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BAYESIAN AND CLASSICAL ESTIMATION FOR THE WEIBULL DISTRIBUTION PARAMETERS UNDER PROGRESSIVE TYPE-II CENSORING SCHEMES

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ABSTRACT

This article discusses the estimation of the Weibull parameters using the maximum product spacing (MPS), the maximum likelihood (ML) and Bayesian estimation methods. The estimation is done under progressive type-II censored samples and a comparative study among the three methods is made using Monte Carlo Simulation. A real data is used to study the performance of the estimation process under this optimal scheme in practice. We also discuss the construction of progressively censored sampling plans for the Weibull distribution to determine the optimal progressive censoring plans. Lindley's approximation has been employed to compute the Bayes estimators of the Weibull distribution.

Keywords: *Weibull distribution, Maximum Product Spacing, Bayesian estimation, progressive Type-II censoring, Lindley approximation.*

1. INTRODUCTION

The Weibull distribution is commonly used for modeling lifetime data with monotone failure rates. A random variable X has the Weibull distribution with parameters α, and θ if its cumulative distribution function (cdf), probability density function (pdf) and the quantile function are given by

$$F(x;\theta,\alpha) = 1 - e^{-\left(\frac{x}{\alpha}\right)^{\theta}}, \tag{1.1}$$

$$f(x;\theta,\alpha) = \frac{\theta}{\alpha}\left(\frac{x}{\alpha}\right)^{\theta-1} e^{-\left(\frac{x}{\alpha}\right)^{\theta}}, \tag{1.2}$$

and

$$x = \alpha(-\ln(1-u))^{\frac{1}{\theta}} \quad ; 0 < u < 1, \tag{1.3}$$

respectively, where $\alpha, \theta > 0$ and $x > 0$.

Many authors have estimated the parameters of the Weibull distribution under different sampling schemes. Shuo-Jye Wu (2002) found the maximum likelihood estimators of the Weibull distribution under different progressive censored data and found an exact confidence interval as well as exact joint confidence interval for the distribution parameters. Debasis Kundu (2008) dealt with the Bayesian inference of unknown parameters of the progressively censored Weibull distribution. Kundu assumed that the shape parameter has a log-concave prior density function and for the given shape parameter, the scale parameter has a conjugate prior distribution and used Lindley's approximation to compute the Bayes estimates and the Gibbs sampling procedure to calculate the credible intervals. Pak et al. (2013) estimated the Weibull parameters assuming that collected data might be imprecise and are represented in the form of fuzzy numbers. They included the maximum likelihood estimation, Bayesian estimation and method of moments. Alizadeh *et. al*, (2015) have derived analytical expressions for the bias and the mean squared error when estimating the Weibull distribution based on maximum likelihood (MLE), percentile, least squares and weight least squares estimation methods. For more about the weibull distribution the reader can be directed to the book written by Rinne (2008).

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The maximum product of spacings estimation (MPS) method was introduced by Cheng and Amin (1979, 1983) and independently discussed by Ranneby (1984) as an alternative to maximum likelihood estimation (MLE) method for the estimation of parameters of continuous univariate distributions. It was shown that for some distributions such as the three-parameter Gamma, Lognormal or Weibull distributions where the MLE method breaks down due to unboundedness of the likelihood, the MPS method produces consistent and asymptotically efficient estimators. In situations like mixtures of normal distributions where the MLE method is known to produce inconsistent estimators (see Ranneby, 1984). For comprehensive content, one can refer to Ekström (2006) and Almetwally and Almongy (2019_b).

Right censoring is one of the censoring techniques used in life-testing experiments. A sample is said to be censored if while it is drawn from a complete population, the item values of some of its members are unknown. Kundu and Pradhan (2009) discussed the two most common censoring schemes termed as type-I and type-II censoring schemes. Ng *et al.* (2012) introduced estimation of parameters for a three-parameter Weibull distribution based on progressively Type-II right censored samples using the MLE, corrected MLE, weighted MLE and least squares estimation methods. Singh *et al.* (2016) proposed estimation of generalized inverted exponential distribution based on progressive type-II censored samples. Recently, Almetwally and Almongy (2018_a) compared between the maximum likelihood estimator (MLE) and Bayesian estimation for the shape parameters of the generalized power Weibull (GPW) distribution based on complete censoring data, type-II censoring scheme and type-II progressive censoring scheme. Progressive Type-II censoring scheme can be described as follows: Suppose n units are placed on a life test and the experimenter decides beforehand the quantity m, the number of failures to be observed. Now at the time of the first failure, R_1 of the remaining $n-1$ surviving units are randomly removed from the experiment. At the time of the second failure, R_2 of the remaining $n-R_1-1$ units are randomly removed from the experiment. Finally, at the time of the m-th failure, all the remaining surviving units $R_m = n - m - R_1 - \cdots - R_{m-1}$ are removed from the experiment. Therefore, a progressive Type-II censoring scheme consists of m, and $R_1 \ldots R_m$, such that $R_1 + \cdots + R_m = n - m$. For more example see Almetwally *et al.* (2018).

The aim of this paper is to estimate the parameters of the model from both frequentist and Bayesian perspectives for the Weibull distribution under Progressive Type-II Censoring Schemes. The maximum likelihood estimators (MLEs) and the maximum product of spacing estimation (MPS) method are used as frequentist methods. On the other hand, Bayesian estimators for the Weibull parameters are considered under the assumptions of independent gamma priors are considered under squared error loss function. When computing Bayesian estimators, we considered both the likelihood and the product of spacing geometric mean functions as the joint distribution for evaluating the joint posterior distribution function. To evaluate the performance of the estimators, a simulation study is carried out. The optimal censoring scheme has been suggested using two different optimality criteria (mean squared error (MSE) and Bias) and a Lindley's approximation is utilized for computing the Bayes' estimates.

The paper is organized as follows: section 2 is devoted for the estimation of the Weibull parameters using the MLE method. In section 3 we apply the MPS method while in section 4 the Bayesian approach based on both MLE and MPS probability functions is considered and Lindley's approximation for the Bayes estimators is utilized. In section 5, we present Monte Carlo simulation study to compare the performance of the estimators of the Weibull distribution parameters for all estimation methods used. Finally, we address the results and the conclusion of the current study.

2. THE MAXIMUM LIKELIHOOD ESTIMATION METHOD

Based on the observed sample $x_1 < \cdots < x_m$ from a progressive Type-II censoring scheme, $R_1, \ldots, R_m$ the likelihood function can be written as

$$L_{ML} = A \prod_{i=1}^{m} f(t_i;\, \theta, \alpha)\, (1 - F(t_i;\, \theta, \alpha)^{R_i} \quad ; \; \theta, \alpha > 0, \tag{2.1}$$

where

$$A = n(n - R_1 - 1) \ldots \left(n - \sum_{i=1}^{m-1} R_i - (m-1) \right),$$

where $F(\cdot)$ and $f(\cdot)$ are as defined in Eqs. (1.1) and (1.2) respectively. The likelihood function and the log likelihood are written as

$$L_{ML} = A \left(\frac{\theta}{\alpha}\right)^{m} \left(e^{-\sum_{i=1}^{m} \left(\frac{x_i}{\alpha}\right)^{\theta}} \right)^{R_i+1} \prod_{i=1}^{m} \left(\frac{x_i}{\alpha}\right)^{\theta-1}, \tag{2.2}$$

and
$$\ln L_{ML} = \ln A + m \ln\theta - m\ln\alpha + (\theta - 1)\sum_{i=1}^{m} \ln\left(\frac{x_i}{\alpha}\right) - \sum_{i=1}^{m} (R_i + 1)\left(\frac{x_i}{\alpha}\right)^{\theta}, \tag{2.3}$$

respectively. To obtain the normal equations for the unknown parameters, we differentiate (2.3) partially with respect to the parameters θ and α and equate them to zero. The estimators $\hat{\theta}_{ML}$ and $\hat{\alpha}_{ML}$ can be obtained as the solution of the following equations.

$$\frac{\partial \ln L_{ML}}{\partial \theta} = \frac{m}{\theta} + \sum_{i=1}^{m} \ln\left(\frac{x_i}{\alpha}\right) - \sum_{i=1}^{m} (R_i + 1)\left(\frac{x_i}{\alpha}\right)^{\theta} \ln\left(\frac{x_i}{\alpha}\right) = 0, \tag{2.4}$$

and

$$\frac{\partial \ln L_{ML}}{\partial \alpha} = \frac{-m}{\alpha} + \sum_{i=1}^{m} (R_i + 1)\frac{\theta\left(\frac{x_i}{\alpha}\right)^{\theta}}{\alpha} - \frac{\theta - 1}{\alpha} = 0. \tag{2.5}$$

the MLEs $\hat{\theta}_{ML}$ and $\hat{\alpha}_{ML}$ of θ and α, are the solutions of the nonlinear Eqs. (2.4) and (2.5) that cannot be solved explicitly and numerical determination of maximum likelihood estimators are discussed.

3. The Method of Maximum Product of Spacing

Ng *et al.* (2012) introduced the product spacing function under progressive type-II censoring scheme and Almetwally and Almongy (2019_a) discussed parameter estimation of the Generalized Power Weibull under progressive type-II censored samples using the maximum product spacing (MPS) method as follows**:**

$$S = \prod_{i=1}^{m+1} \left(F(t_i;\ \theta, \alpha) - F(t_{i-1};\ \theta, \alpha)\right) \prod_{i=1}^{m} \left(1 - F(t_i;\ \theta, \alpha)\right)^{R_i}. \tag{3.1}$$

According to Cheng and Amin (1983), Eq. (3.1) is rewritten in the form

$$L_{MPS} = A\, G \prod_{i=1}^{m} \left(1 - F(t_i;\ \theta, \alpha)^{R_i}\right) \quad ; \quad \theta, \alpha > 0, \tag{3.2}$$

where G is defined as the geometric mean of the product spacing function, where:

$$G = \left(\prod_{i=1}^{m+1} D_i\right)^{\frac{1}{m+1}},$$

and

$$D_i = \begin{cases} D_1 = F(x_1) \\ D_i = F(x_i) - F(x_{i-1});\ i = 2 \ldots m \\ D_{m+1} = 1 - F(x_m) \end{cases} \tag{3.3}$$

such that $\sum D_i = 1$, depending on MPS method that was introduced by Cheng and Amin (1983) and Progressive Type-II Censored scheme that was discussed by Balakrishnan et al. (2004).

$$L_{MPS} = A\left(\left(1 - e^{-\left(\frac{x_1}{\alpha}\right)^{\theta}}\right)\left(e^{-\left(\frac{x_m}{\alpha}\right)^{\theta}}\right)\prod_{i=2}^{m}\left[e^{-\left(\frac{x_{i-1}}{\alpha}\right)^{\theta}} - e^{-\left(\frac{x_i}{\alpha}\right)^{\theta}}\right]\right)\prod_{i=1}^{m}\left(e^{-\left(\frac{x_i}{\alpha}\right)^{\theta}}\right)^{R_i}, \tag{3.4}$$

the natural logarithm of the product spacing function is

$$\ln L_{MPS} = \ln A + \left(\ln\left(1 - e^{-\left(\frac{x_1}{\alpha}\right)^{\theta}}\right) - \left(\frac{x_m}{\alpha}\right)^{\theta} + \sum_{i=2}^{m} \ln\left[e^{-\left(\frac{x_{i-1}}{\alpha}\right)^{\theta}} - e^{-\left(\frac{x_i}{\alpha}\right)^{\theta}}\right]\right) - \sum_{i=1}^{m} R_i \left(\frac{x_i}{\alpha}\right)^{\theta}, \tag{3.5}$$

To obtain the normal equations for the unknown parameters, we differentiate Eq. (3.5) partially with respect to the parameters θ and α and equate them to zero. The estimators $\hat{\theta}_{MPS}$ and $\hat{\alpha}_{MPS}$ of θ and α can be obtained as the solution of the following equations.

$$\begin{aligned}\frac{\partial \ln L_{MPS}}{\partial \theta} = \Bigg(& \frac{e^{-\left(\frac{x_1}{\alpha}\right)^{\theta}}\left(\frac{x_1}{\alpha}\right)^{\theta}\ln\left(\frac{x_1}{\alpha}\right)}{1 - e^{-\left(\frac{x_1}{\alpha}\right)^{\theta}}} - \left(\frac{x_m}{\alpha}\right)^{\theta}\ln\left(\frac{x_m}{\alpha}\right) \\ & + \sum_{i=2}^{m} \frac{e^{-\left(\frac{x_i}{\alpha}\right)^{\theta}}\left(\frac{x_i}{\alpha}\right)^{\theta}\ln\left(\frac{x_i}{\alpha}\right) - e^{-\left(\frac{x_{i-1}}{\alpha}\right)^{\theta}}\left(\frac{x_{i-1}}{\alpha}\right)^{\theta}\ln\left(\frac{x_{i-1}}{\alpha}\right)}{\left[e^{-\left(\frac{x_{i-1}}{\alpha}\right)^{\theta}} - e^{-\left(\frac{x_i}{\alpha}\right)^{\theta}}\right]}\Bigg) - \sum_{i=1}^{m} R_i\left(\frac{x_i}{\alpha}\right)^{\theta}\ln\left(\frac{x_i}{\alpha}\right).\end{aligned} \tag{3.6}$$

Differentiating the log product spacing function in Eq. (3.5) with respect to α is given as

$$\frac{\partial \ln L_{MPS}}{\partial \alpha} = \left(\frac{\theta\left(\frac{x_1}{\alpha}\right)^{\theta}}{\alpha\left(1 - e^{-\left(\frac{x_1}{\alpha}\right)^{\theta}}\right)} + \frac{\theta\left(\frac{x_1}{\alpha}\right)^{\theta}}{\alpha} + \sum_{i=2}^{m} \frac{\frac{\theta}{\alpha}\left(\frac{x_{i-1}}{\alpha}\right)^{\theta} e^{-\left(\frac{x_{i-1}}{\alpha}\right)^{\theta}} - \frac{\theta}{\alpha}\left(\frac{x_i}{\alpha}\right)^{\theta} e^{-\left(\frac{x_i}{\alpha}\right)^{\theta}}}{\left[e^{-\left(\frac{x_{i-1}}{\alpha}\right)^{\theta}} - e^{-\left(\frac{x_i}{\alpha}\right)^{\theta}}\right]}\right) - \sum_{i=1}^{m} R_i\, \frac{\theta\left(\frac{x_1}{\alpha}\right)^{\theta}}{\alpha}. \tag{3.7}$$

The above nonlinear equations can't be solved analytically so, the MPSs $\hat{\theta}_{MPS}$ and $\hat{\alpha}_{MPS}$ of θ and α, can then be obtained as the numerical solution of Eqs. (3.7) and (3.6).

4. BAYESIAN ESTIMATION

In this section we consider the Bayesian estimation of the unknown parameters α and θ. The Bayes estimates is considered under the assumption that the random variables α and θ have an independent gamma distribution is a conjugate prior to the Weibull distributions. Assumed that $\theta \sim Gamma(a,b)$ and $\alpha \sim Gamma(c.d)$, then, the joint prior density of α and θ can be written as

$$g(\alpha,\theta) \propto \theta^{a-1}\, e^{-\frac{\theta}{b}}\, \alpha^{c-1}\, e^{-\frac{\alpha}{d}}; \quad a,b,c, and\ d > 0, \tag{4.1}$$

where all the hyper parameters $a, b, c\ and\ d$ are known and non-negative. To more example see ALMETWALY and ALMONGY (2018_b).

4.1. Bayesian Estimation based on likelihood function

Based on the likelihood function in Eq. (2.2) and the joint prior density in Eq. (4.1), the joint posterior of Progressive Type-II censored of Weibull distribution with parameters α and θ is

$$g(\alpha,\theta|x)_{ML} = K_{ML}\, \theta^{m+a-1}\, \alpha^{c-m-1} e^{-\frac{\theta}{b}-\frac{\alpha}{d}} \left(e^{-\sum_{i=1}^{m}\left(\frac{x_i}{\alpha}\right)^{\theta}}\right)^{R_i+1} \prod_{i=1}^{m}\left(\frac{x_i}{\alpha}\right)^{\theta-1}, \tag{4.2}$$

where the normalizing constant K is

$$K_{ML} = \left[\iint_0^{\infty} \theta^{m+a-1}\, \alpha^{c-m-1} e^{-\frac{\theta}{b}-\frac{\alpha}{d}} \left(e^{-\sum_{i=1}^{m}\left(\frac{x_i}{\alpha}\right)^{\theta}}\right)^{R_i+1} \prod_{i=1}^{m}\left(\frac{x_i}{\alpha}\right)^{\theta-1} d\theta\, d\alpha\right]^{-1}. \tag{4.3}$$

Under squared error loss function the Estimators $\hat{\theta}_{BML}, \hat{\alpha}_{BML}$ of θ and α are obtained as the expectations of the joint posterior distribution and are given by

$$\hat{\theta}_{BML} = \iint_0^{\infty} K_{ML}\, \theta^{m+a}\, \alpha^{c-m-1} e^{-\frac{\theta}{b}-\frac{\alpha}{d}} \left(e^{-\sum_{i=1}^{m}\left(\frac{x_i}{\alpha}\right)^{\theta}}\right)^{R_i+1} \prod_{i=1}^{m}\left(\frac{x_i}{\alpha}\right)^{\theta-1} d\theta\, d\alpha, \tag{4.4}$$

and

$$\hat{\alpha}_{BML} = \iint_0^{\infty} K_{ML}\, \theta^{m+a-1}\, \alpha^{c-m} e^{-\frac{\theta}{b}-\frac{\alpha}{d}} \left(e^{-\sum_{i=1}^{m}\left(\frac{x_i}{\alpha}\right)^{\theta}}\right)^{R_i+1} \prod_{i=1}^{m}\left(\frac{x_i}{\alpha}\right)^{\theta-1} d\theta\, d\alpha. \tag{4.5}$$

4.2. MPS Estimation based on Progressive Type-II Censoring Scheme

Based on Product spacing function of given in Eq. (3.4) and the joint prior density in Eq. (4.2), the joint posterior of Progressive Type-II censored of Weibull distribution is

$$\begin{aligned} g(\alpha,\theta|x)_{MPS} &= K_{MPS}\theta^{a-1}\, \alpha^{c-1}\, e^{-\frac{\theta}{b}-\frac{\alpha}{d}} \prod_{i=1}^{m}\left(e^{-\left(\frac{x_i}{\alpha}\right)^{\theta}}\right)^{R_i} \\ &\left(\left(1-e^{-\left(\frac{x_1}{\alpha}\right)^{\theta}}\right)\left(e^{-\left(\frac{x_m}{\alpha}\right)^{\theta}}\right)\prod_{i=2}^{m}\left[e^{-\left(\frac{x_{i-1}}{\alpha}\right)^{\theta}} - e^{-\left(\frac{x_i}{\alpha}\right)^{\theta}}\right]\right), \end{aligned} \tag{4.6}$$

where the normalizing constant K is

$$\begin{aligned} K_{MPS} = \Bigg[\iint_0^{\infty} &\theta^{a-1}\, \alpha^{c-1}\, e^{-\frac{\theta}{b}-\frac{\alpha}{d}} \prod_{i=1}^{m}\left(e^{-\left(\frac{x_i}{\alpha}\right)^{\theta}}\right)^{R_i} \\ &\left(\left(1-e^{-\left(\frac{x_1}{\alpha}\right)^{\theta}}\right)\left(e^{-\left(\frac{x_m}{\alpha}\right)^{\theta}}\right)\prod_{i=2}^{m}\left[e^{-\left(\frac{x_{i-1}}{\alpha}\right)^{\theta}} - e^{-\left(\frac{x_i}{\alpha}\right)^{\theta}}\right]\right) d\theta\, d\alpha\Bigg]^{-1}. \end{aligned} \tag{4.7}$$

Under the square error loss function, the estimators $\hat{\theta}_{BMPS}$ and $\hat{\alpha}_{BMPS}$ are given by

$$\begin{aligned} \hat{\theta}_{BMPS} = \iint_0^{\infty} &K_{MPS}\theta^{a}\, \alpha^{c-1}\, e^{-\frac{\theta}{b}-\frac{\alpha}{d}} \prod_{i=1}^{m}\left(e^{-\left(\frac{x_i}{\alpha}\right)^{\theta}}\right)^{R_i} \\ &\left(\left(1-e^{-\left(\frac{x_1}{\alpha}\right)^{\theta}}\right)\left(e^{-\left(\frac{x_m}{\alpha}\right)^{\theta}}\right)\prod_{i=2}^{m}\left[e^{-\left(\frac{x_{i-1}}{\alpha}\right)^{\theta}} - e^{-\left(\frac{x_i}{\alpha}\right)^{\theta}}\right]\right) d\theta\, d\alpha, \end{aligned} \tag{4.8}$$

and

$$\hat{\alpha}_{BMPS} = \iint_0^\infty K_{MPS}\theta^{a-1}\,\alpha^c\, e^{-\frac{\theta}{b}-\frac{\alpha}{d}} \prod_{i=1}^{m}\left(e^{-\left(\frac{x_i}{\alpha}\right)^\theta}\right)^{R_i} \left(\left(1-e^{-\left(\frac{x_1}{\alpha}\right)^\theta}\right)\left(e^{-\left(\frac{x_m}{\alpha}\right)^\theta}\right)\prod_{i=2}^{m}\left[e^{-\left(\frac{x_{i-1}}{\alpha}\right)^\theta}-e^{-\left(\frac{x_i}{\alpha}\right)^\theta}\right]\right)\, d\theta\, d\alpha. \tag{4.9}$$

Equations (4.4, 4.5, 4.9 and 4.9) are performed numerically using a nonlinear optimization algorithm to obtain the Bayesian Estimators of θ and α.

4.3. Lindley's approximation

The Bayes estimators of any function of θ and α say $U(\alpha,\theta)$, can be rewritten as

$$\widetilde{U}(\alpha,\theta) = \frac{\int_0^\infty\int_0^\infty U(\alpha,\theta)\, e^{\ln L(\alpha,\theta)+\ln g(\alpha,\theta)}\, d\theta\, d\alpha}{\int_0^\infty\int_0^\infty e^{\ln L(\alpha,\theta)+\ln g(\alpha,\theta)}\, d\theta\, d\alpha} \tag{4.10}$$

The ratio of the two integrals given by Eq. (4.10) cannot be obtained in a closed form. Lindley's procedure developed by Lindley (1980) is used to obtain the ratio of integral. For the two parameter case, $\delta = (\delta_1,\ \delta_2) = (\alpha,\ \theta)$, Lindley's approximation of Eq. (4.10) takes the form

$$\widetilde{U}(\delta) = U(\delta) + \frac{1}{2}\sum_{i\neq j} u_{ij}\tau_{ij} + \sum_j U_j\rho_j + \frac{1}{2}\,L_{30}\tau_{11}U_1 + \frac{1}{2}\,L_{21}(2\,\tau_{12}U_1 + \tau_{11}U_2) + \frac{1}{2}\,L_{12}(\tau_{22}U_1 + 2\tau_{12}U_2) + \frac{1}{2}L_{03}\tau_{22}U_2 \tag{4.11}$$

where $L_\zeta = \frac{\partial^{\zeta+} L(\theta)}{\partial^\zeta\,\delta_1\partial\delta_2}$, $\zeta,\ = 0,1,2,3$ and $\zeta +\ = 3$, $u_i = \frac{\partial U(\delta)}{\partial\delta_i}$, $u_{ij} = \frac{\partial^2 U(\delta)}{\partial\delta_i\partial\delta_j}$ for $i,j = 1,2$

$U_j = \sum_i u_i\tau_{i,j}$ where $\tau_{i,j}$ is the inverse of the element of the fisher information matrix, and $\rho_j = \frac{\partial\rho(\delta)}{\partial\rho_j}$. Applying Lindley's approximation we have

$$L_{20} = \frac{\partial^2 \ln L}{\partial\theta^2} = \frac{-m}{\hat{\theta}^2} - \sum_{i=1}^{m}(R_i+1)\left(\frac{x_i}{\hat{\alpha}}\right)^{\hat{\theta}}\left(\ln\left(\frac{x_i}{\hat{\alpha}}\right)\right)^2$$

$$L_{02} = \frac{\partial^2 \ln L}{\partial\alpha^2} = \frac{m}{\hat{\alpha}^2} + \frac{m(\hat{\theta}-1)}{\hat{\alpha}^2} - \frac{\hat{\theta}(\hat{\theta}+1)}{\hat{\alpha}^2}\sum_{i=1}^{m}(R_i+1)\left(\frac{x_i}{\hat{\alpha}}\right)^{\hat{\theta}}$$

$$L_{11} = \frac{\partial^2 \ln L}{\partial\theta\,\partial\alpha} = \frac{-m}{\hat{\theta}} - \frac{\hat{\theta}}{\hat{\alpha}}\sum_{i=1}^{m}(R_i+1)\left(\frac{x_i}{\hat{\alpha}}\right)^{\hat{\theta}}\ln\left(\frac{x_i}{\hat{\alpha}}\right) + \frac{2}{\hat{\alpha}}\sum_{i=1}^{m}(R_i+1)\left(\frac{x_i}{\hat{\alpha}}\right)^{\hat{\theta}}$$

$$L_{30} = \frac{\partial^3 \ln L}{\partial\theta^3} = \frac{2m}{\hat{\alpha}^3} - \sum_{i=1}^{m}(R_i+1)\left(\frac{x_i}{\hat{\alpha}}\right)^{\hat{\theta}}\left(\ln\left(\frac{x_i}{\hat{\alpha}}\right)\right)^3$$

$$L_{03} = \frac{\partial^3 \ln L}{\partial\alpha^3} = \frac{-2m}{\hat{\alpha}^3} - \frac{2m(\hat{\theta}-1)}{\hat{\alpha}^3} + \frac{\hat{\theta}(\hat{\theta}^2+3\hat{\theta}+2)}{\hat{\alpha}^3}\sum_{i=1}^{m}(R_i+1)\left(\frac{x_i}{\hat{\alpha}}\right)^{\hat{\theta}}$$

$$L_{21} = \frac{\partial^3 \ln L}{\partial\theta^2\partial\alpha} = \frac{2}{\hat{\alpha}}\sum_{i=1}^{m}(R_i+1)\left(\frac{x_i}{\hat{\alpha}}\right)^{\hat{\theta}}\ln\left(\frac{x_i}{\hat{\alpha}}\right) + \frac{\hat{\theta}}{\hat{\alpha}}\sum_{i=1}^{m}(R_i+1)\left(\frac{x_i}{\hat{\alpha}}\right)^{\hat{\theta}}\left(\ln\left(\frac{x_i}{\hat{\alpha}}\right)\right)^2$$

$$L_{12} = \frac{\partial^3 \ln L}{\partial\theta\,\partial\alpha^2} = \frac{m}{\hat{\alpha}^2} + \left(\frac{\hat{\theta}}{\hat{\alpha}}\right)^2\sum_{i=1}^{m}(R_i+1)\left(\frac{x_i}{\hat{\alpha}}\right)^{\hat{\theta}}\ln\left(\frac{x_i}{\hat{\alpha}}\right) + \frac{\hat{\theta}}{\hat{\alpha}^2}\sum_{i=1}^{m}(R_i+1)\left(\frac{x_i}{\hat{\alpha}}\right)^{\hat{\theta}} - 2(\hat{\theta}+1)\hat{\alpha}^{-\hat{\theta}-2}\sum_{i=1}^{m}(R_i+1)\,x_i^{\hat{\theta}}$$

The elements of the inverse of Fisher information matrix $\tau_{i,j}$ are obtained as

$$\tau_{11} = \frac{V}{WV-H^2}, \tau_{22} = \frac{W}{WV-H^2}, and\ \tau_{12} = \tau_{21} = \frac{-H}{WV-H^2}$$

where

$$W = \frac{m}{\hat{\theta}^2} + \sum_{i=1}^{m}(R_i+1)\left(\frac{x_i}{\hat{\alpha}}\right)^{\hat{\theta}}\left(\ln\left(\frac{x_i}{\hat{\alpha}}\right)\right)^2,$$

$$V = \frac{-m}{\hat{\alpha}^2} - \frac{m(\hat{\theta}-1)}{\hat{\alpha}^2} + \frac{\hat{\theta}(\hat{\theta}+1)}{\hat{\alpha}^2}\sum_{i=1}^{m}(R_i+1)\left(\frac{x_i}{\hat{\alpha}}\right)^{\hat{\theta}},$$

and

$$H = \frac{m}{\hat{\theta}} + \frac{\hat{\theta}}{\hat{\alpha}}\sum_{i=1}^{m}(R_i+1)\left(\frac{x_i}{\hat{\alpha}}\right)^{\hat{\theta}}\ln\left(\frac{x_i}{\hat{\alpha}}\right) - \frac{2}{\hat{\alpha}}\sum_{i=1}^{m}(R_i+1)\left(\frac{x_i}{\hat{\alpha}}\right)^{\hat{\theta}}.$$

Respectively. Using the joint prior density function in Eq. (4.1), we obtain

$$\rho(\alpha,\theta) \propto (\alpha-1)\ln(\theta) + (c-1)\ln(\alpha) - b\theta - d\alpha$$

and differentiating $\rho(\alpha,\theta)$ with respect to θ and α we get

$$\rho_1 = \frac{\partial\,\rho(\alpha,\theta)}{\partial\alpha} = \frac{(c-1)}{\alpha} - d \text{ and } \rho_2 = \frac{\partial\,\rho(\alpha,\theta)}{\partial\theta} = \frac{(a-1)}{\theta} - b$$

Under square error loss function, $U(\alpha,\theta)=\alpha,\ u_1 = \frac{\partial U(\alpha,\theta)}{\partial\alpha} = 1,\ u_2 = u_{12} = u_{21} = 0,\ U_1 = \tau_{11},$ and $U_2 = \tau_{12}$ we have

$$\tilde{\alpha} = \hat{\alpha} + \rho_1\tau_{11} + \rho_2\tau_{12} + \frac{1}{2}\,L_{30}\tau_{11}^2 + \frac{3}{2}L_{21}\tau_{11}\tau_{12} + \frac{1}{2}L_{12}(\tau_{11}\tau_{22} + 2\tau_{12}^2) + \frac{1}{2}L_{03}\tau_{12}\tau_{22}$$

Similarly, when $U(\alpha,\theta)=\theta,\ u_2 = \frac{\partial U(\alpha,\theta)}{\partial\theta} = 1,\ u_1 = u_{12} = u_{21} = 0,\ U_1 = \tau_{21},$ and $U_2 = \tau_{22}$ we have

$$\tilde{\theta} = \hat{\theta} + \rho_1\tau_{21} + \rho_2\tau_{22} + \frac{1}{2}\,L_{30}\tau_{11}\tau_{21} + \frac{1}{2}L_{21}(2\tau_{12}^2 + \tau_{11}\tau_{22}) + \frac{3}{2}L_{12}\tau_{22}\tau_{12} + \frac{1}{2}L_{03}\tau_{22}^2$$

Similarly, Lindley's approximations are employed when the joint posterior distribution function is based on the product spacing function.

5. SIMULATION STUDY

In this section; Monte Carlo simulation is done for the comparison between MLEs, maximum MPSs, Bayesian estimators (MLE.Bays) and Bayesian estimators (MPS.Bays) under Progressive Type-II Censoring Scheme. The simulation study is done using the R language.

Simulation Algorithm Scheme

Monte Carlo experiments were carried out based on the data generated form Weibull distribution for the following parameter space $(\theta,\alpha) = (0.5, 1.5), (1, 1.5), and\ (1.5, 1.5)$, and for different sample size $n = 20,\ 25,\ 30,\ and\ 50$, with 1000 replications. Different effective sample sizes (m), and sets of different sample schemes are considered. The parameters of prior distribution are $(a,b) = (1.1, 2)$ and $(c,d) = (1.2, 1.5)$. Balakrishnan and Aggarwala (2000) introduced two optimality criteria that we considered the mean squared error and the bias. Some selected optimal progressive censoring plans are presented according to these optimality criteria. We could define the best scheme as the scheme, which minimizes the mean squared error (MSE) of the estimator. That is, the objective function (to be minimized in this case) would be

$$MSE = Mean(\hat{\delta} - \delta)^2, \tag{5.1}$$

where $\hat{\delta}$ is the estimated value of $\delta = (\theta,\alpha)$ and with bias

$$Bias = \hat{\delta} - \delta \tag{5.2}$$

Tables 1-12 shows the MSEs and biases for the estimators of θ and α under different censoring schemes and for different sample sizes

Table-1: the bias and MSE of the different estimators of θ and α under complete and progressive type-II censoring Scheme when $n = 20$

θ=1 ; α=1.5										
(n,m)	scheme		ML		MPS		ML.Bays		MPS.Bays	
			Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
(20,0)	complete	$\hat{\theta}$	0.0641	0.0445	-0.0729	0.0362	0.0456	0.0391	-0.0336	0.0164
		$\hat{\alpha}$	0.0047	0.1338	0.0430	0.1109	0.0799	0.1315	0.3360	0.1292
(20,5)	(0*4,15) type 2	$\hat{\theta}$	1.1747	3.3910	0.1042	1.2966	0.3818	0.5409	-0.0250	0.0452
		$\hat{\alpha}$	-1.2402	1.5557	-0.9855	1.0400	-0.8371	0.7708	-0.0215	0.0383
	(15,0*4)	$\hat{\theta}$	1.1742	3.3885	0.4533	1.1061	0.7279	1.0301	0.0457	0.0574
		$\hat{\alpha}$	-1.2404	1.5570	-1.2269	1.5247	-1.1389	1.3192	-0.3262	0.1147
	(3*5)	$\hat{\theta}$	1.1742	3.3886	0.2435	0.7065	0.4609	0.6330	0.1358	0.0777
		$\hat{\alpha}$	-1.2404	1.5570	-0.4983	0.7910	-0.8099	0.7333	0.0875	0.0535
(20,10)	(0*9,10) type 2	$\hat{\theta}$	0.6357	0.7325	0.0042	0.1214	0.2275	0.2814	0.3625	0.2036
		$\hat{\alpha}$	-0.9654	0.9657	-0.1014	0.2651	-0.2837	0.2510	0.7756	0.8546
	(10,0*9)	$\hat{\theta}$	0.6357	0.7325	0.2859	0.2846	0.2057	0.6052	0.1645	0.0630
		$\hat{\alpha}$	-0.9654	0.9657	-0.9516	0.9402	-0.8655	0.8208	-0.1900	0.0494
	(1*10)	$\hat{\theta}$	0.6357	0.7325	0.3363	0.3309	0.6727	1.4978	0.7889	0.6900
		$\hat{\alpha}$	-0.9654	0.9657	-0.6832	0.4053	-0.4524	0.3386	0.0842	0.1710
(20,15)	(0*14,5) types 2	$\hat{\theta}$	0.3898	0.2757	-0.3173	0.4809	0.1155	0.1633	0.0930	0.0312
		$\hat{\alpha}$	-0.6062	0.4287	1.4615	3.6140	0.6468	3.0321	1.3010	1.8238
	(5,0*14)	$\hat{\theta}$	0.3898	0.2757	0.1664	0.1142	0.3672	0.3179	0.1680	0.0528
		$\hat{\alpha}$	-0.6062	0.4287	-0.5889	0.4094	-0.5382	0.3551	0.0412	0.0206

Table-2: the bias and MSE of the different estimators of θ and α under complete and progressive type-II censoring Scheme when $n = 20$

θ =1.5 ; α =1.5										
(n, m)	scheme		ML		MPS		ML.Bays		MPS.Bays	
			Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
(20,0)	complete	$\hat{\theta}$	0.1169	0.1016	-0.0923	0.0753	0.0841	0.0526	-0.2326	0.0695
		$\hat{\alpha}$	-0.0112	0.0527	0.0139	0.0491	0.0339	0.0407	0.3206	0.1138
(20,5)	(0*4,15) type 2	$\hat{\theta}$	1.7332	7.2362	0.3007	1.3505	1.1989	1.0218	-0.2716	0.1117
		$\hat{\alpha}$	-1.0437	1.1117	-0.7493	0.6187	-0.4540	0.3634	0.1468	0.0622
	(15,0*4)	$\hat{\theta}$	1.7332	7.2372	0.6566	2.3128	1.0207	1.1393	-0.1819	0.2777
		$\hat{\alpha}$	-1.0437	1.1117	-1.0273	1.0783	-0.6560	0.7397	-0.2342	0.0658
	(3*5)	$\hat{\theta}$	1.7332	7.2367	0.3476	1.4690	0.5092	1.0608	-0.0598	0.0538
		$\hat{\alpha}$	-1.0437	1.1117	-0.4018	0.4006	-0.3823	0.3591	0.2742	0.1197
(20,10)	(0*9,10) type 2	$\hat{\theta}$	0.9536	1.6482	0.1062	0.9731	0.5769	0.7596	-0.1671	0.0557
		$\hat{\alpha}$	-0.7556	0.5997	-0.1880	0.3208	-0.0946	0.2149	0.6284	0.1531
	(10,0*9)	$\hat{\theta}$	0.9536	1.6482	0.4288	0.6403	0.2886	0.4933	-0.0718	0.0382
		$\hat{\alpha}$	-0.7556	0.5997	-0.7426	0.5806	-0.3207	0.2304	-0.0964	0.0216
	(1*10)	$\hat{\theta}$	0.9536	1.6481	0.5010	0.7460	0.3784	0.7001	-0.6621	0.4903
		$\hat{\alpha}$	-0.7556	0.5997	0.5724	0.5292	0.0957	0.4289	0.1235	0.3037
(20,15)	(0*14,5) types 2	$\hat{\theta}$	0.5846	0.6202	-0.3151	0.4754	0.0842	0.3795	-0.1428	0.0406
		$\hat{\alpha}$	-0.3467	0.4971	0.5465	0.4771	0.2833	0.2091	1.1812	0.1926
	(5,0*14)	$\hat{\theta}$	0.5846	0.6202	0.2496	0.2571	0.4802	0.2178	-0.0578	0.0278
		$\hat{\alpha}$	-0.4467	0.2371	-0.4330	0.2254	-0.2385	0.2077	0.0985	0.0236

Table-3: the bias and MSE of the different estimators of θ and α under complete and progressive type-II censoring Scheme when $n = 20$

θ =1.5 ; α =1										
(n, m)	scheme		ML		MPS		ML.Bays		MPS.Bays	
			Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
(20,0)	complete	$\hat{\theta}$	0.1169	0.1016	-0.0923	0.0753	0.0868	0.1139	-0.2252	0.0648
		$\hat{\alpha}$	-0.0074	0.0234	0.0093	0.0230	0.0312	0.0259	0.5610	0.3224
(20,5)	(0*4,15) type 2	$\hat{\theta}$	1.7332	7.2354	0.2852	1.4116	1.0720	1.3044	-0.3631	0.5637
		$\hat{\alpha}$	-0.6958	0.4941	-0.5030	0.2793	-0.2878	0.2539	0.4332	0.2082
	(15,0*4)	$\hat{\theta}$	1.7326	7.2305	0.6563	2.3069	0.4879	1.0931	-0.2721	0.1118
		$\hat{\alpha}$	-0.6958	0.4941	-0.6849	0.4792	-0.4337	0.3219	0.1546	0.0287
	(3*5)	$\hat{\theta}$	1.7332	7.2366	0.3477	0.4705	0.3507	0.3108	-0.1823	0.0740
		$\hat{\alpha}$	-0.6958	0.4941	-0.2679	0.1781	-0.2341	0.1475	0.5465	0.0923
(20,10)	(0*9,10) type 2	$\hat{\theta}$	0.9536	1.6482	0.3055	0.6742	0.5506	0.5645	-0.2549	0.1885
		$\hat{\alpha}$	-0.5037	0.2665	-0.0475	0.1878	-0.0604	0.0925	0.0049	0.0808
	(10,0*9)	$\hat{\theta}$	0.9536	1.6482	0.4287	0.6402	0.2162	0.2664	-0.1402	0.0484
		$\hat{\alpha}$	-0.5037	0.2665	-0.4951	0.2581	-0.2159	0.1446	0.2257	0.0569
	(1*10)	$\hat{\theta}$	0.9536	1.6482	0.5006	0.7459	0.1249	0.3028	0.4790	0.2717
		$\hat{\alpha}$	-0.5037	0.2665	0.1513	0.2391	0.0678	0.2077	0.2340	0.1200
(20,15)	(0*14,5) types 2	$\hat{\theta}$	0.5846	0.6202	-0.2381	0.2004	0.2736	0.1972	-0.2148	0.0629
		$\hat{\alpha}$	-0.2978	0.2054	0.2766	0.1974	0.1869	0.0837	0.1722	0.0896
	(5,0*14)	$\hat{\theta}$	0.5846	0.6202	0.2496	0.2570	0.9865	0.2239	-0.1018	0.0918
		$\hat{\alpha}$	-0.2978	0.1054	-0.2887	0.1002	-0.1535	0.0920	0.1639	0.0898

Table-4: the bias and MSE of the different estimators of θ and α under complete and progressive type-II censoring Scheme when $n = 25$

θ =1 ; α =1.5

(n, m)	scheme		ML		MPS		ML.Bays		MPS.Bays	
			Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
(25,0)	complete	$\hat{\theta}$	0.0536	0.0324	-0.1769	0.0317	0.0389	0.0291	-0.0299	0.0130
		$\hat{\alpha}$	0.0036	0.1035	-0.0527	0.1014	0.0643	0.1007	-0.1377	0.0968
(25,5)	(0*4,20) type 2	$\hat{\theta}$	1.2178	4.2036	0.1770	1.1093	0.3592	0.1201	-0.0695	0.0946
		$\hat{\alpha}$	-1.2934	1.6839	-1.1094	1.2678	-0.9530	0.9584	-0.7117	0.8782
	(20,0*4)	$\hat{\theta}$	1.2178	4.2023	0.4799	1.4598	0.7209	0.9978	0.0025	0.2191
		$\hat{\alpha}$	-1.2934	1.6839	-1.2827	1.6570	-1.2043	1.4651	-0.6619	0.9364
	(4*5)	$\hat{\theta}$	1.2178	4.2033	0.2402	0.8894	0.4065	0.5536	0.1422	0.2103
		$\hat{\alpha}$	-1.2934	1.6839	-0.8214	0.8608	-0.9227	0.7079	-0.3277	0.4115
(25,10)	(0*9,15) type 2	$\hat{\theta}$	0.7042	0.8775	0.0668	0.1555	0.2419	0.2288	0.1462	0.1309
		$\hat{\alpha}$	-1.0933	1.2160	-0.5713	0.4263	-0.6129	0.4063	0.2695	0.2322
	(15,0*9)	$\hat{\theta}$	0.7035	0.8823	0.3368	0.3518	0.5982	0.6303	0.1431	0.3848
		$\hat{\alpha}$	-1.0874	1.2047	-1.0775	1.1837	-1.0389	1.1031	-0.2818	0.5188
(25,15)	(0*14,10) type 2	$\hat{\theta}$	0.5004	0.4140	-0.1537	0.2849	0.1084	0.0922	0.1020	0.0326
		$\hat{\alpha}$	-0.8322	0.7283	0.3453	0.4005	-0.0330	0.1419	0.8607	0.1408
	(10,0*14)	$\hat{\theta}$	0.5004	0.4140	0.2602	0.1826	0.1543	0.1428	0.1875	0.1007
		$\hat{\alpha}$	-0.8322	0.7283	-0.8211	0.7104	-0.7860	0.6539	-0.1154	0.1060
(25,20)	(0*19,5) type 2	$\hat{\theta}$	0.3196	0.1832	-0.1811	0.1229	-0.0141	0.0920	0.0704	0.0256
		$\hat{\alpha}$	-0.5044	0.3108	2.4931	0.2977	0.5880	0.2647	0.5543	0.1340
	(5,0*19)	$\hat{\theta}$	0.3196	0.1832	0.1490	0.1732	0.2941	0.1594	0.1594	0.1450
		$\hat{\alpha}$	-0.5044	0.3108	-0.4896	0.2971	-0.4543	0.2623	0.1978	0.2253

Table-5: the bias and MSE of the different estimators of θ and α under complete and progressive type-II censoring Scheme when $n = 25$

θ =1.5 ; α =1.5

(n, m)	scheme		ML		MPS		ML.Bays		MPS.Bays	
			Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
(25,0)	complete	$\hat{\theta}$	0.0926	0.0847	-0.0833	0.0666	0.0582	0.0576	-0.0300	0.0519
		$\hat{\alpha}$	0.0105	0.0436	0.0318	0.0405	0.0404	0.0390	0.0308	0.0384
(25,5)	(0*4,20) type 2	$\hat{\theta}$	1.1574	5.6705	0.5335	2.9823	0.4824	0.7344	-0.2892	0.1226
		$\hat{\alpha}$	-1.1045	1.2372	-0.8913	0.8285	-0.7860	0.6608	0.0986	0.1302
	(20,0*4)	$\hat{\theta}$	1.1577	6.6986	0.9304	2.2792	0.8756	1.3086	-0.3960	0.4835
		$\hat{\alpha}$	-1.1045	1.2372	-1.0913	1.2086	-1.0412	1.1017	-0.2822	0.4276
	(4*5)	$\hat{\theta}$	1.1581	5.9363	0.5406	2.3315	0.5419	1.1029	-0.1357	0.2665
		$\hat{\alpha}$	-1.1045	1.2372	-0.6694	1.1535	-0.7624	0.6276	0.1409	0.1522
(25,10)	(0*9,15) type 2	$\hat{\theta}$	1.0866	2.0104	0.1113	0.3244	0.3239	0.3008	-0.1878	0.0916
		$\hat{\alpha}$	-0.8699	0.7790	-0.4092	0.2283	-0.4467	0.2026	0.4195	0.0903
	(15,0*9)	$\hat{\theta}$	0.8866	2.0110	0.5366	0.8108	0.5172	0.4840	-0.1759	0.0876
		$\hat{\alpha}$	-0.8699	0.7790	-0.8596	0.7614	-0.8393	0.4262	-0.1830	0.0435
(25,15)	(0*14,10) type 2	$\hat{\theta}$	0.7506	0.9283	-0.2759	0.2618	0.1356	0.1825	-0.1435	0.0413
		$\hat{\alpha}$	-0.6322	0.4266	0.2529	0.3329	-0.1320	0.1656	0.0911	0.0938
	(10,0*14)	$\hat{\theta}$	0.7506	0.9315	0.3902	0.4108	0.6322	0.5621	-0.0450	0.0258
		$\hat{\alpha}$	-0.6332	0.4277	-0.6234	0.4155	-0.6078	0.3956	-0.0357	0.0119
(25,20)	(0*19,5) type 2	$\hat{\theta}$	0.4794	0.4123	-0.2717	0.1416	-0.0368	0.0913	-0.1511	0.0403
		$\hat{\alpha}$	-0.3658	0.1664	1.3543	0.1488	0.3640	0.1149	0.1609	0.0450
	(5,0*19)	$\hat{\theta}$	0.4794	0.4123	0.2233	0.1871	0.1149	0.1279	-0.0601	0.0234
		$\hat{\alpha}$	-0.3658	0.1664	-0.3545	0.1585	-0.3408	0.1481	0.1399	0.0311

Table-6: the bias and MSE of the different estimators of θ and α under complete and progressive type-II censoring Scheme when $n = 25$

θ =1.5 ; α =1										
(n,m)	scheme		ML		MPS		ML.Bays		MPS.Bays	
			Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
(25,0)	complete	$\hat{\theta}$	0.0926	0.0847	-0.0833	0.0666	0.0610	0.0630	-0.2206	0.0615
		$\hat{\alpha}$	0.0070	0.0194	0.0212	0.0192	0.0344	0.0184	0.1671	0.0179
(25,5)	(0*4,20) type 2	$\hat{\theta}$	1.1558	5.5264	0.5254	2.8188	0.4287	0.6664	-0.3816	0.1783
		$\hat{\alpha}$	-0.7364	0.5499	-0.5942	0.3682	-0.5011	0.2722	0.3575	0.1422
	(20,0*4)	$\hat{\theta}$	2.1561	8.5411	0.9383	5.7361	0.8484	1.3118	-0.2903	0.1223
		$\hat{\alpha}$	-0.7364	0.5499	-0.7275	0.5371	-0.6827	0.4752	0.1220	0.0183
	(4*5)	$\hat{\theta}$	1.1564	7.5856	0.5404	3.3513	0.5182	0.9174	-0.2526	0.1030
		$\hat{\alpha}$	-0.7364	0.5499	-0.4463	0.2459	-0.4803	0.2246	0.4461	0.2158
(25,10)	(0*9,15) type 2	$\hat{\theta}$	1.0866	2.0104	0.1107	0.3251	0.3068	0.2742	-0.2772	0.0993
		$\hat{\alpha}$	-0.5799	0.3462	-0.2497	0.2956	-0.2919	0.1089	0.6033	0.0860
	(15,0*9)	$\hat{\theta}$	1.0866	2.0104	0.5367	0.8109	0.3883	0.5642	-0.1529	0.0512
		$\hat{\alpha}$	-0.5799	0.3462	-0.5731	0.3384	-0.5518	0.3186	0.1629	0.0313
(25,15)	(0*14,10) type 2	$\hat{\theta}$	0.7506	0.9314	-0.0895	0.1818	0.1350	0.1763	-0.2294	0.0703
		$\hat{\alpha}$	-0.4221	0.1901	0.4875	0.1141	-0.0184	0.0297	0.9144	0.8642
	(10,0*14)	$\hat{\theta}$	0.7506	0.9314	0.3903	0.4108	0.6308	0.3579	-0.1026	0.0317
		$\hat{\alpha}$	-0.4221	0.1901	-0.4156	0.1847	-0.4025	0.1737	0.2612	0.0734
(25,20)	(0*19,5) type 2	$\hat{\theta}$	0.4794	0.4123	-0.2749	0.1451	-0.0372	0.0913	-0.2172	0.0614
		$\hat{\alpha}$	-0.2439	0.0739	0.9441	0.0691	0.0990	0.0690	0.3082	0.0509
	(5,0*19)	$\hat{\theta}$	0.4794	0.4123	0.2234	0.1872	0.4173	0.1301	-0.0957	0.0267
		$\hat{\alpha}$	-0.2439	0.0739	-0.2363	0.0705	-0.2235	0.0643	0.0926	0.0603

We note the different results according to each method due to the characteristics of each method. The maximum likelihood (ML) function is the most common and famous. In the ML, all of the censoring scheme observations are taken. While the MPS method takes into consideration the observations of the sample from the second observation until the last observation with the cumulative function of the first observation and the survival function on the last observation as the geometric mean, after taking into consideration the censoring schemes. While the Bayesian method is better after taking a suitable prior distribution Gamma distribution for each of the previous methods.

By comparing Tables 1, 2 and 3, we note, that the bias and MSEs of MPSs are smaller than that of MLEs. We note in sometimes, that the bias and MSE's are lower in the progressive censoring scheme than the Type-II censoring scheme. The Bias and MSE's of Bayesian estimation (ML.Bays and MPS.Bays) are smaller than that of Non-Bayesian (MLEs, MPSs). We conclude that the Bayesian estimation based on Gamma prior is better for estimation of the Weibull parameter under censored schemes. Whenever the initial values of θ increases, the bias and MSE of all estimation methods will increase, especially the MLE method. Sometimes MLE is better than MPS in case of type-II censoring in Bayesian and Non-Bayesian estimation.

Table-7: the bias and MSE of the different estimators of θ and α under complete and progressive type-II censoring Scheme when $n = 30$

θ =1.5 ; α =1										
(n,m)	scheme		ML		MPS		ML.Bays		MPS.Bays	
			Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
(30,0)	completa	$\hat{\theta}$	0.0727	0.0599	-0.0807	0.0514	0.0479	0.0501	-0.2182	0.0497
		$\hat{\alpha}$	-0.0087	0.0156	0.0030	0.0154	0.0140	0.0153	0.5555	0.0151
(30,5)	(0*4,25) type 2	$\hat{\theta}$	1.8982	4.9872	0.3989	1.6001	0.9980	0.9678	-0.4239	0.2112
		$\hat{\alpha}$	-0.7738	0.6045	-0.6584	0.4452	-0.4638	0.2635	0.3090	0.1075
	(25,0*4)	$\hat{\theta}$	1.8982	7.9882	0.7692	2.6124	0.5567	0.1936	-0.3346	0.1495
		$\hat{\alpha}$	-0.7738	0.6045	-0.7660	0.5927	-0.5628	0.4029	0.0983	0.0126
	(5*5)	$\hat{\theta}$	1.8981	7.9863	0.3874	1.4720	0.2667	0.7864	-0.3291	0.1446
		$\hat{\alpha}$	-0.7738	0.6045	-0.5427	0.3258	-0.4297	0.2399	0.3789	0.1576
(30,10)	(0*9,20) type 2	$\hat{\theta}$	1.0920	2.1278	0.1274	0.3541	0.3157	0.3316	-0.3145	0.1213
		$\hat{\alpha}$	-0.6390	0.4156	-0.3956	0.1737	-0.3249	0.1617	0.5059	0.2722
	(20,9)	$\hat{\theta}$	1.0920	2.1278	0.5378	0.8671	0.1467	0.5501	-0.1890	0.0639
		$\hat{\alpha}$	-0.6390	0.4156	-0.6332	0.4083	-0.4241	0.3399	0.1195	0.0178
	(4*5,0*5)	$\hat{\theta}$	1.0927	2.1298	0.8560	1.5155	0.7295	0.4460	-0.0669	0.0359
		$\hat{\alpha}$	-0.6389	0.4154	-0.5575	0.3229	-0.3136	0.2993	0.3017	0.1014

(30,15)	(0*14, 15) type 2	$\hat{\theta}$	0.8233	1.0909	0.0030	0.1620	0.3092	1.3640	-0.2569	0.0829
		$\hat{\alpha}$	-0.5106	0.2693	-0.1057	0.0726	-0.1474	0.0753	0.7376	0.0648
	(15,0*14)	$\hat{\theta}$	0.8233	1.0909	0.4476	0.4899	0.1996	0.0615	-0.1226	0.0360
		$\hat{\alpha}$	-0.5106	0.2693	-0.5055	0.2642	-0.3182	0.2112	0.1908	0.0405
	(1*15)	$\hat{\theta}$	0.8343	1.0901	0.5051	0.5670	0.1015	0.2041	0.1346	0.1349
		$\hat{\alpha}$	-0.5106	0.2693	0.3014	0.2639	-0.0459	0.2332	-0.2441	0.1751
(30,20)	(0*19,10) type 2	$\hat{\theta}$	0.6245	0.6097	-0.1314	0.1039	0.0646	0.0953	-0.2266	0.0646
		$\hat{\alpha}$	-0.3767	0.1521	0.1077	0.0696	0.0626	0.0310	0.0169	0.0296
	(10,0*19)	$\hat{\theta}$	0.6245	0.6097	0.3463	0.2861	0.2364	0.1956	-0.0947	0.0254
		$\hat{\alpha}$	-0.3767	0.1521	-0.3717	0.1484	-0.2726	0.2395	0.2829	0.0846
(30,25)	(0*24,5)	$\hat{\theta}$	0.4080	0.2843	-0.3725	0.3629	-0.0808	0.0678	-0.2253	0.0618
		$\hat{\alpha}$	-0.2235	0.0620	0.1566	0.0528	0.1673	0.0511	0.3730	0.0382
	5,0*24)	$\hat{\theta}$	0.4080	0.2843	0.1980	0.1326	0.4218	0.1223	-0.0987	0.0233
		$\hat{\alpha}$	-0.2235	0.0620	-0.2175	0.0594	-0.1879	0.0566	0.4000	0.0550

Table-8: the bias and MSE of the different estimators of θ and α under complete and progressive type-II censoring Scheme when $n = 30$

θ =1 ; α =1.5										
(n, m)	scheme		ML		MPS		ML.Bays		MPS.Bays	
			Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
(30,0)	completa	$\hat{\theta}$	0.0477	0.0276	-0.0547	0.0235	0.0385	0.0216	-0.0267	0.0112
		$\hat{\alpha}$	-0.0066	0.0809	0.0204	0.0804	0.0508	0.0785	0.3362	0.0732
(30,5)	(0*4,25) type 2	$\hat{\theta}$	1.2610	3.8625	0.2753	0.7523	0.4984	0.6996	-0.2962	0.4437
		$\hat{\alpha}$	-1.3292	1.7745	-1.1801	1.4165	-1.0177	1.0840	-0.3776	0.5514
	(25,0*4)	$\hat{\theta}$	1.2462	3.7908	0.5028	1.2720	0.3655	0.7890	-0.6264	0.5439
		$\hat{\alpha}$	-1.3295	1.7753	-1.3208	1.7528	-1.2018	1.4750	-0.3880	0.1550
	(5*5)	$\hat{\theta}$	1.2524	3.3736	0.2499	0.5955	0.6222	0.5881	-0.6096	0.4430
		$\hat{\alpha}$	-1.3279	1.7701	-0.9998	1.0857	-0.9773	1.0050	-0.3118	0.2332
(30,10)	(0*9,20) type 2	$\hat{\theta}$	0.7316	0.9559	0.0905	0.1689	0.2892	0.1474	0.0982	0.1245
		$\hat{\alpha}$	-1.1644	1.3708	-0.7777	0.6632	-0.7605	0.6405	0.1992	0.1907
	(20,9)	$\hat{\theta}$	0.7316	0.9559	0.3606	0.3875	0.8330	0.3676	0.1983	0.1766
		$\hat{\alpha}$	-1.1644	1.3708	-1.1564	1.3327	-1.0898	1.2642	-0.4076	0.1722
	(5*4,0*5)	$\hat{\theta}$	0.5633	0.5314	0.2084	0.1855	0.1577	0.0799	0.0737	0.0749
		$\hat{\alpha}$	-0.9758	0.9756	-0.2266	0.1724	-0.3703	0.1237	0.4112	0.1160
(30,15)	(0*14, 15) type 2	$\hat{\theta}$	0.5941	0.6006	0.1242	0.1250	0.1767	0.1139	0.0808	0.0970
		$\hat{\alpha}$	-0.9727	0.9676	-0.1600	0.7899	-0.3629	0.2208	0.4163	0.1904
	(15,0*14)	$\hat{\theta}$	0.5556	0.4686	0.3047	0.2063	0.5341	0.1950	0.1643	0.1459
		$\hat{\alpha}$	-0.9691	0.9635	-0.9610	0.9482	-0.9246	0.8846	-0.2903	0.1930
	(1*15)	$\hat{\theta}$	0.5561	0.4681	0.3466	0.2397	0.5178	0.1975	0.3512	0.1450
		$\hat{\alpha}$	-0.9686	0.9624	0.7373	0.9151	-0.5497	0.3673	0.4364	0.1542
(30,20)	(0*19,10) type 2	$\hat{\theta}$	0.4163	0.2710	-0.1859	0.0951	0.0546	0.0780	0.2179	0.0695
		$\hat{\alpha}$	-0.7547	0.6020	0.6805	0.5925	0.1600	0.2128	0.1579	0.1646
	(10,0*19)	$\hat{\theta}$	0.4163	0.2710	0.2310	0.1272	0.3919	0.1116	0.2286	0.0745
		$\hat{\alpha}$	-0.7547	0.6020	-0.7457	0.5889	-0.7149	0.5447	-0.2194	0.1138
(30,25)	(0*24,5)	$\hat{\theta}$	0.2720	0.1964	-0.4265	0.1663	0.0854	0.1048	0.3422	0.0631
		$\hat{\alpha}$	-0.4660	0.2649	0.1837	0.2145	0.1352	0.2116	0.3295	0.1109
	5,0*24)	$\hat{\theta}$	0.2720	0.1264	0.1320	0.0590	0.1533	0.0427	0.1958	0.0410
		$\hat{\alpha}$	-0.4660	0.2649	-0.4539	0.2545	-0.4243	0.2275	0.1824	0.1493

Table-9: the bias and MSE of the different estimators of θ and α under complete and progressive type-II censoring Scheme when $n = 30$

θ =1.5 ; α =1.5

(n, m)	scheme		ML		MPS		ML.Bays		MPS.Bays	
			Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
(30,0)	complete	$\hat{\theta}$	0.0715	0.0622	-0.0821	0.0530	0.0450	0.0515	-0.2277	0.0501
		$\hat{\alpha}$	-0.0104	0.0359	0.0076	0.0354	0.0156	0.0351	0.3211	0.0326
(30,5)	(0*4,25) type 2	$\hat{\theta}$	1.8692	8.5280	0.4031	1.6673	0.2613	1.5680	-0.3329	0.1458
		$\hat{\alpha}$	-1.1578	1.3543	-0.9771	0.9827	-0.7030	0.6010	-0.2302	0.1262
	(25,0*4)	$\hat{\theta}$	1.8692	8.5271	0.7539	2.8557	-0.7345	0.2685	-0.2384	0.1979
		$\hat{\alpha}$	-1.1578	1.3543	-1.1460	1.3274	-0.8429	0.9144	-0.3211	0.2096
	(5*5)	$\hat{\theta}$	1.8692	8.5279	0.3714	1.6473	0.4875	0.5732	-0.2188	0.1895
		$\hat{\alpha}$	-1.1578	1.3543	-0.8098	0.7248	-0.6609	0.5598	0.0511	0.1311
(30,10)	(0*9,20) type 2	$\hat{\theta}$	1.0974	2.1507	0.1358	0.3800	0.3552	0.3674	-0.2235	0.1762
		$\hat{\alpha}$	-0.9552	0.9300	-0.5897	0.6891	-0.4480	0.4382	0.2828	0.1135
	(20,9)	$\hat{\theta}$	1.0974	2.1507	0.5408	0.8716	0.1679	0.7119	-0.1060	0.1418
		$\hat{\alpha}$	-0.9552	0.9300	-0.9465	0.9136	-0.2153	0.6707	-0.2464	0.1686
	(4*5,0*5)	$\hat{\theta}$	1.0981	2.1529	0.8584	1.5168	0.3829	1.2715	0.2559	0.1189
		$\hat{\alpha}$	-0.9550	0.9297	-0.8322	0.7215	-0.2963	0.6097	-0.1017	0.1232
(30,15)	(0*14, 15) type 2	$\hat{\theta}$	0.8233	1.0909	0.8038	1.0609	0.4281	0.4163	-0.1702	0.1488
		$\hat{\alpha}$	-0.7659	0.6058	-0.3669	0.5878	-0.2184	0.1710	0.6181	0.1118
	(15,0*14)	$\hat{\theta}$	0.8233	1.0909	0.4476	0.6899	0.4360	0.5338	-0.1561	0.1768
		$\hat{\alpha}$	-0.7659	0.6058	-0.7583	0.5944	-0.4410	0.4288	-0.1329	0.1259
	(1*15)	$\hat{\theta}$	0.8233	1.0909	0.5057	0.5672	0.3545	0.5590	0.8425	0.1491
		$\hat{\alpha}$	-0.7659	0.6058	0.6483	0.5947	-0.1664	0.5409	0.7590	0.1868
(30,20)	(0*19,10) type 2	$\hat{\theta}$	0.6245	0.6097	-0.1289	0.4014	0.0707	0.1604	-0.1431	0.1361
		$\hat{\alpha}$	-0.5651	0.3423	0.4118	0.2705	0.0914	0.1670	0.1054	0.1078
	(10,0*19)	$\hat{\theta}$	0.6245	0.6097	0.3464	0.2862	0.3383	0.1527	-0.1442	0.1203
		$\hat{\alpha}$	-0.5651	0.3423	-0.5576	0.3339	-0.4119	0.2958	-0.0024	0.1089
(30,25)	(0*24,5)	$\hat{\theta}$	0.4080	0.2843	-0.3238	0.1909	-0.0783	0.1248	-0.1625	0.0901
		$\hat{\alpha}$	-0.3353	0.1395	0.4933	0.1214	0.4015	0.0911	0.4455	0.0804
	5,0*24)	$\hat{\theta}$	0.4080	0.2843	0.1980	0.1327	0.4191	0.0975	-0.0671	0.0896
		$\hat{\alpha}$	-0.3353	0.1395	-0.3262	0.1337	-0.2868	0.1296	0.1509	0.0320

Table-10: the bias and MSE of the different estimators of θ and α under complete and progressive type-II censoring Scheme when $n = 50$

θ =1.5 ; α =1

(n, m)	scheme		ML		MPS		ML.Bays		MPS.Bays	
			Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
(50,0)	complete	$\hat{\theta}$	0.0252	0.0144	-0.0445	0.0139	0.0183	0.0137	-0.0262	0.0093
		$\hat{\alpha}$	-0.0090	0.0508	0.0082	0.0506	0.0226	0.0501	0.3372	0.0403
(50,5)	(0*4,45) type 2	$\hat{\theta}$	0.1190	13.1319	0.0738	2.1395	0.3180	0.8018	-0.1929	0.1623
		$\hat{\alpha}$	0.7202	1.9836	0.2742	0.8218	-1.1932	0.6442	-0.2829	0.1881
	(45,0*4)	$\hat{\theta}$	1.8737	4.5281	1.6624	3.7512	0.7170	0.2877	-0.1304	0.1492
		$\hat{\alpha}$	0.6059	0.7888	0.3567	0.4289	-1.3147	1.0370	-0.4202	0.6799
(50,10)	(0*9,40) type 2	$\hat{\theta}$	-0.0090	0.9591	0.1930	0.9184	0.4056	0.7300	-0.3768	0.1616
		$\hat{\alpha}$	0.2160	0.2377	0.1021	0.1577	-0.5636	0.1359	0.3164	0.1092
	(40,0*9)	$\hat{\theta}$	0.8210	1.0859	0.6396	0.7968	0.6437	0.6660	-0.2498	0.1868
		$\hat{\alpha}$	0.2417	0.1550	0.0876	0.0914	-0.6964	0.0754	0.1346	0.0533
(50,15)	(0*14,35)	$\hat{\theta}$	-0.0383	0.3190	0.1111	0.3085	0.2702	0.2933	-0.3118	0.1123
		$\hat{\alpha}$	0.1362	0.1189	0.0523	0.1105	-0.4495	0.1070	0.2097	0.1041
	(35,0*14)	$\hat{\theta}$	0.4431	0.4129	0.2934	0.2898	0.6776	0.2679	-0.1763	0.1505
		$\hat{\alpha}$	0.1416	0.0718	0.0146	0.0474	-0.6430	0.0466	0.0634	0.0464
(50,20)	(0*19,30) types 2	$\hat{\theta}$	-0.0284	0.1852	0.0718	0.1797	0.2074	0.1289	-0.2839	0.0929
		$\hat{\alpha}$	0.1078	0.0806	0.0542	0.0654	-0.3291	0.0596	0.1035	0.0522
	(30,0*19)	$\hat{\theta}$	0.2616	0.2133	0.1321	0.1513	0.6590	0.1281	-0.1299	0.0921
		$\hat{\alpha}$	0.0989	0.0500	-0.0141	0.0367	-0.5750	0.0349	0.1102	0.0347

(50,25)	(0*24,25)	$\hat{\theta}$	-0.0224	0.0981	0.0464	0.0947	0.1366	0.0928	-0.2548	0.0755
		$\hat{\alpha}$	0.0811	0.0557	0.0249	0.0488	-0.1911	0.0419	0.1095	0.0406
	(25,0*24)	$\hat{\theta}$	0.1653	0.1315	0.0508	0.0971	0.6365	0.0886	-0.0998	0.0826
		$\hat{\alpha}$	0.0686	0.0309	-0.0330	0.0247	-0.4989	0.0217	0.0634	0.0204
(50,30)	(0*29,20)	$\hat{\theta}$	-0.0109	0.0832	0.0512	0.0747	0.0603	0.0701	-0.2376	0.0659
		$\hat{\alpha}$	0.0634	0.0366	0.0274	0.0311	-0.0424	0.0173	0.0973	0.0110
	(20,0*29)	$\hat{\theta}$	0.1043	0.0997	0.0011	0.0825	0.5888	0.0524	-0.0828	0.0478
		$\hat{\alpha}$	0.0515	0.0248	-0.0427	0.0218	-0.4152	0.0201	0.0217	0.0191
(50,35)	(0*34,15)	$\hat{\theta}$	-0.0049	0.0689	0.0374	0.1014	-0.0179	0.0845	-0.2340	0.0660
		$\hat{\alpha}$	0.0521	0.0300	-0.0092	0.0867	0.1218	0.0318	0.0917	0.0108
	(15,0*34)	$\hat{\theta}$	0.0661	0.0777	-0.0286	0.0687	0.4976	0.0684	-0.0774	0.0456
		$\hat{\alpha}$	0.0449	0.0227	-0.0443	0.0206	-0.3378	0.0192	0.0856	0.0186
(50,40)	(0*39,10)	$\hat{\theta}$	-0.0020	0.0583	0.0542	0.0655	-0.0957	0.0464	-0.2254	0.0447
		$\hat{\alpha}$	0.0473	0.0236	0.0152	0.0203	0.0831	0.0201	0.0174	0.0105
	(10,0*39)	$\hat{\theta}$	0.0376	0.0663	-0.0480	0.0623	0.3965	0.0639	-0.0806	0.0445
		$\hat{\alpha}$	0.0389	0.0186	-0.0428	0.0171	-0.2483	0.0166	0.3539	0.0156

Table-11: the bias and MSE of the different estimators of θ and α under complete and progressive type-II censoring Scheme when $n = 50$

θ =1 ; α =1.5										
(n, m)	scheme		ML		MPS		ML.Bays		MPS.Bays	
			Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
(50,0)	complete	$\hat{\theta}$	0.0409	0.0309	-0.0636	0.0296	0.0262	0.0287	-0.2097	0.0204
		$\hat{\alpha}$	-0.0044	0.0096	0.0032	0.0094	0.0089	0.0096	0.5529	0.3090
(50,5)	(0*4,45) type 2	$\hat{\theta}$	-0.1126	0.8528	0.1514	0.8200	0.2681	0.4644	-0.4990	0.2737
		$\hat{\alpha}$	1.0804	4.4631	0.3562	0.8320	-0.6688	0.4577	0.2171	0.3538
	(45,0*4)	$\hat{\theta}$	-0.0739	0.0936	0.2350	0.0879	0.7855	0.0812	-0.4153	0.0808
		$\hat{\alpha}$	0.2046	0.2684	-0.0454	0.1676	-0.7999	0.1441	0.0567	0.0121
(50,10)	(0*9,40) type 2	$\hat{\theta}$	-0.0415	0.1559	0.1057	0.1320	0.2535	0.1308	-0.3824	0.1295
		$\hat{\alpha}$	0.3240	0.5349	0.1365	0.3351	-0.5624	0.3165	0.3151	0.3083
	(40,0*9)	$\hat{\theta}$	-0.0269	0.0470	0.1504	0.0468	0.1051	0.0401	-0.1068	0.0400
		$\hat{\alpha}$	0.0975	0.1242	-0.0909	0.0981	-0.1117	0.0974	0.0344	0.0932
(50,15)	(0*14,35)	$\hat{\theta}$	-0.0321	0.0604	0.0253	0.0605	0.0336	0.0601	-0.1195	0.0598
		$\hat{\alpha}$	0.2043	0.2675	0.0665	0.2594	-0.4580	0.2198	0.4416	0.2047
	(35,0*14)	$\hat{\theta}$	-0.0108	0.0300	0.1122	0.0203	0.1419	0.0201	-0.1758	0.0187
		$\hat{\alpha}$	0.0869	0.0936	-0.0775	0.0758	-0.1357	0.0734	0.0643	0.0664
(50,20)	(0*19,30) types 2	$\hat{\theta}$	-0.0215	0.0357	0.0220	0.0416	0.1293	0.0409	-0.0780	0.0403
		$\hat{\alpha}$	0.1617	0.1814	0.0810	0.1466	-0.1342	0.1238	0.1756	0.1120
	(30,0*19)	$\hat{\theta}$	-0.0042	0.0235	0.0902	0.0206	0.0423	0.0201	-0.1261	0.0195
		$\hat{\alpha}$	0.0855	0.0817	-0.0635	0.0660	-0.0667	0.0620	0.1089	0.0519
(50,25)	(0*24,25)	$\hat{\theta}$	-0.0148	0.0193	0.0063	0.0189	0.0521	0.0175	-0.0513	0.0136
		$\hat{\alpha}$	0.1216	0.1253	0.0253	0.1219	-0.1965	0.0518	0.0241	0.0494
	(25,0*24)	$\hat{\theta}$	-0.0012	0.0175	0.0746	0.0165	0.0625	0.0103	-0.0951	0.0101
		$\hat{\alpha}$	0.0764	0.0678	-0.0589	0.0562	-0.1002	0.0549	0.0607	0.0503
(50,30)	(0*29,20)	$\hat{\theta}$	-0.0056	0.0136	-0.0028	0.0279	0.0721	0.0239	-0.2352	0.0210
		$\hat{\alpha}$	0.0781	0.0675	-0.0512	0.0617	-0.0483	0.0596	0.0909	0.0579
	(20,0*29)	$\hat{\theta}$	0.0011	0.0163	0.0657	0.0153	0.0986	0.0144	-0.0788	0.0116
		$\hat{\alpha}$	0.0650	0.0502	-0.0610	0.0431	-0.0589	0.0413	0.0988	0.0396
(50,35)	(0*34,15)	$\hat{\theta}$	-0.0037	0.0114	0.0210	0.0104	-0.0117	0.0101	-0.0279	0.0096
		$\hat{\alpha}$	0.0709	0.0530	0.0228	0.0456	0.1144	0.0428	0.0827	0.0412
	(15,0*34)	$\hat{\theta}$	0.0002	0.0144	0.0565	0.0139	0.0070	0.0122	-0.0753	0.0115
		$\hat{\alpha}$	0.0578	0.0461	-0.0613	0.0411	-0.3358	0.0408	0.0865	0.0399
(50,40)	(0*39,10)	$\hat{\theta}$	-0.0003	0.0105	0.0243	0.0103	-0.0975	0.0101	-0.0251	0.0098
		$\hat{\alpha}$	0.0565	0.0411	-0.0062	0.0412	0.0927	0.0408	0.0270	0.0397
	(10,0*39)	$\hat{\theta}$	0.0007	0.0126	0.0493	0.0120	0.0946	0.0108	-0.0814	0.0103
		$\hat{\alpha}$	0.0590	0.0409	-0.0518	0.0359	-0.0430	0.0326	0.0578	0.0311

Table-12: the bias and MSE of the different estimators of θ and α under complete and progressive type-II censoring Scheme when $n = 50$

θ =1.5 ; α =1.5										
(n,m)	scheme		ML		MPS		ML.Bays		MPS.Bays	
			Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
(50,0)	complete	$\hat{\theta}$	0.0409	0.0509	-0.0635	0.0296	0.0246	0.0286	-0.0184	0.0246
		$\hat{\alpha}$	-0.0066	0.0216	0.0048	0.0210	0.0082	0.0207	0.0214	0.0205
(50,5)	(0*4,45) type 2	$\hat{\theta}$	-0.1688	1.9192	0.1125	1.2985	0.3915	0.5990	-0.4106	0.4986
		$\hat{\alpha}$	1.0804	4.4630	0.3660	1.1670	-1.0396	1.0998	-0.1778	0.8439
	(45,0*4)	$\hat{\theta}$	-0.1108	0.2106	0.3519	0.2103	0.8073	0.2045	-0.3187	0.1970
		$\hat{\alpha}$	0.2046	0.2684	-0.0454	0.1677	-1.2085	0.1581	-0.3940	0.1563
(50,10)	(0*9,40) type 2	$\hat{\theta}$	-0.0622	0.3508	0.1320	0.3436	0.3738	0.3384	-0.2938	0.3281
		$\hat{\alpha}$	0.3240	0.5349	0.1411	0.3423	-0.8823	0.3371	0.3014	0.3274
	(40,0*9)	$\hat{\theta}$	-0.0404	0.1058	0.2255	0.1019	0.1496	0.1004	-0.1660	0.0940
		$\hat{\alpha}$	0.0975	0.1242	-0.0909	0.0981	-0.0971	0.0942	-0.3763	0.0925
(50,15)	(0*14,35)	$\hat{\theta}$	-0.0481	0.1358	0.0356	0.1313	0.2987	0.1284	-0.2321	0.0910
		$\hat{\alpha}$	0.2043	0.2675	0.0528	0.3740	-0.6969	0.3648	0.1980	0.3381
	(35,0*14)	$\hat{\theta}$	-0.0162	0.0675	0.1683	0.0632	0.0979	0.0611	-0.0932	0.0588
		$\hat{\alpha}$	0.0869	0.0936	-0.0775	0.0758	-0.1841	0.0744	-0.1132	0.0695
(50,20)	(0*19,30) types 2	$\hat{\theta}$	-0.0323	0.0803	0.0330	0.0804	0.2270	0.0763	-0.1915	0.0707
		$\hat{\alpha}$	0.1617	0.1814	0.0812	0.1470	-0.2038	0.1460	0.1970	0.1385
	(30,0*19)	$\hat{\theta}$	-0.0063	0.0528	0.1353	0.0502	0.0578	0.0490	-0.0518	0.0486
		$\hat{\alpha}$	0.0855	0.0817	-0.0635	0.0660	-0.0734	0.0633	-0.2402	0.0624
(50,25)	(0*24,25)	$\hat{\theta}$	-0.0222	0.0435	0.0110	0.0378	0.1570	0.0324	-0.1059	0.0317
		$\hat{\alpha}$	0.1216	0.1253	0.0207	0.2370	-0.2977	0.2253	0.1112	0.2188
	(25,0*24)	$\hat{\theta}$	-0.0017	0.0394	0.1119	0.0362	0.0868	0.0309	-0.0296	0.0304
		$\hat{\alpha}$	0.0764	0.0678	-0.0589	0.0562	-0.0586	0.0540	-0.0628	0.0514
(50,30)	(0*29,20)	$\hat{\theta}$	-0.0134	0.0367	0.0275	0.0302	0.0746	0.0321	-0.0510	0.0310
		$\hat{\alpha}$	0.0951	0.0823	0.0411	0.0710	-0.0736	0.0704	0.0465	0.0701
	(20,0*29)	$\hat{\theta}$	0.0017	0.0366	0.0985	0.0306	0.0992	0.0301	-0.0228	0.0295
		$\hat{\alpha}$	0.0650	0.0502	-0.0610	0.0431	-0.6394	0.0429	-0.0809	0.0416
(50,35)	(0*34,15)	$\hat{\theta}$	-0.0084	0.0306	0.0166	0.0298	-0.0124	0.0276	-0.0451	0.0249
		$\hat{\alpha}$	0.0781	0.0675	-0.0026	0.0618	0.0689	0.0609	0.0102	0.0601
	(15,0*34)	$\hat{\theta}$	0.0003	0.0324	0.0847	0.0326	0.0027	0.0319	-0.0282	0.0216
		$\hat{\alpha}$	0.0578	0.0461	-0.0613	0.0411	-0.0132	0.0405	0.0050	0.0395
(50,40)	(0*39,10)	$\hat{\theta}$	-0.0056	0.0257	0.0315	0.0249	-0.0965	0.0238	-0.0863	0.0219
		$\hat{\alpha}$	0.0709	0.0530	0.0228	0.0456	0.0218	0.0414	0.4184	0.0401
	(10,0*39)	$\hat{\theta}$	0.0010	0.0284	0.0740	0.0260	0.0952	0.0238	-0.0469	0.0220
		$\hat{\alpha}$	0.0590	0.0409	-0.0518	0.0359	-0.0747	0.0342	0.0976	0.0319

6. APPLICATION

In this section, we discuss an application of Weibull distribution using real data set to illustrate that Weibull distribution provides significant improvements over. The economic data set, consists of 31 yearly time series observations [1980:2010] on response variable: GDP growth (x). The main reasons for selecting the economic data for the present study may due to the fact that, economic is an important sector for many developed and developing countries. Thus, the government is interested in increasing GPA growth. The data are (10.011329, 3.7561, 9.907171, 7.401136, 6.091518, 6.602036, 2.646586, 2.51941, 7.930073, 4.972375, 5.701749, 1.078837, 4.431994, 2.900787, 3.973172, 4.6424, 4.988731, 5.491124, 4.036373, 6.105463, 5.367998, 3.535252, 2.37046, 3.192285, 4.089940, 4.478960, 6.853908, 7.090271, 7.157617, 4.673845, and 5.145106). Figure 1 shows the maximum distance between the empirical and the theoretical cdf as well as a Q-Q plot and a p-p plot

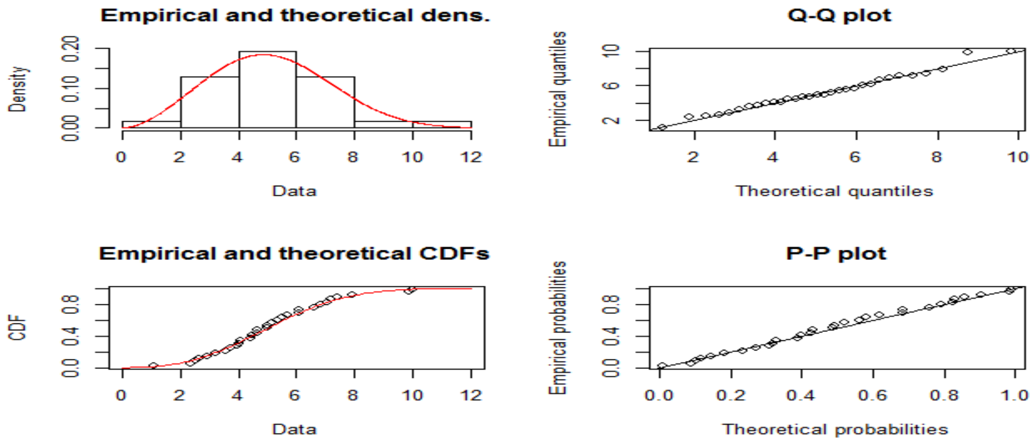

Figure-1: Plot the Maximum Distance between the Empirical and Theoretical cdf

Table 13 provides the results of the one sample Kolmogorov-Smirnov goodness of fit and (AIC and BIC) test for that data and table 14 shows the bias and MSEs

Table-13: Goodness of Fit

	D	P-Value	AIC	BIC
Weibull	0.0625	0.9991	135.3818	138.2497
Gamma	0.0713	0.9942	135.8115	138.6794
Normal	0.0784	0.9832	136.4884	139.3564
Lnorm	0.0996	0.8879	138.7808	141.6487
Gen.Exp	0.0912	0.9379	137.3278	140.1957

Table-14: estimators of θ and α under complete and progressive type-II censoring Scheme for the breakdown data

	Scheme		ML	ML Bays	MPS	MPS Bays
(31,0)	complete	$\hat{\alpha}$	2.6742	2.2451	2.4401	2.4109
		$\hat{\theta}$	5.7751	4.8414	5.9781	5.9616
(31,5)	(0*4,26) type 2	$\hat{\alpha}$	3.6328	3.8090	2.9131	3.3563
		$\hat{\theta}$	4.6983	4.4124	5.3664	5.3154
	(26,0*4)	$\hat{\alpha}$	5.7512	4.0596	4.7720	4.4742
		$\hat{\theta}$	2.5604	2.7277	2.7347	2.9197
	(1*4,22)	$\hat{\alpha}$	3.7772	3.7894	3.0447	3.4078
		$\hat{\theta}$	4.5140	4.3651	5.1061	5.1764
(31,10)	(0*9,21) type 2	$\hat{\alpha}$	2.7484	3.6526	2.7854	2.9263
		$\hat{\theta}$	4.9043	4.7584	5.7812	5.8448
	(21,0*9)	$\hat{\alpha}$	4.4779	3.9186	3.9660	3.8637
		$\hat{\theta}$	3.3273	3.3681	3.4754	3.5517
	(1*9,12)	$\hat{\alpha}$	3.4586	3.6674	3.1385	3.1732
		$\hat{\theta}$	4.9631	4.6465	5.1796	5.3360
(31,15)	(0*14,16) type 2	$\hat{\alpha}$	3.2869	3.5080	3.0746	3.0757
		$\hat{\theta}$	5.3452	4.8142	5.4855	5.5748
	(16,0*14)	$\hat{\alpha}$	4.3917	3.9498	4.0016	3.8952
		$\hat{\theta}$	3.8467	3.8543	3.9563	3.9920
	(1*14,2)	$\hat{\alpha}$	4.2053	3.9051	3.9047	3.7906
		$\hat{\theta}$	4.5966	4.5755	4.7111	4.8015
(31,20)	(0*19,11) type 2	$\hat{\alpha}$	3.1722	3.0633	3.0021	2.9795
		$\hat{\theta}$	4.7270	4.8485	5.5765	5.6218
	(11,0*19)	$\hat{\alpha}$	4.1791	3.9069	3.8280	3.7397
		$\hat{\theta}$	4.3194	4.3118	4.4265	4.4457
(31,25)	(0*24,6) type 2	$\hat{\alpha}$	2.8277	2.4485	2.6860	2.6629
		$\hat{\theta}$	5.7044	4.8493	5.8342	5.8451
	(6,0*24)	$\hat{\alpha}$	3.5883	3.4449	3.2813	3.2269
		$\hat{\theta}$	4.8732	4.7170	5.0058	5.0154

The Gini index is strictly linked to the representation of income inequality through the Lorenz Curve. In particular, it measures the ratio of the area between the Lorenz Curve and the equidistribution line (henceforth, the concentration area) to the area of maximum concentration. The Gini coefficient measures the inequality among values of a frequency distribution (for example, levels of income). A Gini coefficient of zero expresses perfect equality, where all values are the same (for example, where everyone has the same income). A Gini coefficient of 1 (or 100%) expresses maximal inequality among values (e.g., for a large number of people, where only one person has all the income or consumption, and all others have none, the Gini coefficient will be very nearly one). The Gini coefficient is given by

$$G = \frac{1}{\mu}\int_0^{\infty} F(x)\big(1 - F(x)\big)dx$$

For weibull distribution

$$G = 1 - 2^{-1/\alpha}$$

Table-15: The Gini Coefficient for Data

index	x
G	0.2225

The Gini index for GDP growth with value 0.2225, being near zero, but it is more fair than exports of goods and services, indicates that the distribution of income in Egypt 1980:2010 is far from a fair one. In economics, the Lorenz curve is a graphical representation of the distribution of income or of wealth.

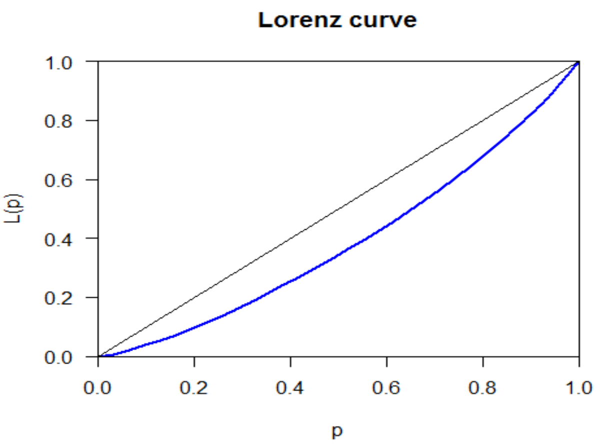

Figure-2: The Lorenz Curve for data

The generalized entropy index has been proposed as a measure of income inequality in a population. It is derived from information theory as a measure of redundancy in data. In information theory a measure of redundancy can be interpreted as non-randomness or data compression; thus this interpretation also applies to this index. In additional interpretation of the index is as biodiversity as entropy has also been proposed as a measure of diversity, to more information see Shorrocks (1980). Theil index (TI) is calculated for $\alpha = 1$, the mean log deviation (LD) for α =0 and Entropy index when $\alpha = 0.5$.

Table-16: The Generalized Entropy Index for Data

GEI	x
$\alpha = 0$	0.0908
$\alpha = 0.5$	0.0844
$\alpha = 1$	0.0808

7. CONCLUSION

In this paper, we show that under different censoring schemes, Bayesian estimation based on MPS and the MLE functions works better than the frequents MPS and MLE methods when the priors are gamma when estimating the Weibull distribution parameters. The Bayes estimators based on MPS function also behave quite better than the Bayes estimators based on the MLE function in the case of the two parameter Weibull distribution, where the bias and MSE decreases than the other methods. We have also noted that increasing the value of the shape parameter θ will lead to increase in the bias and MSE of the different estimators of the Weibull parameters. We can conclude that the MPS method can be a good alternative to the usual MLE method in many situation and even when Bayesian estimation is carried out.

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