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HARMONIC RAYLEIGH-RITZ METHOD FOR THREE-PARAMETER EIGENVALUE PROBLEMS

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ABSTRACT

This paper discusses Harmonic Rayleigh-Ritz method for three-parameter eigenvalue problems. Finally some numerical results are presented.

***Key words**: multiparameter, eigenvalue, eigenvector, Rayleigh-Ritz method.*

1.1 INTRODUCTION

Multiparameter eigenvalue problems are generalization of one-parameter eigenvalue problems and can be found when the method of separation of variables is applied to certain boundary value problems associated with partial differential equations. Much more works have been done in the field of one-parameter eigenvalue problems, both theoretically and numerically compared to two-parameter or more than two-parameter eigenvalue problems. Some works have been done theoretically in the field of multiparameter eigenvalue problems. Few authors have dealt with the multiparameter eigenvalue problems numerically mainly in two-parametic cases. Numerical methods applied to a three-parameter problems are very limited and hence some contribution in this area are always is needed.

1.2 Three-parameter eigenvalue problem and its reduction to a system of one-parameter problems

A three-parameter eigenvalue problems in matrix form is as follows

$$\begin{aligned} &A_{10}\, x - \lambda_1 A_{11} x - \lambda_2 A_{12} x - \lambda_3 A_{13} x = 0 \\ &A_{20} y - \lambda_1 A_{21} y - \lambda_2 A_{22} y - \lambda_3 A_{23} y = 0 \\ &A_{30} z - \lambda_1 A_{31} z - \lambda_2 A_{32} z - \lambda_3 A_{33} z = 0 \end{aligned} \tag{1.2.1}$$

where $\lambda_i \in C, i = 1,2,3$

$$\begin{aligned} &x \in C^n \backslash \{0\};\ A_{10}, A_{11}, A_{12}\,, A_{13} \in C^{n\times n} \\ &y \in C^m \backslash \{0\};\ A_{20}, A_{21}, A_{22}, A_{23} \in C^{m\times m} \\ &z \in C^p \backslash \{0\};\ A_{30}, A_{31}, A_{32} A_{13} \in C^{p\times p} \end{aligned}$$

Problem (1.2.1) can be reduced to a system of three one-parameter problems:

$$\begin{aligned} &\Delta_1 u = \lambda_1 \Delta_0 u \\ &\Delta_2 u = \lambda_2 \Delta_0 u \\ &\Delta_3 u = \lambda_3 \Delta_0 u \end{aligned} \tag{1.2.2}$$

where $\Delta_0, \Delta_1, \Delta_2, \Delta_3$ are $mnp \times mnp$ dimensional matrices defined as

$$\Delta_0 = A_{11} \otimes A_{22} \otimes A_{33} - A_{11} \otimes A_{23} \otimes A_{32} + A_{12} \otimes A_{23} \otimes A_{31} - A_{12} \otimes A_{21} \otimes A_{33} + A_{13} \otimes A_{21} \otimes A_{32} - A_{13} \otimes A_{22} \otimes A_{31} \tag{1.2.3}$$

$$\Delta_1 = A_{10} \otimes A_{22} \otimes A_{33} - A_{10} \otimes A_{23} \otimes A_{32} + A_{12} \otimes A_{23} \otimes A_{30} - A_{12} \otimes A_{20} \otimes A_{33} + A_{13} \otimes A_{20} \otimes A_{32} - A_{13} \otimes A_{22} \otimes A_{30} \tag{1.2.4}$$

$$\Delta_2 = A_{11} \otimes A_{20} \otimes A_{33} - A_{11} \otimes A_{23} \otimes A_{30} + A_{10} \otimes A_{23} \otimes A_{31} - A_{10} \otimes A_{21} \otimes A_{33} + A_{13} \otimes A_{21} \otimes A_{30} - A_{13} \otimes A_{20} \otimes A_{31} \tag{1.2.5}$$

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$\Delta_3 = A_{11} \otimes A_{22} \otimes A_{30} - A_{11} \otimes A_{20} \otimes A_{32} + A_{12} \otimes A_{20} \otimes A_{31} - A_{12} \otimes A_{21} \otimes A_{30} + A_{10} \otimes A_{21} \otimes A_{32} - A_{10} \otimes A_{22} \otimes A_{31}$ (1.2.6)

and $u = x \otimes y \otimes z$

with $\otimes$ denoting the Kronecker product (or Tensor) product of two matrices discussed in (1.3).

Theorem: Let $(\lambda_1, \lambda_2, \lambda_3)$ be an eigenvalue and (x, y, z) a corresponding eigenvector of the system (1.2.1) then $(\lambda_1, \lambda_2, \lambda_3)$ is an eigenvalue of the system (1.2.2) and $u = x \otimes y \otimes z$ is the corresponding eigenvector.

1.2 THE KRONECKER PRODUCT

The Kronecker product of two matrix $A \in M^{p,q}$ with the matrix $B \in M^{r,s}$ is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1q}B \\ \vdots & \ddots & \vdots \\ a_{p1}B & \cdots & a_{pq}B \end{bmatrix}$$

where we use the standard notation $(A)_{ij} = a_{ij}$

The Kronecker product is a special case of the tensor product, and as such it inherits the properties of bilinearity and associativity, i.e.

$$(kA) \otimes B = A \otimes (kB) = k(A \otimes B)$$
$$A \otimes (B + C) = A \otimes B + A \otimes C$$
$$(A + B) \otimes C = A \otimes C + B \otimes C$$

a famous property of the Kronecker product, from [9] is given bellow

Lemma (Mixed product property): Let $A \in C^{m \times n}, B \in C^{p \times q}, C \in C^{n \times k}, D \in C^{q \times r}$. Then

$$(A \otimes B)(C \otimes D) = (AC \otimes BD)$$

In particular, if $A, B \in C^{m \times m}$ and $x, y \in C^m$ then

$$(A \otimes B)(x \otimes y) = Ax \otimes By$$

1.3 Rayleigh Ritz Method for Three-parameter Eigenvalue Problems.

consider the three-parameter eigenvaliue problem (1.2.1). The main objective of this method is to find an approximation $((\boldsymbol{\theta}, \boldsymbol{\phi}, \boldsymbol{\psi}), \boldsymbol{u} \otimes \boldsymbol{v} \otimes \boldsymbol{w})$ to the eigenpair $((\lambda_1, \lambda_2, \lambda_3), x \otimes y \otimes z)$, where approximate eigenvectors u, v, w should be in a given search space U_k, V_k and W_k of law dimension k and $(\boldsymbol{\theta}, \boldsymbol{\phi}, \boldsymbol{\psi})$ should be in the neighbourhood of the target $(\eta, \tau, \zeta) \in C$. Since $u \in U_k, v \in V_k$ and $w \in W_k$ so one can write $u = u_k c, v = v_k d$ and $w = w_k e$, where u_k, v_k and w_k form an orthonormal basis for U_k, V_k and W_k respectively and c, d, e are vectors in C^k of unit norm. Applying Ritz-Galerkin condition on the residual

$$r_1 = A_{10}u - \theta A_{11}u - \emptyset A_{12}u - \psi A_{13}u \perp U_k$$
$$r_2 = A_{20}v - \theta A_{21}v - \emptyset A_{22}v - \psi A_{23}v \perp V_k$$
$$r_3 = A_{30}w - \theta A_{31}w - \emptyset A_{32}w - \psi A_{33}w \perp W_k$$

i.e

$$u_k^* A_{10} u_k c = \theta u_k^* A_{11} u_k c + \phi u_k^* A_{12} u_k c + \psi u_k^* A_{13} u_k c$$
$$v_k^* A_{20} v_k d = \theta v_k^* A_{21} v_k d + \phi v_k^* A_{22} v_k d + \psi v_k^* A_{23} v_k d$$
$$w_k^* A_{30} w_k e = \theta w_k^* A_{31} w_k e + \phi w_k^* A_{32} w_k e + \psi w_k^* A_{33} w_k e$$

It will become a three-parameter eigenvalue problem of lower dimensional. We solve this problem using kronecker product method and we will get the approximate eigenvalue $(\boldsymbol{\theta}, \boldsymbol{\phi}, \boldsymbol{\psi})$ and corresponding eigenvectors u, v and w.

But the problem with this method is that even if there is a Ritz value $(\theta, \phi, \psi) \approx (\lambda_1, \lambda_2, \lambda_3)$, we donot have the gurantee that the two norm $\| r \|$ is small, which reflects the fact that the approximate eigenvector may be poor. This method works well for exterior eigenvalue, but is generally less favourable for interior ones.

1.5 NUMERICAL EXAMPLE:

Consider the three-parameter eigenvalue problem

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} X - \lambda_1 \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -.5 \\ 1 & -.5 & 2 \end{bmatrix} X - \lambda_2 \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} X - \lambda_3 \begin{bmatrix} 4 & 1 & 2 \\ 1 & 5 & 0 \\ 2 & 0 & 4 \end{bmatrix} X = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} Y - \lambda_1 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} Y - \lambda_2 \begin{bmatrix} 3 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 4 \end{bmatrix} Y - \lambda_3 \begin{bmatrix} 6 & 1 & -.5 \\ 1 & 6 & 1 \\ -.5 & 1 & 7 \end{bmatrix} Y = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} Z - \lambda_1 \begin{bmatrix} 2 & -.5 & 0 \\ -.5 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} Z - \lambda_2 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix} Z - \lambda_3 \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} Z = 0$$

Consider $u_k = \begin{bmatrix} -.8473 & .4179 \\ .2679 & .9086 \\ .4586 & -.0066 \end{bmatrix}$, $v_k = \begin{bmatrix} -.2847 & .1696 \\ .9267 & .0733 \\ -.2452 & .0800 \end{bmatrix}$ and $w_k = \begin{bmatrix} .0044 & -.2 \\ -.3582 & .1036 \\ .9336 & .0407 \end{bmatrix}$

Then

$$u_k^* A_{10} u_k c = \theta u_k^* A_{11} u_k c + \phi u_k^* A_{12} u_k c + \psi u_k^* A_{13} u_k c$$
$$v_k^* A_{20} v_k d = \theta v_k^* A_{21} v_k d + \phi v_k^* A_{22} v_k d + \psi v_k^* A_{23} v_k d$$
$$w_k^* A_{30} w_k e = \theta w_k^* A_{31} w_k e + \phi w_k^* A_{32} w_k e + \psi w_k^* A_{33} w_k e$$ gives

$$\begin{bmatrix} .7179 & -.3541 \\ -.3541 & .1746 \end{bmatrix} c = \theta \begin{bmatrix} .7178 & -.6521 \\ -.6521 & 3.5859 \end{bmatrix} c + \phi \begin{bmatrix} -.9854 & -.0457 \\ -.0457 & .7419 \end{bmatrix} c + \psi \begin{bmatrix} 2.0635 & -.4748 \\ -.4748 & 5.5749 \end{bmatrix} c$$

$$\begin{bmatrix} .8588 & .0679 \\ .0679 & .0054 \end{bmatrix} d = \theta \begin{bmatrix} -.7013 & .0602 \\ .0602 & .0989 \end{bmatrix} d + \phi \begin{bmatrix} 2.4967 & -.0202 \\ -.0202 & .2460 \end{bmatrix} d + \psi \begin{bmatrix} 5.0079 & .2052 \\ .2052 & .2726 \end{bmatrix} d$$

$$\begin{bmatrix} .8716 & .0380 \\ .0380 & .0017 \end{bmatrix} e = \theta \begin{bmatrix} 1.4609 & .0090 \\ .0090 & .1447 \end{bmatrix} e + \phi \begin{bmatrix} 2.0794 & .0437 \\ .0437 & .0064 \end{bmatrix} e + \psi \begin{bmatrix} 1.3279 & .1551 \\ .1551 & .0318 \end{bmatrix} c$$

Solving these three-parameter eigenvalue problem we have

$$\theta = 1.1260, .7714, .3821, .0301, .003, -.0401, -.0369, -.0360$$
$$\phi = .5334, .4043, .2432, .0103, .0002, -.2300, -.2614, -.2633$$
$$\psi = -.4979, -.1536, .2199, -.0746, -.0194, -.002, .2817$$

1.6 CONCLUSION

The exact eigenvalues of the above problems are

$$\lambda_1 = 0, 1.1569, .9410, .9487, .7953, -.6141, .4354, .3821, -.2823, -.2391, -.2215, .1651, -.1175, .1159, .0732, -.0044, -.0408, -.0360$$
$$\lambda_2 = 0, 1.3237, 1.2018, .7968, -.5188, .3824, .3715, .2608, .2434, .1624, .0958, .0103, .0298, -.3954, -.1382, -.3281, -.2739, -.2666, -.2543, -.2463$$
$$\lambda_3 = 0, -.6283, -.4524, -.3883, .3871, -.2062, -.1655, .3189, .2875, .29, .2199, .1525, -.0862, .0858, -.0262, .0603, .0334, .0261, .0103, .0082$$

It has seen that the eigenvalues using Rayleigh Ritz method are very close to the exact eigenvalues. In this method the dimension of the matrix can be reduced to lower dimensional. So one can use this method easily to solve higher order matrix eigenvalue problems.

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