(b))(f*(a)) = nₓ(f*(b), f*(a)) hold. Obviously f* is bijective. Thus f* is isomorphism.

Definition 2.3: A function f: X→Y is said to be contra πβc-irresolute [4] (resp.contra-δβc-irresolute) if the inverse image of every πβc-open (resp. δβc-open) set in Y is πβc-closed(resp. δβc-closed) set in X.

Definition 2.4: For a topological space (X,τ), we define the following collection of functions:
con-δβcch(X,τ) = {f | f: (X,τ)→(X,τ) is a contra-δβc-irresolute bijection and f⁻¹ is contra-δβc-irresolute}.

Remark 2.2: If f and g are contra-δβc-irresolute, then so is fog.

Remark 2.3: If f is δβc-irresolute and g is contra-δβc-irresolute, then gof is contra-δβc-irresolute.

Theorem 2.6: The union of two collections, δβcch(X,τ)∪con-δβcch(X,τ) forms a group under the composite of functions.

Proof: Let Bₓ = δβcch(X,τ)∪con-δβcch(X,τ). A binary operation Wₓ: Bₓ × Bₓ → Bₓ is well defined by Wₓ(a,b) = boa where, boa: X→X is a composite function of functions a and b. Indeed let (a, b) ∈ Bₓ; if aε δβcch(X,τ) and bε con-δβcch(X,τ), then boa: X→ X is a contra-δβ-irresolute bijection (and boa)⁻¹ is also contra-δβ-irresolute and so Wₓ(a,b) = boa∈ δβch(X,τ)⊆ Bₓ (Remark 2.3). If a, bε con-δβcch(X,τ), then boa: (X,τ)→ (X,τ) is a contra-δβc irresolute bijection and so aεcon-δβcch(X,τ)⊆ Bₓ (By Remark 2.2). If a,bε δβcch(X,τ), then boa: (X,τ)→ (X,τ) is a δβc irresolute bijection and so aε δβcch(X,τ)⊆ Bₓ (By Remark 2.3). By similar arguments of Theorem 2.3, it is claimed that binary operation Wₓ: Bₓ × Bₓ → Bₓ satisfies the axiom of group, for the identity element e of
BX, e = 1X: (X, τ) → (X, τ) (the identity function). Thus, the pair (BX, WX) forms a group under the composite of functions, i.e., δβch(X; τ) ∪ con-δβch(X; τ) is a group.

**Theorem 2.7:** The group δβch(X; τ) is a subgroup of δβch(X; τ) ∪ con-δβch(X; τ).

**Proof:** The group δβch(X; τ) is non empty from Remark 2.2. Using the binary operation in Theorem 2.6, it is shown that WX(a, b⁻¹) = b⁻¹oa ∈ δβch(X; τ) for any a, b ∈ δβch(X; τ) and so δβch(X; τ) is a subgroup of δβch(X; τ) ∪ con-δβch(X; τ).

**Theorem 2.8:** If (X, τ) and (Y, σ) are homeomorphic, then there exists isomorphisms: δβch(X; τ) ∪ con-δβch(X; τ) ∼ δβch(Y; σ) ∪ con-δβch(Y; σ).

**Proof:** Let f: (X, τ) → (Y, σ) be a homeomorphism. We put BX = δβch(X; τ) ∪ con-δβch(X; τ) (resp. BV = δβch(Y; σ) ∪ con-δβch(Y; σ)). For a topological space (X, τ) (resp. (Y, σ)). First we have a well defined function f*: BX → BV by (f* (a)) = foafo⁻¹ for every a ∈ BX. Indeed by Theorem 2.2, f and f⁻¹ are δβ- irresolute, the bijections foafo⁻¹ and (foafo⁻¹)⁻¹ are δβ-irresolute or contra δβ-irresolute and so f* is well defined. The induced function f* is a homomorphism. Indeed, f*(WX(a,b)) = fobo f⁻¹ofoafo⁻¹ = (f* (b))o(f* (a)) = WY(f* (a), f* (a)) hold. WX: BX × BX → BX and WY: BY × BY → BY are binary operations defined in Theorem 2.5, obviously f* is bijective. Thus we have the isomorphism. Also since identity function is δβ-irresolute, f*(1X) = 1Y holds.

**REFERENCES**


Source of support: Nil, Conflict of interest: None Declared.

[Copy right © 2019. This is an Open Access article distributed under the terms of the International Journal of Mathematical Archive (IJMA), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.]