International Journal of Mathematical Archive-10(8), 2019, 36-38
Available online through www.ijma.info ISSN 2229 – 5046

COVERING PATHS IN HYPERCUBES: CONJECTURE ABOUT LINK LENGTH BOUNDED FROM BELOW

MARCO RIPÀ*

sPIqr Society, World Intelligence Network, Rome, Italy.

(Received On: 13-07-19; Revised & Accepted On: 26-08-19)

ABSTRACT

In 1994 Kranakis et al. published a conjecture about the minimal length of a rectilinear (polygonal) covering path in a k-dimensional n × ... × n points grid. In this paper we consider the general Line-Cover problem, where the line-segments are not required to be axis-parallel, showing that, given n = 3 < k, the known lower bound is not greater than the upper bound of the original Kranakis' conjecture only if exists a multiplicative constant c ≥ 1.5 for the lower order terms.

Keywords: *graph theory, topology, combinatorics, segment, connectivity, outside the box, upper bound, lower bound, point.*

2010 Mathematics Subject Classification: *91A43, 05C40.*

1. INTRODUCTION

The present paper takes into account a multidimensional extension of a well known puzzle involving lateral thinking [4, 7, 8].

Let P be a finite set of n^k points in $\mathbb{R}^k$, we need to visit all of them (at least once) with a polygonal path that has the minimum number of line segments connected at their end-points [3, 6]. We define as $h_l(n,k) \le h(n,k) \le h_u(n,k)$ the length of the Minimum-link Covering Path [10].

In 1994, Evangelos Kranakis *et al.* [6] conjectured that, for any $k \ge 3$, $\frac{k}{k-1} \cdot n^{k-1} \le h(n,k) \le \frac{k}{k-1} \cdot n^{k-1} + O(n^{k-2})$, under two main additional constraints: all line-segments are axis-parallel [2], and every point of the k-dimensional grid cannot be visited twice.

Thus, Kranakis' expected value of the minimal covering path length should be greater (or equal) than any proved lower bound for the same minimization problem, and this proposition would automatically be confirmed if we remove the aforementioned additional constraints (e.g., for the generalized case it has been showed that $h(n=4,\ k=3) \le 23 < \frac{k}{k\text{-}1} \cdot n^{k\text{-}1}$ [11, 13] and also $h(n=5,\ k=3) \le 36 < \frac{k}{k\text{-}1} \cdot n^{k\text{-}1}$ [9]).

2. CURRENT UPPER BOUND VS BEST THEORETICAL SOLUTION

Let $k \ge 3$, for any $n \ge 3$, it has been proved [11] the lower bound

$$h_l(n,k) = \left\lceil \frac{n^k + \frac{k}{2} \cdot (n\text{-}2)^2 \text{-} n^2 + 3 \cdot n \text{-} 4}{n\text{-}1} \right\rceil + 1. \tag{1}$$

Given $k \ge n\text{-}1 > 2$, the lower bound (1) improves Kranakis' conjectured $\frac{k}{k\text{-}1} \cdot n^{k\text{-}1} \le h(n,k)$, confirming and extending (under the aforementioned constraint) his lower bound for axis-aligned spanning paths [1] to arbitrary paths with a minimum number of links as well.

Corresponding Author: Marco Ripà*
sPIqr Society, World Intelligence Network, Rome, Italy.

Since [12]

$$h(n,k) \leq \left(\left\lfloor\frac{3}{2}\cdot n^2\right\rfloor - \left\lfloor\frac{n\text{-}1}{4}\right\rfloor + \left\lfloor\frac{n+1}{4}\right\rfloor - \left\lfloor\frac{n+2}{4}\right\rfloor + \left\lfloor\frac{n}{4}\right\rfloor + n\text{-}1\right)\cdot n^{k\text{-}3}\text{-}1 = h_u(n,k) \tag{2}$$

we see that

$$\lim_{k\to\infty}\frac{n^k}{h_u(n,k)} = \lim_{k\to\infty}\frac{n^k}{\left(\left\lfloor\frac{3}{2}\cdot n^2\right\rfloor - \left\lfloor\frac{n\text{-}1}{4}\right\rfloor + \left\lfloor\frac{n+1}{4}\right\rfloor - \left\lfloor\frac{n+2}{4}\right\rfloor + \left\lfloor\frac{n}{4}\right\rfloor + n\text{-}1\right)\cdot n^{k\text{-}3}\text{-}1} \geq \lim_{k\to\infty}\frac{n^k}{\left(\frac{3\cdot n^2}{2} - \frac{n\text{-}1}{4} + \frac{n+1}{4} - \frac{n+2}{4} + \frac{n}{4} + n\text{-}1\right)\cdot n^{k\text{-}3}\text{-}1}$$

$$= \lim_{k\to\infty}\frac{2\cdot n^k}{n^{k\text{-}3}\cdot(3\cdot n^2+2\cdot n\text{-}2)\text{-}2} = \frac{\mathbf{2\cdot n^3}}{\mathbf{3\cdot n^2+2\cdot n\text{-}2}} \tag{3}$$

and

$$\lim_{k\to\infty}\frac{n^k}{h_l(n,k)} = \lim_{k\to\infty}\left(\frac{n^k}{\frac{n^k+\frac{k}{2}(n\text{-}2)^2\text{-}n^2+3\cdot n\text{-}4}{n\text{-}1}+1}\right) = \boldsymbol{n\text{-}1}. \tag{4}$$

It is also trivial to check that, for any $n \geq 3$,

$$\lim_{k\to\infty}\frac{n^k}{h_u(n,k)} = \lim_{k\to\infty}\frac{n^k}{\left(\left\lfloor\frac{3}{2}\cdot n^2\right\rfloor - \left\lfloor\frac{n\text{-}1}{4}\right\rfloor + \left\lfloor\frac{n+1}{4}\right\rfloor - \left\lfloor\frac{n+2}{4}\right\rfloor + \left\lfloor\frac{n}{4}\right\rfloor + n\text{-}1\right)\cdot n^{k\text{-}3}\text{-}1} < n\text{-}1. \tag{5}$$

It follows that, on average, as k approaches infinity, the efficiency loss for each link is equal to $L(n)$ unvisited points, where

$$L(n) := \lim_{k\to\infty}\frac{n^k}{h_l(n,k)} - \lim_{k\to\infty}\frac{n^k}{h_u(n,k)} \leq \lim_{k\to\infty}\left(n\text{-}1 + \frac{1}{h_l(n,k)}\right) - \frac{2\cdot n^3}{3\cdot n^2+2\cdot n\text{-}2} = \frac{\mathbf{n^3\text{-}n^2\text{-}4\cdot n+2}}{\mathbf{3\cdot n^2+2\cdot n\text{-}2}}. \tag{6}$$

This is reasonable because, from [5], we know that $n\text{-}1 + \frac{1}{h_l(n,k)}$ is the maximum average number of "new" visited points for all the links of a connected path, since the highest theoretical number of points covered by h straight line segments (connected at their end-points) is $\boldsymbol{h\cdot(n\text{-}1)+1}$.

Let $n = 3$, we can improve (1) as [11]

$$h_l(3,k) = \left\lceil\frac{3^k+k\text{-}3}{2}\right\rceil. \tag{7}$$

Thus, for any $k \geq 3$, we have that

$$h(3,k) > 3^{k\text{-}1} + \frac{3}{2}\cdot 3^{k\text{-}2} \tag{8}$$

and Kranakis' conjectured upper bound $h_u(n \geq 3,\ k \geq 4) = \frac{k}{k\text{-}1}\cdot n^{k\text{-}1} + O(n^{k\text{-}2})$, for rectilinear walks with minimal link length visiting all the n^k points, implies the existence of a constant $c \geq \frac{3}{2}$ such that $h_u = \frac{k}{k\text{-}1}\cdot n^{k\text{-}1} + c\cdot n^{k\text{-}2}$.

Moreover, we know [11, 14] that $h_u(3,4) = 41$, so $h(3,k) \leq 42\cdot 3^{(k\text{-}4)}\text{-}1$.

Therefore,

$$\lim_{k\to\infty}\frac{3^k}{h_u(3,k)} = \lim_{k\to\infty}\frac{3^k}{42\cdot 3^{k\text{-}4}\text{-}1} = \frac{27}{14} \approx 1.929 \tag{9}$$

and the efficiency loss for each link can be reduced to only

$L(3) = \lim_{k\to\infty}\frac{3^k}{h_l(3,k)} - \lim_{k\to\infty}\frac{3^k}{h_u(3,k)} = 3 - 1 - \frac{27}{14} = \frac{\mathbf{1}}{\mathbf{14}}$ unvisited points, instead of $\frac{8}{31}$ as given by (6).

Theorem 2.1: For any arbitrarily large k-dimensional grid, the links of the best covering path joins (on average) less than $n\text{-}1$ new points, for any $(n \geq 3,\ k \geq 2)$ except $\{(3,2)\}$.

Proof: Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$ be such that $n \geq 3$ and $k \geq 3$, the only solution of $\frac{n^k}{h_l(n,k)} \geq n\text{-}1$ would imply that $n = 3$ and $k = 3$, since $\frac{n^k}{\frac{n^k+\frac{k}{2}(n\text{-}2)^2\text{-}n^2+3\cdot n\text{-}4}{n\text{-}1}+1} \geq n\text{-}1 \Rightarrow \frac{k}{2}\cdot(n\text{-}2)^2\text{-}n^2 + 4\cdot n\text{-}5 \leq 0$, but we already know [11] that $\frac{3^3}{h_l(3,3)} = \frac{27}{14} < 2$.

Thus, we have proved that, $\forall(n \geq 3, k \geq 2) - \{(3,2)\}$, the links of the best covering path joins (on average) less than $n\text{-}1$ new points [5]. □

Finally, from $\lim_{k\to\infty}\frac{n^k}{h_l(n,k)} = n\text{-}1$, we get the asymptotic formula for k arbitrarily large:

$$(n\text{-}1)\cdot h(n,k) > n^k \Longrightarrow h(n,k) > \frac{n^k}{n\text{-}1}. \tag{10}$$

and this is coherent with Kranakis' conjecture too.

3. CONCLUSION

Since any upper bound for the link length of a rectilinear walk cannot fall below than the corresponding lower bound (for every pair $(n \geq 3, k \geq 4)$), especially considering the generalized (arbitrary) covering paths taken into account in [11, 12], Kranakis' conjectured upper bound can be rewritten as $h(n,k) \leq \frac{k}{k-1} \cdot n^{k-1} + c \cdot n^{k-2}$, where $c \geq \frac{3}{2}$. On the contrary, the upper bound proved by Bereg *et al.* [1] of $\frac{k}{k-1} \cdot 3^{k-1} + \frac{3}{2} \cdot 3^{k-\frac{3}{2}}$ definitely holds for every finite set of 3^k points in $\mathbb{R}^k$.

REFERENCES

1. Bereg, S., Bose, P., Dumitrescu, A. et al., Traversing a Set of Points with a Minimum Number of Turns. Discrete Comput. Geom., 41(4), (2009), 513-532.
2. Collins, M.J., Moret, M.E., Improved lower bounds for the link length of rectilinear spanning paths in grids. Inf. Process. Lett. 68(6), (1998), 317–319.
3. Collins, M.J., Covering a set of points with a minimum number of turns, International Journal of Computational Geometry & Applications 14(1-2), (2004), 105–114.
4. Kihn, M., Outside the Box: The Inside Story. FastCompany (1995).
5. Keszegh, B., Covering Paths and Trees for Planar Grids, arXiv (2013), Available online: https://arxiv.org/abs/1311.0452
6. Kranakis, E., Krizanc, D., Meertens, L., Link length of rectilinear Hamiltonian tours in grids. Ars Comb. 38 (1994), 177–192.
7. Loyd, S., Cyclopedia of Puzzles. The Lamb Publishing Company (1914), p. 301.
8. Lung, C.T., Dominowski, R.L., Effects of strategy instructions and practice on nine-dot problem solving. Journal of Experimental Psychology: Learning, Memory, and Cognition 11(4), (1985), 804–811.
9. Ripà, M., Solving the n_1 X n_2 X n_3 Points Problem for n_3 < 6, viXra (2019), Available online: http://vixra.org/pdf/1906.0501v4.pdf
10. Ripà, M. (2014). The Rectangular Spiral or the $n_1 \times n_2 \times \ldots \times n_k$ Points Problem. Notes on Number Theory and Discrete Mathematics 20(1), (2019), 59-71.
11. Ripà, M., The 3 x 3 x ... x 3 Points Problem solution. Notes on Number Theory and Discrete Mathematics 25(2), (2019), 68-75.
12. Ripà, M., Bencini, V., n X n X n Dots Puzzle: An Improved "Outside The Box" Upper Bound, viXra, (2018), Available online: http://vixra.org/pdf/1807.0384v2.pdf
13. Sloane, N.J.A., The Online Encyclopedia of Integer Sequences (2013). Web. 9 Jul. 2019, Available online: http://oeis.org/A225227
14. Sloane, N.J.A., The Online Encyclopedia of Integer Sequences (2015). Web. 9 Jul. 2019, Available online: http://oeis.org/A261547