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A HARMONIC AND HERON MEAN INEQUALITIES FOR ARGUMENTS IN DIFFERENT INTERVALS

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ABSTRACT

The Mathematical verification of inequalities for Harmonic mean $H < H' < H^c$ and Heron mean $H_e < H_e' < H_e^c$ for two positive arguments respectively in $a, b \in (1, 3/2]$, $a', b' \in (3/2, 2]$ and $a^c, b^c \in (3, \infty)$ are discussed.

1. INTRODUCTION

The Hand book of Means and their Inequalities, by Bullen [1], gave the tremendous work on Mathematical means and the corresponding inequalities involving huge number of means. The authors in [2, 3, 4] discussed about the relations between the well-known means and series. The generalization of the means is discussed in [5, 6, 18, 19]. Relevant to this paper the authors in [12-16] established the good number of inequalities, double inequalities, introduces new means, studied homogenous functions as application, inequalities are obtained. The set of arbitrary non negative real numbers $y \in (0, 1/2]$ and $y' = (1 - y) \in [1/2, 1)$ is represented as a function in the form given by [1].

$$f(y) = \begin{cases} y, & \text{for } 0 < y \leq \frac{1}{2} \\ (1-y), & \text{for } \frac{1}{2} \leq y < 1 \end{cases}$$

In the discussion of the famous inequalities due to Ky Fan, the following are the standard notations in *n* variables.

$$A_n = A_n\ (y_1,\ y_2, \ldots\ldots\ldots y_n) \qquad A'_n = A'_n\ (1-y_1,\ 1-y_2, \ldots\ldots\ldots 1-y_n)$$
$$G_n = G_n\ (y_1,\ y_2, \ldots\ldots\ldots y_n) \qquad G'_n = G'_n\ (1-y_1,\ 1-y_2, \ldots\ldots\ldots 1-y_n)$$
$$H_n = H_n\ (y_1,\ y_2, \ldots\ldots\ldots y_n) \qquad H'_n = H'_n\ (1-y_1,\ 1-y_2, \ldots\ldots\ldots 1-y_n)$$

has been introduced and later on strengthened by several authors namely Rooin etal. , Sandoor *et al.* and others [20-28]. This work motivates us to develop two double inequalities in this paper. The following are the few definitions of means from the above survey papers.

For given n arbitrary non negative real numbers $y_1,\ y_2, \ldots\ldots\ldots y_n \in (0, 1/2]$ unweighted Arithmetic mean, Geometric mean and Harmonic means are represented respectively by A_n, G_n and H_n are given by

$$A_n = \frac{1}{n}\sum_{i=1}^{n} y_i \qquad G_n = \prod_{i=1}^{n} \sqrt[n]{y_i} \qquad H_n = \frac{n}{\sum_{i=1}^{n}\frac{1}{y_i}}$$

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Also, the Arithmetic, Geometric and Harmonic means of the set of elements$1-y_1,\ 1-y_2, \dots \dots \dots 1-y_n$. Represented by $A'_{n,}$ G'_n and H'_n are given by;

$$A'_n = \frac{1}{n}\sum_{i=1}^{n} 1-y_i \qquad G'_n = \prod_{i=1}^{n} \sqrt[n]{1-y_i} \qquad H'_n = \frac{n}{\sum_{i=1}^{n}\frac{1}{1-y_i}}$$

It is of main importance to consider an interval to define index and conjugate index sets. Such a consideration can be methodically deduced starting from the complete set of reals. Let R be the set of index numbers which is nothing but the set of real numbers. Let $a \in R^+$, $a \neq 1$, the conjugate index of a is denoted by a^c and is defined in [1] as $a^c = \frac{a}{a-1}$ and $b^c = \frac{b}{b-1}$. It is clear that for $a = 1, b = 1$, a^c, b^c are not defined so we study for $a, b \in R^+ - (1)$. Further for $a, b \in (0, 1)$, a^c and b^c are negative and the mean definition does not hold. Therefore, we shall consider $a, b \in (1, \infty)$.

Recall some definitions and propositions which are essential to develop this paper.

Definition 1.1: [1] For any $a, b \in (1, \infty)$, then $a^c = \frac{a}{a-1}$ and $b^c = \frac{b}{b-1}$ are the conjugates of a and b.

Definition 1.2: [1] For two real numbers $a, b \in (1, \infty)$, then Harmonic mean and Heron means are respectively given by $H = \frac{2ab}{a+b}$ and $H_e = \frac{a+\sqrt{ab}+b}{3}$.

Definition 1.3: The set of arbitrary non negative real numbers $\mathrm{y} \in (1, 3/2]$ and $\mathrm{y}' = (3 - \mathrm{y}) \in [3/2, 2)$ is represented as a function in the form given by;

$$f(y) = \begin{cases} y & 1 < y \leq \frac{3}{2} \\ (3-y) & \frac{3}{2} \leq y < 2 \end{cases}$$

Proposition 1.1: Let $a_i \in R^+ - (0, 1]$ and $a_i^c = \frac{a_i}{a_i - 1}$ is conjugate of a_i, then

(i) $(a_i^c)^c = a_i$
(ii) $a_i + a_i^c = a. a_i^c$
(iii) if $a_i \epsilon (1, \infty)$ then $a_i^c \ \epsilon \ (1, \infty)$

Proposition 1.2: Let $a_i \in (1, 2]$ and the conjugate of $a_i^c = \frac{\mathrm{a}}{a_i - 1}$, then

(i) $a_i > a_i^c$ if $a_i > 2$
(ii) $a_i < a_i^c$ if a$< a_i < 2$
(iii) $a_i = a_i^c$ if $a_i = 2$

2. MAIN RESULTS

In this section, the inequalities for Harmonic Mean and Heron Mean for the two arguments in $a, b \in (1,3/2]$, $a', b' \in (3/2, 2]$ and a^c, $b^c \in (3, \infty)$ are established.

Theorem 2.1: The Harmonic mean for the arguments in $a, b \in (1, 3/2]$, $a', b' \in (3/2, 2]$ and a^c, $b^c \in (3, \infty)$ are respectively denoted by $H \leq H' \leq H^c$ holds.

Proof: Let the Harmonic mean for two arguments

$H = \frac{2ab}{a+b}$ for $a, b \in (1,3/2]$

$H' = \frac{2a'b'}{a'+b'}$ for $a', b' \in (3/2, 2]$, $a' = 3 - a$, $b' = 3 - b$, then

$$H' = \frac{2(3-a)(3-b)}{3 - \mathrm{a} + 3 - b}$$

and

$H^c = \frac{2a^c b^c}{a^c + b^c}$ for a^c, $b^c \in (3, \infty)$, $a = \frac{a}{\mathrm{a}-1}$, $\mathrm{b}= \frac{b}{\mathrm{b}-1}$

$$H^c = \frac{2\frac{a}{\mathrm{a}-1}\frac{b}{\mathrm{b}-1}}{\frac{a}{\mathrm{a}-1} + \frac{b}{\mathrm{b}-1}} = \frac{2ab}{2ab - a - b}$$

Now consider $H - H' = \frac{2ab}{a+b} - \frac{2(3-a)(3-b)}{3-a+3-b}$

$$H - H' = \frac{2}{(a+b)(6-a-b)}[ab(6-a-b) - (3-a)(3-b)(a+b)] \tag{2.1}$$

Let $\delta = ab(6-a-b) - (3-a)(3-b)(a+b)$ which simplifies as follows

$$\delta = 6ab - a^2b - ab^2 - 9a - 9b + 3ab + 3b^2 + 3a^2 + 3ab - a^2b - ab^2$$

$$\delta = 3a^2 + 3b^2 + 12ab - 9a - 9b - 2a^2b - 2ab^2$$

$$\delta = 3(a+b)^2 + 6ab - 9(a+b) - 2ab(a+b)$$

$$\delta = 12A^2 + 6G^2 - 18A - 4AG^2$$

$$\delta = 2(6A^2 + 3G^2 - 9A - 2AG^2) \text{ , since } = \frac{a+b}{2}, \; G = \sqrt{ab}\,, G^2 = AH$$

$$\delta = 2(6A^2 + 3AH - 9A - 2A^2H)$$

$$\delta = 2[2A^2(3-H) + 3A(H-3)]$$

$$\delta = 2(3-H)A(2A^2 - 3\mathrm{A})$$

$$\delta = 2(3-H)A(2A-3)$$

Therefore, from eqn (2.1) $H - H' = \frac{4(3-H)A(2A-3)}{(a+b)(6-a-b)}$

Thus $H - H' = \frac{4(3-H)A(2A-3)}{(a+b)(6-a-b)} \leq 0$, since $(2A-3) \leq 0$.

This proves that $H - H' \leq 0$.

Again consider $H' - H^c = \frac{2(3-a)(3-b)}{3-a+3-b} - \frac{2ab}{2ab-a-b}$ on simplify leads to

$$H' - H^c = 2\left[\frac{(3-a)(3-b)(2ab-a-b) - ab(6-a-b)}{(6-a-b)(2ab-a-b)}\right] \tag{2.2}$$

Let $\tau = (3-a)(3-b)(2ab-a-b) - ab(6-a-b)$ which simplifies as follows;

$$\tau = 18ab - 9a - 9b - 6ab^2 + 3ab + 3b^2 - 6a^2b + 3a^2 + 3ab + 2a^2b^2 - a^2b - ab^2 - 6ab + a^2b + ab^2$$

$$\tau = 18ab - 9(a+b) - 6ab(a+b) + 3(a^2+b^2) + 2a^2b^2$$

$$\tau = 12ab - 9(a+b) - 6ab(a+b) + 3(a^2+b^2+2ab) + 2a^2b^2$$

$$\tau = 12G^2 - 18A - 6.G^2.2A + 3(a+b)^2 + 2G^4$$

$$\tau = 2A(6H - 9 - 6AH + 6A + AH^2)$$

$$\tau = 2A[2H(3-2A) + AH(H-2) + 3(2A-3)]$$

$$\tau = 2A[4H(3/2 - A) + AH(H-2) + 6(A - 3/2)]$$

$$\tau = 2A[(3/2 - A)(4H-6) + AH(H-2)], \quad \text{since } (2H-3) \leq 0 \quad \text{and } (H-2) \leq 0$$

$$\tau = 2A[(3/2 - A)(4H-6) + AH(H-2)] < 0.$$

Therefore from eqn (2.2) $H' - H^c = 2\left[\frac{(3-a)(3-b)(2ab-a-b) - ab(6-a-b)}{(6-a-b)(2ab-a-b)}\right] \leq 0$

This proves that $H' - H^c \leq 0$

Hence the proof of the inequality $H \leq H' \leq H^c$ of theorem 2.1 completes.

Theorem 2.2: The Heron mean for the arguments in $a, b \in (1, 3/2]$, $a', b' \in (3/2, 2]$ and a^c, $b^c \in (3, \infty)$ are respectively denoted by H_e, H'_e, H_e^c, then the inequality, then $H_e \leq H'_e \leq H_e^c$ holds.

Proof: Let the Heron mean for two arguments

$$H_e = \frac{a+\sqrt{ab}+b}{3} \quad \text{for } a, b \in \left(1, \frac{3}{2}\right]$$

$$H'_e = \frac{a'+\sqrt{a'b'}+b'}{3} \quad \text{for } a', b' \in \left(\frac{3}{2}, 2\right], \quad a' = 3-a, \quad b' = 3-b$$

and

$$H_e^c = \frac{a^c+\sqrt{a^c b^c}+b^c}{3} \quad \text{for } a^c,\ b^c \in (3, \infty), \quad a^c = \frac{a}{a-1}, \ b^c = \frac{b}{b-1}$$

Now consider $H_e - H'_e = \frac{a+\sqrt{ab}+b}{3} - \frac{a'+\sqrt{a'b'}+b'}{3}$

$$= \frac{a+\sqrt{ab}+b}{3} - \frac{(3-a)+\sqrt{(3-a)(3-b)}+3-b}{3}$$

$$= \frac{a+\sqrt{ab}+b}{3} - \frac{(6-a-b)+\sqrt{(3-a)(3-b)}}{3}$$

$$= \frac{1}{3}\left[a+\sqrt{ab}+b-6+a+b-\sqrt{9-3b-3a+ab}\ \right]$$

$$= \frac{1}{3}\left[4A+G-6-\sqrt{9-6A+G^2}\ \right] \tag{2.3}$$

Let $4A+G-6 < \sqrt{9-6A+G^2}$ squaring on both sides gives

$(4A+G-6)^2 < 9-6A+G^2$ which is equivalent to

$16A^2+G^2+8AG+36-48A-12G < 9-6A+G^2$ on simplifying further gives

$16A^2-48A+27+4G(2A-3)+6A < 0$ or

$16A^2-42A+27+4G(2A-3) < 0$

Since $16A^2-42A+27 = (A-3/2)(A-9/8) < 0,\ \ 2A-3 < 0$

Therefore from eqn (2.3) $H_e - H'_e = \frac{1}{3}\left[4A+G-6-\sqrt{9-6A+G^2}\ \right] < 0$

Thus $H_e - H'_e < 0$

Again consider $H'_e - H_e^c = \frac{a'+\sqrt{a'b'}+b'}{3} - \frac{a^c+\sqrt{a^c b^c}+b^c}{3}$

$$= \frac{1}{3}\left[\ 6-a-b+\sqrt{(3-a)(3-b)} - \frac{a}{a-1} - \sqrt{\frac{ab}{(a-1)(b-1)}} - \frac{b}{b-1}\right] \tag{2.4}$$

Let us assume $6-a-b-\frac{a}{a-1}-\frac{b}{b-1} < 0$ implies

$$6-a-b < \frac{a}{a-1}+\frac{b}{b-1}$$

$$(6-a-b)(a-1)(b-1) < a(b-1)+b(a-1)$$

$$6ab-6a-6b+6+a^2+b^2-a^2b-ab^2 < 0$$

$$6a(b-1)+6(1-b)+a^2(1-b)+b^2(1-a) < 0$$

$$b^2(1-a)+(b-1)[6a-6-a^2] < 0,$$

$b^2(1-a)+(1-b)[a^2-6a+6] < 0$ since $1-a < 0,\ \ 1-\mathrm{b} < 0,\ \ [a^2-6a+6] > 0$

Hence the assumption $6 - a - b - \frac{a}{a-1} - \frac{b}{b-1} < 0.$ is true

Similarly consider $\sqrt{(3-a)(3-b)} - \sqrt{\frac{ab}{(a-1)(b-1)}} < 0$

Squaring on both the sides gives $(3-a)(3-b) < \frac{ab}{(a-1)(b-1)}$

$$(3-a)(3-b)(a-1)(b-1) < ab$$

$$15ab - 12(a+b) + 9 - 4a^2b - 4ab^2 + 3(a^2+b^2) + a^2b^2 < 0$$

$$9ab - 12(a+b) + 9 - 4ab(a+b) + 3(a+b)^2 + a^2b^2 < 0$$

$$9G^2 - 24A + 9 - 8G^2A + 12A^2 + G^4 < 0$$

$$9AH - 24A + 9 - 8A^2H + 4A^2 + A^2H^2 < 0$$

$$9AH - 12\mathrm{A} - 12\mathrm{A} + 9 - 4A^2H - 4A^2H + 4A^2 + A^2H^2 < 0$$

$$3\mathrm{A}(3\mathrm{H} - 4) + 3(3 - 4\mathrm{A}) + A^2H(H-4) + 4A^2(1 - \mathrm{H}) < 0$$

Since $(3H-4) < 0,\ (3-4\mathrm{A}) < 0,\ (\mathrm{H}-4) < 0,\ (1-\mathrm{H}) < 0$

Hence our assumption $\sqrt{(3-a)(3-b)} - \sqrt{\frac{ab}{(a-1)(b-1)}} < 0$ is true

Thus eqn (2.4) $H'_e - H_e^c < 0$

Hence the proof of the inequality $H_e \leq H'_e \leq H_e^c$ of theorem 2.2 completes.

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