FIXED POINTS FOR NON-SELF MAPPINGS ON CONVEX VECTOR METRIC SPACES

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ABSTRACT

We introduce the concept of convex structure on a vector metric space and obtain some fixed point theorems for a class of non-self mappings satisfying certain contractive conditions in the setting of convex vector metric spaces.

Keywords and phrases: Convex vector metric space, non-self mapping, fixed point.

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1. INTRODUCTION:

Deterministic fixed point theorems are generally proved for self mappings only. In 1970, W. Takahashi [5] had introduced the concept of convexity in a metric space and obtained some important fixed point theorems previously proved for Banach spaces. Afterwards, Gajić [2] obtained an important fixed point theorem for a class of non-self mappings in Takahashi convex metric spaces. In this paper, our attempt is to introduce the notion of convex structure on a vector metric space with relevant definitions with their properties. Finally, we prove some fixed point theorems for a class of non-self mappings over a subset of a convex vector metric space.

2. DEFINITIONS AND BASIC FACTS:

In this section, we recall some basic definitions and important results for vector metric spaces that will be needed in the sequel.

Definition: 2.1 [4] Let \( X \) be a non empty set. Then a function \( \sigma : X \times X \rightarrow \mathbb{R}^+ \) is called a vector metric on \( X \) if the following conditions are satisfied:

(i) \( \sigma(x, y) = 0 \) if and only if \( x = y \),
(ii) \( \sigma(x, y) = \sigma(y, x) \) for all \( x, y \in X \),
(iii) \( \sigma(x, z) + \sigma(z, y) \geq \sigma(x, y) \) for all \( x, y, z \in X \).

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The pair \((X,V)\) is called a vector metric space. We may verify that \(V(x,y)\) is a continuous function of its arguments.

**Theorem:** 2.2\[4\] If \(V(x,y) = \{d_n(x,y)\}\) be a vector metric then each \(d_n(x,y)\) is a quasi metric function; conversely if each \(d_n(x,y)\) is a quasi metric and the relations \(d_n(x,y) = 0\) for all \(n\) imply \(x = y\), then \(V(x,y) = \{d_n(x,y)\}\) is a vector metric.

**Remark:** 2.3 If \((X,d)\) is a metric space and if \(V(x,y) = \{d(x,y), d(x,y), \cdots\}\), then \((X,V)\) is a vector metric space. So any metric space is a vector metric space and for the converse we can say from Theorem 2.2 that a vector metric space is a quasi metric space.

**Example:** 2.4 Let \(X = R^n\). If we define \(V\) on \(X \times X\) by
\[
V(x,y) = \left\{ \left| x_1 - y_1 \right|, \left| x_2 - y_2 \right|, \cdots, \left| x_n - y_n \right|, 0, \cdots \right\}
\]
where \(x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in X\) then \((X,V)\) is a vector metric space.

**Example:** 2.5\[3\] Let \(X = [0,1]\). If we define \(V\) on \(X \times X\) by
\[
V(x,y) = \left\{ \left| x - y \right|, 0, \left| 1 - \frac{x - y}{x - y} \right|, 0, 2\left| x - y \right|, 0, \frac{2\left| x - y \right|}{1 + 2\left| x - y \right|}, 0, \cdots \right\}
\]
then \((X,V)\) is a vector metric space.

**Definition:** 2.6\[4\] Let \((X,V)\) be a vector metric space. A sequence \(\{x_k\}\) in \(X\) is said to converge to an element \(x \in X\) i.e. \(\lim k \infty x_k = x\) if \(V(x_k,x) \rightarrow \theta\) as \(k \rightarrow \infty\) which is interpreted as
\[
d_n(x_k,x) \rightarrow 0\] as \(k \rightarrow \infty\) for all \(n\) where \(V(x_k,x) = \{d_n(x_k,x)\}\).

**Definition:** 2.7\[3\] A sequence \(\{x_k\}\) in \((X,V)\) is said to be a vector Cauchy sequence if \(V(x_k,x_{k+p}) \rightarrow \theta\) as \(k \rightarrow \infty\) for each \(p\), i.e. \(d_n(x_k,x_{k+p}) \rightarrow 0\) as \(k \rightarrow \infty\) for each \(p\) and for each \(n\).

In other words, a sequence \(\{x_k\}\) in \((X,V)\) is said to be a vector Cauchy sequence if \(V(x_k,x_j) \rightarrow \theta\) as \(k, j \rightarrow \infty\), i.e. \(d_n(x_k,x_j) \rightarrow 0\) as \(k, j \rightarrow \infty\) and for each \(n\).

**Definition:** 2.8\[3\] A vector metric space \((X,V)\) is said to be complete if every vector Cauchy sequence in \(X\) converges to an element in \(X\). Otherwise \(X\) is called incomplete.

It is to be noted that every complete metric space may be considered as a complete vector metric space.

**Definition:** 2.9 Let \((X,V)\) be a vector metric space and \(A \subset X\). Then \(A\) is called closed if and only if every sequence \(\{x_n\}\) in \(A\) with \(\lim n \rightarrow \infty x_n = x\) implies that \(x \in A\).

It is easy to verify that every closed subset of a complete vector metric space is complete.

**Definition:** 2.10 Let \((X,V)\) be a vector metric space, \(x_0 \in X\) and \(r\) be a positive member of \(S\). Then the set denoted by \(B(x_0,r) = \{x \in X: V(x_0,x) < r\}\) is called an open ball centered at \(x_0\) and radius \(r\) in \(X\).

**Definition 2.11** Let \((X,V)\) be a vector metric space and \(A \subset X\). A point \(x \in X\) is called a boundary point of \(A\) if every open ball centered at \(x\) intersects both \(A\) and \(X - A\). The set of all boundary points of \(A\), denoted by \(\partial A\), is called the boundary of \(A\).
**Definition: 2.12** A subset $A$ of a vector metric space $(X, V)$ is called bounded if there exists a positive member $K$ in $S$ such that $V(x, y) \leq K$ for all $x, y \in A$.

**Definition: 2.13** Diameter of a bounded set $A$ in a vector metric space $(X, V)$, denoted by $Diam(A)$ or $\delta(A)$, is defined as

$$Diam(A) = \sup_{x, y \in A} V(x, y) = \left\{ \sup_{x, y \in A} a_1(x, y), \sup_{x, y \in A} a_2(x, y), \ldots, \sup_{x, y \in A} a_n(x, y), \ldots \right\}$$

where $V(x, y) = \{a_i(x, y)\}$ and for each $i$, $\sup_{x, y \in A} a_i(x, y) < +\infty$ as $A$ is bounded.

In this case we write $\delta(A) < +\infty$.

**3. MAIN RESULTS:**

Let $(X, V)$ be a vector metric space, and $I$ denote the closed unit interval $[0, 1]$ of reals.

**Definition: 3.1** A mapping $W : X \times X \times I \to X$ is said to be a convex structure on $X$ if for all $x, y, u \in X$, $\lambda \in I$,

$$V(u, W(x, y, \lambda)) \leq \lambda V(u, x) + (1 - \lambda) V(u, y).$$

Then $X$ together with a convex structure $W$ is called a convex vector metric space.

**Remark: 3.2** It is observed that $W(x, y, 1) = x$ and $W(x, y, 0) = y$.

**Remark: 3.3** Any convex subset of a normed linear space is a convex vector metric space with convex structure $W(x, y, \lambda) = \lambda x + (1 - \lambda) y$.

**Definition: 3.4** Let $(X, V)$ be a convex vector metric space with a convex structure $W$. For $x, y \in X$, we define

$$Seg [x, y] = \{W(x, y, \lambda) : \lambda \in [0, 1]\}.$$

Clearly $x, y \in Seg [x, y]$, since $x = W(x, y, 1), y = W(x, y, 0)$.

**Proposition: 3.5** If $(X, V)$ is a convex vector metric space with convex structure $W$, then for every $x, y \in X$ and every $\lambda \in [0, 1]$,

$$V(x, y) = V(x, W(x, y, \lambda)) + V(W(x, y, \lambda), y).$$

**Proof:** For every $x, y \in X$ and every $\lambda \in [0, 1]$, we have

$$V(x, y) \leq V(x, W(x, y, \lambda)) + V(W(x, y, \lambda), y) \leq \lambda V(x, x) + (1 - \lambda) V(x, y) + \lambda V(y, x) + (1 - \lambda) V(y, y) = V(x, y)$$

This implies that $V(x, y) = V(x, W(x, y, \lambda)) + V(W(x, y, \lambda), y)$.

**Proposition: 3.6** Let $K$ be a non-empty closed subset of the convex vector metric space $(X, V)$ with convex structure $W$ continuous on third variable. Let $x \in K$ and $y \notin K$. Then there exists a $\lambda^* \in [0, 1]$ such that $W(x, y, \lambda^*) \in Seg [x, y] \cap \partial K$ where $\partial K$ is the boundary of $K$. 

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Proof: Let us consider the set  
\[ A = \{ q : q \geq 0, \ W(x, y, \eta) \in K \text{ for all } q \leq \eta \leq 1\} \]. Then  \( A \) is non-empty, since  \( W(x, y, 1) = x \in K \). We put  \( \lambda = \inf_{q \geq 0} q \) and let  \( \{ q_n \}_{n \in \mathbb{N}} \subset A \) be a sequence satisfying  \( \lim_{n \to \infty} q_n = \lambda \). Then, for  \( n \in \mathbb{N}, W(x, y, q_n) \in K \). By using the continuity of  \( W \) on the third variable and since  \( K \) is closed, we have  
\[ W(x, y, \lambda) = \lim_{n \to \infty} W(x, y, q_n) \in K. \]
Clearly  \( \lambda > 0 \) since  \( W(x, y, 0) = y \notin K \).

Now we prove that  \( W(x, y, \lambda) \in \partial K \). For any positive member  \( \varepsilon \in \mathbb{S}, \ B(W(x, y, \lambda), \varepsilon) \cap K \neq \emptyset \). So we have to show that for any positive member  \( \varepsilon \in \mathbb{S}, \ B(W(x, y, \lambda), \varepsilon) \cap (X - K) \neq \emptyset \).

Since  \( \lambda > 0 \), we can find  \( \lambda_1 \in (0, 1) \) such that  \( \lambda - \lambda_1 \). Using continuity of  \( W \) on third variable, we obtain  
\[ V(W(x, y, \lambda_1), W(x, y, \lambda)) < \varepsilon. \]

Thus  \( W(x, y, \lambda_1) \in B(W(x, y, \lambda), \varepsilon) \cap (X - K) \).

This completes the proof.

Theorem: 3.7 Let  \((X, \mathcal{V})\) be a complete convex vector metric space with convex structure  \( W \) which is continuous on third variable,  \( C \) be a non-empty closed subset of  \( X \) and  \( T : C \to X \) be a non-self mapping satisfying the contractive type condition  
\[ V(T(x), T(y)) \leq q \max \{ V(x, y), V(x, T(x)), V(y, T(y)), V(x, T(y)), V(y, T(x)) \} \]
for all  \( x, y \in C \) and  \( 0 < q < 1 \). If  \( T \) has the additional property  \( T(\partial C) \subset C \) where  \( \partial C \) is the boundary of  \( C \), then  \( T \) has a unique fixed point in  \( C \).

Proof: For  \( x \in \partial C, \) we put  \( x_0 = x \). Then  \( T(\partial C) \subset C \) implies that  \( T(x_0) \notin C \). Let us put  \( x_1 = T(x_0) \). We define  \( x_2 = T(x_1) \). If  \( y_2 \in C \), let us take  \( x_2 = y_2 \). If  \( y_2 \notin C \), then by Proposition 3.6, there exists  \( x_2 \in \partial C \cap \text{Seg} [x_1, y_2] \).

Continuing in this way we can obtain a sequence  \( \{ x_n \} \) such that  
\[ x_n = T(x_{n-1}), \text{ if } T(x_{n-1}) \in C, \]
\[ x_n \in \partial C \cap \text{Seg} [x_{n-1}, y_n], \text{ if } y_n = T(x_{n-1}) \notin C. \]

We show that the sequences  \( \{ x_n \} \text{ and } \{ T(x_n) \} \) are bounded.

For  \( n \in \mathbb{N}, \) we define  
\[ A_n = \{ x_i \}_{i=0}^{n-1} \cup \{ T(x_i) \}_{i=0}^{n-1} \] and  \( \alpha_n = \text{diam} A_n \).

We shall show that  \( \alpha_n = \max \{ V(x_0, T(x_i)) : 0 \leq i \leq n - 1 \} \).

We now discuss the following possible cases.

Case I: Suppose  \( \alpha_n = \max \{ V(x_0, T(x_i)) : 0 \leq i, j \leq n - 1 \} \). The case  \( i = 0 \) is trivial. So we suppose that  \( i \geq 1 \).

(1) If  \( x_i = T(x_{i-1}) \in C \), then  
\[ V(T(x_{i-1}), T(x_i)) \leq q \max \left\{ V(x_{i-1}, x_i), V(x_{i-1}, T(x_{i-1})), V(x_i, T(x_i)), V(x_{i-1}, T(x_i)), V(x_i, T(x_{i-1})) \right\} \]
\[ \leq q \alpha_n \]

So, \( \alpha_n = \max \{ V(x_i, T(x_i)) : 0 \leq i, j \leq n-1 \text{ and } i \geq 1 \} \leq q\alpha_n < \alpha_n, \quad \text{a contradiction}. \]

(2) If \( x_i \neq T(x_{i-1}) \) i.e., \( T(x_{i-1}) \notin C \), then
\[ x_i \in \partial C \cap \text{Seg} [x_{i-1}, T(x_{i-1})]. \]

So \( x_i = W(x_{i-1}, T(x_{i-1}), \lambda) \) for some \( \lambda \in [0, 1] \) and \( x_{i-1} = T(x_{i-2}) \).

Now
\[ V(x_i, T(x_j)) = V(W(x_{i-1}, T(x_{i-1}), \lambda), T(x_j)) \]
\[ \leq \lambda V(x_{i-1}, T(x_j)) + (1 - \lambda) V(T(x_{i-1}), T(x_j)) \]
\[ \leq \lambda q \alpha_n + (1 - \lambda) q \alpha_n \]
\[ = q \alpha_n. \]

So, \( \alpha_n = \max \{ V(x_i, T(x_j)) : 0 \leq i, j \leq n-1 \text{ and } i \geq 1 \} \leq q\alpha_n < \alpha_n, \quad \text{a contradiction}. \]

Case II: Suppose \( \alpha_n = \max \{ V(x_i, x_j) : 0 \leq i, j \leq n-1 \} \).

If \( x_j = T(x_{j-1}) \), we have Case I again.

If \( x_j \neq T(x_{j-1}) \), then \( x_j \in \partial C \cap \text{Seg} [x_{j-1}, T(x_{j-1})] \), \( j \geq 2 \) and \( x_{j-1} = T(x_{j-2}) \), so this is similar to Case I(2).

By an argument similar to that used above, the case
\[ \alpha_n = \max \{ V(T(x_i), T(x_j)) : 0 \leq i, j \leq n-1 \} \]

is also impossible. Thus, it must be the case that

\[ \alpha_n = \max \{ V(x_0, T(x_i)) : 0 \leq i \leq n-1 \}. \]

If \( \alpha_n = \theta \), then \( x_0 \) is the fixed point of \( T \), so we can suppose that \( \alpha_n \neq \theta \) for any \( n \in N \).

Further, for \( 0 \leq i \leq n-1 \),
\[ V(x_0, T(x_i)) \leq V(x_0, T(x_0)) + V(T(x_0), T(x_i)) \]
\[ \leq V(x_0, T(x_0)) + q \alpha_n. \]

So, \( \alpha_n = \max \{ V(x_0, T(x_i)) : 0 \leq i \leq n-1 \} \leq V(x_0, T(x_0)) + q \alpha_n \)

i.e.,
\[ \alpha_n \leq \frac{1}{1-q} V(x_0, T(x_0)) \tag{3.1} \]

If \( \alpha_n = \{ \alpha_n^k \}_k \) and \( V(x_0, T(x_0)) = \{ d_k(x_0, T(x_0)) \}_k \), then it follows from (3.1) that
\[ \alpha_n^k \leq \frac{1}{1-q} d_k(x_0, T(x_0)) \text{ for all } k. \]

For fixed \( k \), \( \{ \alpha_n^k \}_n \) is non-decreasing, so there exists \( c_k \in R \) such that
\[ c_k = \lim_n \alpha_n^k \text{ and } c_k \leq \frac{1}{1-q} d_k(x_0, T(x_0)). \]

So there exists \( c = (c_1, c_2, \cdots, c_k, \cdots) \in S \) such that
\[ c \leq \frac{1}{1-q} V(x_0, T(x_0)). \]
We define
\[ B_n = \{ x_i \}_{i=1}^{2n} \cup \{ T(x_i) \}_{i=1}^{2n}, \text{ and } \beta_n = \text{diam } B_n \text{ for integer } n \geq 2. \]
Then by the same technique as given above, one can show that
\[ \beta_n^i = \sup \{ V(x_j, T(x_j)) : j \geq n \} \text{ and that if } \beta_n^i = \{ \beta_n^{i_k} \}_{k=1}^{\infty}, \text{ then for fixed } k, \{ \beta_n^{i_k} \}_{n} \text{ is non-increasing and bounded.} \]
So there exists a limit and it must be zero (see [2]). Therefore, \( \lim_{n \to \infty} \beta_n^i = 0. \) So \( \{ x_n \} \) and \( \{ T(x_n) \} \) are vector Cauchy sequences. Since \((X, V)\) is a complete vector metric space and \( C \) is closed, there exists \( z \in C \) such that \( z = \lim_{n \to \infty} x_n. \)

Again, \( V(x_n, T(x_n)) \leq \beta_n, \beta_n \to \theta, n \to \infty \) implies that
\[ V(x_n, T(x_n)) \to \theta \text{ as } n \to \infty. \]
Now
\[ V(T(x_n), z) \leq V(T(x_n), x_n) + V(x_n, z) \to \theta \text{ as } n \to \infty, \]
gives that \( \lim_{n \to \infty} T(x_n) = z. \)

Assume that \( z \neq T(z). \) Then
\[ V(T(x_n), T(z)) \leq q \max \{ V(x_n, z), V(x_n, T(x_n)), V(z, T(z)), V(x_n, T(z)), V(z, T(x_n)) \}. \]
For \( n \to \infty, \) we have
\[ V(z, T(z)) \leq q V(z, T(z)) < V(z, T(z)), \]
which is a contradiction. Therefore \( z = T(z). \) The uniqueness follows from contractive condition.

The following example shows that the contractive type condition in Theorem 3.7 can not be relaxed in order that Theorem 3.7 is true.

**Example: 3.8** Let \( X = [0, \infty) \) with usual metric \( d \). Then \((X, d)\) is a complete metric space. So \((X, V)\) is a complete vector metric space where \( V(x, y) = \{ d(x, y), d(x, y), \cdots \}. \) We define \( W : X \times X \times I \to X \) by
\[ W(x, y, \lambda) = \lambda x + (1 - \lambda) y \]
for all \( x, y \in X \) and \( \lambda \in I = [0,1]. \)

For \( x, y, u \in X \) and \( \lambda \in I, \) it is easy to check that
\[ V(u, W(x, y, \lambda)) \leq \lambda V(u, x) + (1 - \lambda) V(u, y). \]
Thus \((X, V)\) is a complete convex vector metric space with convex structure \( W \) which is continuous on the third variable.

Let \( C = [0,1] \) be the closed unit interval of reals. Then \( C \subset X \) and let \( T : C \to X \) be a non-self mapping defined by
\[ T(x) = \frac{1}{2} \text{ for } 0 \leq x \leq 1 \text{ except } x = \frac{1}{70}, \frac{1}{2^i} (i = 1, 2, 3, \cdots) \]
\[ = 0 \text{ for } x = \frac{1}{70}, \]
and \( T \left( \frac{1}{2^i} \right) = 2^{i+1} \) for \( i \geq 1. \)
Then \( T(\partial C) \subset C, \) but \( T \) has no fixed point in \( C. \)

We now verify that for \( x = \frac{1}{3}, y = \frac{1}{70}, T \) does not satisfy the contractive condition
\[ V(T(x), T(y)) \leq q \max \{ V(x, y), V(x, T(x)), V(y, T(y)), V(x, T(y)), V(y, T(x)) \} \]
for \( 0 < q < 1. \)
Now, \( V(T(x), T(y)) = \{d(T(x), T(y)), d(T(x), T(y)), \ldots\} = \left\{ \frac{1}{2}, \frac{1}{2}, \ldots \right\} \).

Similarly,

\[
V(x, T(x)) = \left\{ \frac{1}{6}, \frac{1}{6}, \ldots \right\};
\]

\[
V(y, T(y)) = \left\{ \frac{1}{70}, \frac{1}{70}, \ldots \right\};
\]

\[
V(x, y) = \left\{ \frac{67}{210}, \frac{67}{210}, \ldots \right\};
\]

\[
V(x, T(y)) = \left\{ \frac{1}{3}, \frac{1}{3}, \ldots \right\};
\]

\[
V(y, T(x)) = \left\{ \frac{17}{35}, \frac{17}{35}, \ldots \right\}.
\]

Therefore, \( q \max \{V(x, y), V(x, T(x)), V(y, T(y)), V(x, T(y)), V(y, T(x))\} \)

\[
= q \left\{ \frac{17}{35}, \frac{17}{35}, \ldots \right\}
\leq \left\{ \frac{17}{35}, \frac{17}{35}, \ldots \right\}
\leq \left\{ \frac{1}{2}, \frac{1}{2}, \ldots \right\}
= V(T(x), T(y)), \text{ for } 0 < q < 1.
\]

**Theorem: 3.9** Let \((X, V)\) be a complete convex vector metric space with convex structure \(W\) which is continuous on third variable, \(C\) be a non-empty closed subset of \(X\) and \(T_i : C \rightarrow X\) be a non-self mapping satisfying the condition

\[
V(T_i(x), T_i(y)) \leq q \max \{V(x, y), V(x, T_i(x)), V(y, T_i(y)), V(x, T_i(y)), V(y, T_i(x))\}
\]

for all \(x, y \in C\) and \(0 < q < 1\) with the additional property \(T_i(\partial C) \subseteq C\). If \(u_i\) is the fixed point of \(T_i\) for \(i = 1, 2, 3, \ldots\) and \(T(x) = \lim_{i \to \infty} T_i(x)\) for all \(x \in C\) with \(T(\partial C) \subseteq C\), then \(T\) has a unique fixed point \(u\) in \(C\) iff \(u = \lim_{i \to \infty} u_i\).

**Proof:** We have \(T(x) = \lim_{i \to \infty} T_i(x)\) for \(x \in C\). For \(x, y \in C\), we have

\[
V(T_i(x), T_i(y)) \leq q \max \{V(x, y), V(x, T_i(x)), V(y, T_i(y)), V(x, T_i(y)), V(y, T_i(x))\} \text{ where } 0 < q < 1.
\]

As \(i \to \infty\),

\[
V(T(x), T(y)) \leq q \max \{V(x, y), V(x, T(x)), V(y, T(y)), V(x, T(y)), V(y, T(x))\} \text{ where } 0 < q < 1.
\]

Hence by Theorem 3.7, \(T\) has a unique fixed point, say \(u \in C\). By hypothesis \(u_i = T_i(u_i)\) for \(i = 1, 2, 3, \ldots\)
Now for $0 < q < 1$, we have

$$V(u, u_i) = V(T(u), T_i(u_i))$$
$$\leq V(T(u), T_i(u)) + V(T_i(u), T_i(u_i))$$
$$\leq V(T(u), T_i(u)) + q \max \left\{ V(u, u_i), V(u, T_i(u)), V(u_i, T_i(u_i)), V(u_i, T_i(u)) \right\}$$
$$\leq V(T(u), T_i(u)) + q \max \left\{ V(u, u_i), V(u, T_i(u)), V(u_i, u_i) + V(u, T_i(u)) \right\}$$

which implies that,

$$V(u, u_i) \leq \frac{1 + q}{1 - q} V(T(u), T_i(u)) \to 0 \text{ as } i \to \infty.$$ 

Therefore, $u = \lim_{i \to \infty} u_i$.

Conversely, let $u = \lim_{i \to \infty} u_i$, then

$$V(T_i(u), u_i) \leq q \max \left\{ V(u, u_i), V(u, T_i(u)), V(u_i, T_i(u)) \right\}$$

which implies that,

$$V(T_i(u), u_i) \leq q \max \left\{ V(u, u_i), V(u, T_i(u)), V(u_i, T_i(u)) \right\}.$$

Taking limit as $i \to \infty$, we have

$$V(T(u), u) \leq q V(u, T(u)).$$

So, $(1 - q) V(u, T(u)) \leq \theta$ which gives that $T(u) = u$.

We now prove a fixed point theorem for multivalued mappings. For this purpose we need the following notations.

Notations: Let $(X, V)$ be a vector metric space and let $A, B$ be any two subsets of $X$. We denote

$$D(A, B) = \inf_{x \in A, y \in B} V(x, y) = \left( \inf_{x \in A, y \in B} a_1(x, y), \inf_{x \in A, y \in B} a_2(x, y), \ldots, \inf_{x \in A, y \in B} a_n(x, y) \right),$$

$$\rho(A, B) = \sup_{x \in A, y \in B} V(x, y) = \left( \sup_{x \in A, y \in B} a_1(x, y), \sup_{x \in A, y \in B} a_2(x, y), \ldots, \sup_{x \in A, y \in B} a_n(x, y) \right)$$

where $V(x, y) = \{a_i(x, y)\}$.

$$BN(X) = \{A : \emptyset \neq A \subset X \text{ and } \delta(A) < +\infty\}.$$ 

Theorem: 3.10 Let $(X, V)$ be a complete convex vector metric space with convex structure $W$ which is continuous on third variable, $C$ be a non-empty closed subset of $X$ and $F : C \to BN(X)$ be a multivalued mapping satisfying the condition

$$\rho(F(x), F(y)) \leq q \max \left\{ V(x, y), \rho(x, F(x)), \rho(y, F(y)), D(x, F(y)), D(y, F(x)) \right\}$$

for all $x, y \in C$ and $0 < q < 1$. If $F$ has the additional property $F(\partial C) \subset C$ where $F(\partial C) = \bigcup \{F(x) : x \in \partial C\}$, then $F$ has a unique fixed point $u$ in $C$ and $F(u) = \{u\}$.

Proof: Take $0 < \alpha < 1$ and define a single valued mapping $T : C \to X$ as follows: For each $x \in C$, let $T(x)$ be a point of $F(x)$, which satisfies the condition

$$V(x, T(x)) \geq q^\alpha \rho(x, F(x)).$$

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Then $T$ has the property $T(\partial C) \subset C$ since $F(\partial C) \subset C$.

Now for every $x, y \in C$, we have

$$V(T(x), T(y)) \leq \rho(F(x), F(y))$$

$$\leq q \max \left\{ V(x, y), \rho(x, F(x)), \rho(y, F(y)), D(x, F(y)), D(y, F(x)) \right\}$$

$$= q \max \left\{ q^n V(x, y), q^n \rho(x, F(x)), q^n \rho(y, F(y)), \right\}$$

$$\leq q^{1-a} \max \left\{ V(x, y), V(x, T(x)), V(y, T(y)), \right\}$$

for $0 < q < 1$.

Hence by Theorem 3.7, $T$ has a unique fixed point in $C$. Let $u \in C$ be such that $u = T(u)$.

Clearly $u = T(u)$ implies $u \in F(u)$. Since $F$ satisfies (3.2), $u \in F(u)$ implies

$$\rho(F(u), F(u)) \leq q \rho(u, F(u)).$$

This may happen only if $F(u) = \{u\}$. Therefore, $u \in C$ is a fixed point of $T$ if $u$ is a fixed point of $F$. Thus $F$ has a unique fixed point $u$ in $C$ and $F(u) = \{u\}$.

REFERENCES:


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