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COMMON FIXED POINT THEOREM FOR TWO PAIRS OF WEAKLY COMPATIBLE MAPPINGS IN RANDOM M-FUZZY METRIC SPACE

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ABSTRACT

In this paper, we introduce a definition analogous to the definition given by Gupta et al. [13] and prove a common fixed point theorem for two pairs of weakly compatible mappings in Random M-fuzzy metric space.

Mathematics Subject Classification: *47H10, 54H25*

Keywords: *fuzzy metric space, weakly compatible mappings, Metric Space, Random M –fuzzy metric space, Common fixed point.*

1. INTRODUCTION:

After introduction of fuzzy sets by Zadeh [9], Kramosil and Michalek [8] introduced the concept of fuzzy metric space in 1975.Consequently in due course of time many researchers have defined fuzzy metric space in different ways. Researchers like George and Veeramani [1], Grabiec [10], Subrahmanyam [12], Vasuki [14] used this concept to generalize some metric fixed point results .Recently Sedghi and Shobe [15] introduced M-fuzzy metric space which is based on D*-metric concept. In due course of time Random nonlinear analysis was studied by various researchers, which is mainly concerned with the study of random nonlinear operators and their properties and is much needed for the study of various classes of random equations. Of course famously random methods have revolutionized the financial markets. Random fixed point theorems for random contraction mappings on separable complete metric spaces were proved by various researchers like Spacek [2] and Hans [11]. The survey article by Bharucha-Reid [3] in 1976 arrtacted the attention of several mathematician and gave wings to this theory. Itoh [16] extended Spacek's and Hans's theorem to multi valued contraction mapping. Now this theory has become the full fledged research area and various ideas associated with random fixed point theory are used to obtain the solution of nonlinear random system (see [4]). Recently Xu [6], Beg and Shahzad [7] and many other authors have studied the random fixed points of random operator.

The concept of Fuzzy-random-variable was introduced as an analogous notion to random- variable in order to extend statistical analysis to situations when the outcomes of some random experiment are fuzzy sets. But in contrary to the classical statistical methods no unique definition has been established before the work of Volker [17]. He presented set theoretical concept of fuzzy-random-variables using the method of general topology and drawing some results from topological measure theory and the theory of analytic spaces. In this paper we introduce a definition analogous to the definition given by Gupta et al. [13] and prove a common fixed point theorem for four weakly compatible mappings in random M-fuzzy metric space. First we give some known definitions and results in M-fuzzy metric space given by Sedghi and Shobe [15] and then some definitions related to Random space in reference to fuzzy and M-fuzzy metric space and then prove our main result.

2. PRELIMINARIES:

Let (Ω, Σ) be a measurable space (Σ-Sigma algebra) and C a nonempty subset of a arbitrary set X.

A mapping g: $\Omega \to X$ is measurable if $g^{-1}(U) \in \Sigma$ for each open subset U of X. A mapping T: $\Omega\times C \to C$ is a random map if and only if for each fixed $x \in C$, the mapping $T(\cdot, x)$: $\Omega \to C$ is measurable, and it is continuous if for each $\omega \in \Omega$, the mapping $T(\omega, \cdot)$: $C \to X$ is continuous.

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A measurable mapping g: $\Omega \rightarrow X$ is a random fixed point of a random map T: $\Omega \times C \rightarrow X$ if T (ω, g (ω)) = g (ω) for each $\omega \in \Omega$.

Definition: 2.1 ([5]) A binary operation $*$: [0, 1] × [0, 1] → [0, 1] is a continuous t-norm if it satisfies the following conditions

(1) $*$ is associative and commutative,

(2) $*$ is continuous,

(3) $a * 1 = a$ for all $a \in [0, 1]$,

(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each a , b, c, d ∈ [0,1].

Two typical examples of continuous t-norm are $a*b = a\,b$ and $a*b = \min\{a, b\}$.

Definition: 2.2 ([15]) A 3-tuple (X, M,$*$) is called a M-fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^3 \times (0, \infty)$, satisfying the following conditions for each x, y, z, a. X and t, s >0,

(1) M(x, y, z, t) > 0,

(2) M(x, y, z, t) = 1 if and only if x = y = z,

(3) M(x, y, z, t) = M (p{x, y, z}, t), (symmetry) where p is a permutation function,

(4) $M(x, y, a, t) * M(a, z, z, s) \leq M(x, y, z, t + s)$,

(5) M(x, y, z,.): $(0, \infty) \rightarrow [0, 1]$ is continuous.

Remark: 2.1 ([15]) Let (X, M,$*$) be a M-fuzzy metric space. Then for every t > 0 and for every x, y ∈ X. we have M(x, x, y, t) = M(x, y, y, t).

Definition: 2.3 ([15]) A sequence {xn} in X converges to x if and only if M(x, x, xn, t) → 1 as $n \rightarrow \infty$, for each t > 0. It is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and t > 0, there exists n0 ∈ **N** such that M (xn, xn, xm, t) > 1 - ε for each n, m ≥ n0.

The M-fuzzy metric space (X, M, $*$) is said to be complete if every Cauchy sequence is convergent.

Lemma: 2.1([15]) Let (X, M,$*$) be a M-fuzzy metric space. Then M(x, y, z, t) is non decreasing with respect to t, for all x, y, z in X.

Lemma: 2.2([15]) Let (X, M, $*$) be a M-fuzzy metric space. Then M is continuous function on $X^3 \times (0, \infty)$.

Definition: 2.4 ([8]) Let f and g be two self maps of (X, M, $*$).Then f and g are said to be weakly compatible if there exists u in X with fu = gu implies fgu = gfu.

Definition: 2.5 [13] Let (Ω, Σ) be a measurable space and g: $\Omega \rightarrow X$ is measurable is measurable selector. X is any non empty set.* is continuous t-norm. M is fuzzy set in $X^2 \times [0, \infty)$, then (X, Ω, M, *) is said to be randomized fuzzy metric spaces if the following are true:

For all gx , gy , gz ∈ X and s ,t > 0,

RFM-1: M (gx , gy, 0) = 0

RFM-2: M (gx , gy, t) = 1, for all t > 0 iff x=y,

RFM-3: M(gx, gy, t) = M(gy, gx, t)

RFM-4: M (gx, gz, t+s) ≥ M (gx, gy, t) *M (gz, gy, s).

RFM-5: M (gx, gy, a): [0, 1) → [0, 1] is left continuous.

RFM-6: M (gx, gy, t) = 1 as lim t$\to \infty$ for all gx, gy $\in$ X.

In analogous to the definition used by Gupta et al. [13], we extend the same to Random M-fuzzy metric space as follow

Definition: 2.6 Let (Ω, Σ) be a measurable space and g: $\Omega \to X$ is measurable selector. X is any non empty set.* is continuous t-norm. M is fuzzy set in $X^3 \times [0, \infty)$, then (X, Ω, M, *) is said to be randomized fuzzy metric spaces if the following are true For all gx , gy, gz $\in$ X and s,t >0,

RMFM-1: M (gx, gy, gz, t) > 0

RMFM-2: M (gx, gy, gz, t) = 1, for all t > 0 iff x = y = z,

RMFM-3: M (gx, gy, gz, t) =M (p {gx, gy, gz}, t), (symmetry), where p is permutation function.

RMFM-4: M (gx, gy, a, t) * M (a, gz, gz, s) $\le$ M (gx, gy, z, t + s),

RMFM-5: M (gx, gy, gz.): $(0, \infty) \to [0, 1]$ is continuous.

Remark: 2.2 Here in this paper B (w, g (w)) = Bgw for each w$\in \Omega$ where B: $\Omega \times C \to X$ is a random map.

3. MAIN RESULTS

Theorem: 3.1 Let (X, M, Ω, $\ast$) be complete Random M-fuzzy metric space. Let S and T are two continuous self mappings on this space. Let A and B be two self mappings of X satisfying

(I) $A(X) \cup B(X) \subseteq S(X) \cap T(X)$,

(II) {A, T} and {B, S} are weakly compatible pairs, and

(III) a M(Tgx, Sgy,Bgz, t) + bM(Tgx, Agx , Sgz, t) + cM(Sgy, Bgz, Bgy, t)
+ max{M(Agx, Sgz, Bgz,t),M(Bgy, Tgx, Sgy, t)} $\le$ qM(Agx, Sgz, Bgz, t) for all gx, gy, gz $\in$ X and t > 0,

where a, b, c $\ge$ 0 with 0 < q < a+ b + c < 1. Then A, B, S and T have a unique common fixed point.

Proof: Let $gx_0 \in X$ be any arbitrary point. Since $A(X) \subseteq S(X)$, there must exists a point $gx_1 \in X$ such that $Agx_0 = Sgx_1$, Also since $B(X) \subseteq T(X)$, then there exists another point $gx_2 \in X$ such that $Bgx_1 = Tgx_2$ and so on. In general, we get a sequence $\{gy_n\}$ recursively as y_{2n} = gx2n = Sgx2n+1 = Agx_{2n} and y2n+1 = gx2n+1 = Tgx2n+2 = Bgx2n+1, n = 0, 1, 2, 3…

Let Mn = M (y_n, y_{n+1}, y_{n+2}, t) for all n. Putting gx = gx_{2n}, gy = gx_{2n+1} and gz = gx_{2n+2} in (III) we get

am (Tgx_{2n}, Sgx2n+1,Bgx2n+2 , t) + bM(Tgx_{2n} , Agx_{2n}, Sgx2n+1, t) + c M(Sgx2n+1, Bgx2n+2, Bgx_{2n+1}, t)
+ max {M (Agx_{2n}, Sgx2n+2, Bgx2n+2, t), M (Bgx2n+1 , Tgx_{2n}, Sgx2n+1, t) } $\le$ q M(Agx_{2n} , Sgx2n+2, Bgx2n+2, t)

i.e., a M (y2n−1, y2n, y2n +2 , t) + b M(y2n−1, y2n, y2n+1 , t) + c M(y2n, y2n +2, y2n+1 , t)
+max {M (y2n, y2n+1, y2n +2, t), M (y2n+1, y2n-1, y2n, t)} $\le$ q M (y2n, y2n+1, y2n +2, t),

i.e., (a + b) M2n−1 + (c+ 1) M2n $\le$ qM2n,

i.e., (q − c-1) M2n $\ge$ (a + b) M2n−1, (a + b) M2n−1 $\le$ (q − c-1) M2n < (q-c) M2n, (q-c) M2n > (a + b) M2n−1,
M2n > {(a + b) / (q-c)} M2n−1

Let (a + b) / (q-c) = r then r >1 which implies M2n > r M2n -1 > M2n -1. (1)

Thus {M2n, n $\ge$ 0} is an increasing sequence of positive real numbers in [0, 1] and therefore tends to a limit m $\le$ 1. We claim m = 1, for m < 1, taking limit in (1) we get m < m, which is a contradiction. Therefore m = 1.

For any positive integer r,
M (yn, yn, yn+r, t) $\ge$ M(yn, yn ,yn+1, t /r) $\ast \cdots \ast$ M(yn+1, yn+1 ,yn+2, t /r) $\ast \ldots \ast$ M (yn+r-1, yn+r -1, yn+r, t /r)
> (1-ε) $\ast$ (1-ε) $\ast$ (1-ε) $\ast$ ….r times = (1-ε).

Thus M (yn, yn, yn+r, t) > (1-ε) which implies M (yn, yn, yn + s, t) > (1-ε) for all n, s $\ge$ n_0 where $n_0 \in$ N.

Thus {yn} is a Cauchy sequence in X. Since X is complete, there is a point gw ∈ X such that yn → gw, therefore the subsequences $\{Ax_{2n}\}$, {Bx2n+1}, {Sx2n+1} and {Tx2n+2} are Cauchy and converge to same limit, say gw.

Now we will prove that gw is a fixed point of A, B, S and T.

For this first we prove that gw is coincidence point of A, B, S and T under the given condition of weak compatibility.

Since A(X) ⊆ S(X) and A(X) ⊆ T(X) so there must exists a point gu, gv ∈ X such that gw = Sgu and gw= Tgv .

Put gx = gx2n, gy = gu and gz = gu in (III)

$$am(Tgx_{2n}, Sgu, Bgu, t) + bM(Tgx_{2n}, Agx_{2n}, Sgu, t) + cM(Sgu, Bgu, Bgu, t) + \max\{M(Agx_{2n}, Sgu, Bgu, t), M(Bgu, Tgx_{2n}, Sgu, t)\} \le qM(Agx_{2n}, Sgu, Bgu, t),$$

$$am(gw, gw, Bgu, t) + bM(gw, gw, gw, t) + cM(gw, Bgu, Bgu, t) + \max\{M(gw, gw, Bgu, t), M(Bgu, gw, gw, t)\} \le qM(gw, gw, Bgu, t), \text{ implies}$$

$$(a+ c + 1)\, qM(gw, gw, Bgu, t) + b \le q\, M(gw, gw, Bgu, t),$$

Implies M (gw, gw, Bgu, t) ≥ b/ (q-a-c-1 > 1, therefore Bgu = gw = Sgu.

Put gx = gv, gy = gx2n+1, gz = gu in (III)

$$am(Tgv, Sgx2n+1, Bgu, t) + bM(Tgv, Agv, Sgu, t) + c\,M(Sgx2n+1, Bgu, Bgx2n+1, t)\ \max\{M(Agv, Sgu, Bgu, t), M(Bg\,x2n+1, Tgv, Sgx2n+1, t)\} \le qM(Agv, Sgu, Bgu, t),$$

implies $am(gw, gw, gw, t)+ bM(gw, Agv, gw, t) + cM(gw, gw, gw, t) + \max\{M(Agv, gw, gw, t), M(gw, gw, gw, t)\} \le qM(Agv, gw, gw, t)$, implies

$$a +1+(b + c)\, M(gw, Agv, gw, t) \le q\, M(gw, Agv, gw, t),$$

implies M (Agv, gw, gw, t) ≥ {(a + 1)/ (q-b-c)} > 1, implies Agv = gw .

Therefore Agv = Tgv = gw.

Since {A, T} and {B, S} are weakly compatible therefore ATgv =TAgv which gives

Agw = Tgw and BSgu = SBgu which gives Bgw = Sgw, which proves that gw is the coincidence point of A, B, S and T.

Now, we will show that gw is common unique fixed point of A, B, S and T.

First we prove that gw is a fixed point of B.

Put g x = gv, gy =g u, gz = gw in (III) ,we get

$$am(Tgv, Sgu, Bgw, t) + bM(Tgv, Agv, Sgw, t) + cM(Sgu, Bgw, Bgu, t) + \max\{M(Agv, Sgw, Bgw, t), M(Bgu, Tgv, Sgu, t) \le qM(Agv, Sgw, Bgw, t)$$

Implies $am(gw, gw, Bgw, t)+bM(gw, gw, Bgw, t)+cM(gw, Bgw, gw, t) +\max\{M(gw, Bgw, Bgw, t), M(gw, gw, gw, t)\} \le qM(gw, Bgw, Bgw, t)$, implies

(a+b +c) M(gw, gw, Bgw, t)+1 ≤ qM(gw, gw, Bgw, t), implies M (gw, gw, Bgw, t) ≥{1/ (q-a-b-c)} >1.

Therefore Bgw = gw and consequently Sgw = gw. Thus gw is a common fixed point of B and S.

Similarly gw is a common fixed point of A and T. Hence gw is a common fixed point of A, B, S and T.

Now for the uniqueness of gw, suppose gm is another common fixed point of A, B, S and T other than gw. Then

Agm = Bgm = gm = Sgm = Tgm and Agw = Bgw = w= Sgw = Tgw.

Then put gx = gw, gy = gw, gz = gm in (III) we have

$$am(Tgw, Sgw, Bgm, t) + bM(Tgw, Agw, Sgm, t) + cM(Sgw, Bgm, Bgw, t) + \max\{M(Agw, Sgm, Bgm, t), M(Bgw, Tgw, Sgw, t)\} \le qM(Agw,Sgm,Bgm, t)$$

which implies

$$am(gw, gw, gm, t) + bM(gw, gw, gm, t) + cM(gw, gm, gw, t) + \max\{M(gw, gm, gm, t), M(gw, gw, gw, t)\} \le q\, M(gw, gm, gm, t)$$

which implies $(a + b + c)\, M(gw, gw, gm, t) + 1 \le qM(gw, gw, gm, t)$

implies $M(gw, gw, gm, t)(q\text{-}a\text{-}b\text{-}c) \ge 1$ implies $M(gw, gw, gm, t) \ge 1/(q\text{-}a\text{-}b\text{-}c) > 1$.Therefore gw = gm.

Hence gw is unique common fixed point of A, B, S and T.

Corollary: 3.1 Let $(X, M, \Omega, *)$ be complete Random M- fuzzy metric space. Let S and T are two continuous self

mappings on this space. Let A be a self mapping of X satisfying

(1) $A(X) \subseteq S(X) \cap T(X)$,

(2) {A, T} and {A, S} are weakly compatible pairs, and

(3) $aM(Tgx, Sgy, Agz, t) + bM(Tgx, Agx, Sgz, t) + cM(Sgy, Agz, Agy, t) + \max\{M(Agx, Sgz, Agz, t), M(Agy,Tgx, Sgy, t)\} \le qM(Agx, Sgz, Agz, t)$

for all gx, gy, gz $\in$ X and $t > 0$, where $a, b, c \ge 0$ with $0 < q < a+ b + c < 1$. Then A, S and T have a unique common *fixed point*

Proof: Taking B = A in theorem 3.1, we get the required result.

Corollary: 3.2 Let $(X, M, \Omega, *)$ be complete Random M-fuzzy metric space. Let T be continuous self mappings on this space. Let A and B be two self mappings of X satisfying

(1) $A(X) \cup B(X) \subseteq T(X)$,

(2) {A, T} and {B, T} are weakly compatible pairs, and

(3) $aM(Tax, Tgy, Bgz, t) + bM(Tgx, Agx, Tgz, t) + cM(Tgy, Bgz, Bgy, t) + \max\{M(Agx,Tgz, Bgz, t), M(Bgy,Tgx,Tgy, t)\} \le qM(Agx,Tgz,Bgz, t)$

for all gx, gy, gz $\in$ X and $t > 0$, where $a, b, c \ge 0$ with $0 < q < a+ b + c < 1$. Then A, B and T have a unique common fixed point.

Proof: Taking S= T in theorem 3.1, we get the required result.

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