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FIXED POINTS FOR NON-SELF MAPPINGS ON CONVEX VECTOR METRIC SPACES

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ABSTRACT

We introduce the concept of convex structure on a vector metric space and obtain some fixed point theorems for a class of non-self mappings satisfying certain contractive conditions in the setting of convex vector metric spaces.

Keywords and phrases: *Convex vector metric space, non-self mapping, fixed point.*

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1. INTRODUCTION:

Deterministic fixed point theorems are generally proved for self mappings only. In1970, W. Takahashi [5] had introduced the concept of convexity in a metric space and obtained some important fixed point theorems previously proved for Banach spaces. Afterwards, Gajić [2] obtained an important fixed point theorem for a class of non-self mappings in Takahashi convex metric spaces. In this paper, our attempt is to introduce the notion of convex structure on a vector metric space with relevant definitions with their properties. Finally, we prove some fixed point theorems for a class of non-self mappings over a subset of a convex vector metric space.

Let S be the vector space of all real sequences $\alpha = \{a_n\}$ and θ stands for the zero vector $\{0\}$. We can define a partial ordering $\leq$ on S by $\alpha \leq \beta$ $(equivalently, \beta \geq \alpha)$ if and only if $a_n \leq b_n$ for all n, where $\alpha = \{a_n\}, \beta = \{b_n\} \in S$. We write $\alpha < \beta$ (equivalently, $\beta > \alpha$) *if and only if* $a_n < b_n$ *for all n.* An element $\alpha \in S$ is called non-negative if $\alpha \geq \theta$. An element $\alpha = \{a_n\} \in S$ is called positive if $a_n > 0$ *for all* n. Also for any $\alpha = \{a_n\} \in S$ and any real number t we define $t\alpha = \{ta_n\}$. If $\alpha = \{a_n\}, \beta = \{b_n\} \in S$ then
$\alpha + \beta = \{a_n + b_n\}$ *and* $\alpha = \beta$ *if* $a_n = b_n$ *for all* n.

For $\alpha^i = \{a_n^i\}_n \in S$; $i = 1, 2, \cdots, k$, we define

$$\max\{\alpha^i : i = 1, 2, \cdots, k\} = \{\max_{1\leq i\leq k}(a_1^i), \max_{1\leq i\leq k}(a_2^i), \cdots, \max_{1\leq i\leq k}(a_n^i), \cdots\}.$$

Clearly, $\max\{\alpha^i : i = 1, 2, \cdots, k\} \in S$.

2. DEFINITIONS AND BASIC FACTS:

In this section, we recall some basic definitions and important results for vector metric spaces that will be needed in the sequel.

Definition: 2.1 [4] Let X be a non empty set. Then a function $V : X \times X \to S$ is called a vector metric on X if the following conditions are satisfied:

(i) $V(x, y) \geq \theta$ *for all* $x, y \in X$ *and* $V(x, y) = \theta$ *if and only if* $x = y$,
(ii) $V(x, y) = V(y, x)$ *for all* $x, y \in X$,
(iii) $V(x, y) \leq V(x, z) + V(z, y)$ *for all* $x, y, z \in X$.

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The pair (X,V) is called a vector metric space. We may verify that $V(x,y)$ is a continuous function of its arguments.

Theorem: 2.2[4] If $V(x,y)=\{d_n(x,y)\}$ be a vector metric then each $d_n(x,y)$ is a quasi metric function; conversely if each $d_n(x,y)$ is a quasi metric and the relations $d_n(x,y)=0$ for all n imply $x=y$, then $V(x,y)=\{d_n(x,y)\}$ is a vector metric.

Remark: 2.3 If (X,d) is a metric space and if $V(x,y)=\{d(x,y),\ d(x,y),\ \cdots\}$, then (X,V) is a vector metric space. So any metric space is a vector metric space and for the converse we can say from Theorem 2.2 that a vector metric space is a quasi metric space.

Example: 2.4 Let $X=R^n$. If we define V on $X\times X$ by

$$V(x,y)=\{|x_1-y_1|,\ |x_2-y_2|,\ \cdots,|x_n-y_n|,\ 0,\ \cdots\}$$

where $x=(x_1,x_2,\cdots,x_n),\ y=(y_1,y_2,\cdots,y_n)\in X$, then (X,V) is a vector metric space.

Example: 2.5[3] Let $X=[0,1]$. If we define V on $X\times X$ by

$$V(x,y)=\left\{|x-y|,\ 0,\ \frac{|x-y|}{1+|x-y|},\ 0,\ 2|x-y|,\ 0,\ \frac{2|x-y|}{1+2|x-y|},\ 0,\ \cdots\right\},$$ then (X,V) is a vector metric space.

Definition: 2.6[4] Let (X,V) be a vector metric space. A sequence $\{x_k\}$ in X is said to converge to an element $x\in X$ i.e. $\lim_k x_k = x$ *if* $V(x_k,x)\to\theta$ *as* $k\to\infty$ which is interpreted as

$$d_n(x_k,x)\to 0 \text{ as } k\to\infty \text{ for all } n \text{ where } V(x_k,x)=\{d_n(x_k,x)\}.$$

Definition: 2.7[3] A sequence $\{x_k\}$ in (X,V) is said to be a vector Cauchy sequence if $V(x_k,x_{k+p})\to\theta$ *as* $k\to\infty$ *for each* p, *i.e.* $d_n(x_k,x_{k+p})\to 0$ *as* $k\to\infty$ *for each* p *and for each* n.

In other words, a sequence $\{x_k\}$ in (X,V) is said to be a vector Cauchy sequence if $V(x_k,x_j)\to\theta$ *as* $k,\ j\to\infty$,

$$i.e.\ d_n(x_k,x_j)\to 0 \text{ as } k,\ j\to\infty \text{ and for each } n.$$

Definition: 2.8[3] A vector metric space (X,V) is said to be complete if every vector Cauchy sequence in X converges to an element in X. Otherwise X is called incomplete.

It is to be noted that every complete metric space may be considered as a complete vector metric space.

Definition: 2.9 Let (X,V) be a vector metric space and $A\subset X$. Then A is called closed if and only if every sequence $\{x_n\}$ in A with $\lim_{n\to\infty} x_n = x$ *implies that* $x\in A$.

It is easy to verify that every closed subset of a complete vector metric space is complete.

Definition: 2.10 Let (X,V) be a vector metric space, $x_0\in X$ *and* r be a positive member of S. Then the set denoted by $B(x_0,r)=\{x\in X:V(x_0,x)<r\}$ is called an open ball centered at x_0 and radius r in X.

Definition 2.11 Let (X,V) be a vector metric space and $A\subset X$. A point $x\in X$ is called a boundary point of A if every open ball centered at x intersects both A *and* $X-A$. The set of all boundary points of A, denoted by ∂A, is called the boundary of A.

Definition: 2.12 A subset A of a vector metric space (X,V) is called bounded if there exists a positive member K in S such that $V(x,y) \le K$ *for all* $x, y \in A$.

Definition: 2.13 Diameter of a bounded set A in a vector metric space (X,V), denoted by $Diam(A)$ *or* $\delta(A)$, is defined as

$$Diam(A) = \sup_{x,y\in A} V(x,y) = \left\{ \sup_{x,y\in A} a_1(x,y),\ \sup_{x,y\in A} a_2(x,y),\ \cdots,\ \sup_{x,y\in A} a_n(x,y), \cdots \right\}$$

$$where\ V(x,y) = \{a_i(x,y)\}\ and\ \ for\ each\ i,\ \sup_{x,y\in A} a_i(x,y) < +\infty\ as\ A\ is\ bounded.$$

In this case we write $\delta(A) < +\infty$.

3. MAIN RESULTS:

Let (X,V) be a vector metric space, and I denote the closed unit interval $[0,1]$ of reals.

Definition: 3.1 A mapping $W : X \times X \times I \to X$ is said to be a convex structure on X if for all $x, y, u \in X$, $\lambda \in I$;

$$V(u, W(x,y,\lambda)) \le \lambda\, V(u,x) + (1-\lambda)\, V(u,y).$$

Then X together with a convex structure W is called a convex vector metric space.

Remark: 3.2 It is observed that $W(x,y,1) = x$ *and* $W(x,y,0) = y$.

Remark: 3.3 Any convex subset of a normed linear space is a convex vector metric space with convex structure $W(x,y,\lambda) = \lambda x + (1-\lambda) y$.

Definition: 3.4 Let (X,V) be a convex vector metric space with a convex structure W. *For* $x, y \in X$, *we define*

$$Seg\,[x,y] = \{W(x,y,\lambda) : \lambda \in [0,1]\}.$$

Clearly $x, y \in Seg\,[x,y]$, since $x = W(x,y,1),\ y = W(x,y,0)$.

Proposition: 3.5 If (X,V) is a convex vector metric space with convex structure W, then for every $x, y \in X$ *and every* $\lambda \in [0,1]$,

$$V(x,y) = V(x, W(x,y,\lambda)) + V(W(x,y,\lambda), y).$$

Proof: *For every* $x, y \in X$ *and every* $\lambda \in [0,1]$, we have

$$\begin{aligned} V(x,y) &\le V(x, W(x,y,\lambda)) + V(W(x,y,\lambda), y) \\ &\le \lambda\, V(x,x) + (1-\lambda)\, V(x,y) + \lambda\, V(y,x) + (1-\lambda)\, V(y,y) \\ &= V(x,y) \end{aligned}$$

This implies that $V(x,y) = V(x, W(x,y,\lambda)) + V(W(x,y,\lambda), y)$.

Proposition: 3.6 Let K be a non-empty closed subset of the convex vector metric space (X,V) with convex structure W continuous on third variable. Let $x \in K$ *and* $y \notin K$. Then there exists a $\lambda^* \in [0,1]$ such that

$$W(x,y,\lambda^*) \in Seg\,[x,y] \cap \partial K$$

where ∂K is the boundary of K.

Proof: Let us consider the set $A = \{q : q \geq 0,\ W(x, y, \eta) \in K \ for\ all\ q \leq \eta \leq 1\}$. Then A is non-empty, since $W(x, y, 1) = x \in K$. We put $\lambda = \inf_{q \in A} q$ and let $\{q_n\}_{n \in N} \subset A$ be a sequence satisfying $\lim_{n \to \infty} q_n = \lambda$. Then, for $n \in N, W(x, y, q_n) \in K$. By using the continuity of W on the third variable and since K is closed, we have

$$W(x, y, \lambda) = \lim_{n \to \infty} W(x, y, q_n) \in K.$$

Clearly $\lambda > 0$ since $W(x, y, 0) = y \notin K$.

Now we prove that $W(x, y, \lambda) \in \partial K$. For any positive member $\varepsilon \in S$, $B(W(x, y, \lambda), \varepsilon) \cap K \neq \Phi$. So we have to show that for any positive member $\varepsilon \in S$, $B(W(x, y, \lambda), \varepsilon) \cap (X - K) \neq \Phi$.

Since $\lambda > 0$, we can find $\lambda_1 \in (0, 1)$ such that $\lambda_1 < \lambda$ and by definition of $\lambda, W(x, y, \lambda_1) \notin K$.
Using continuity of W on third variable, we obtain

$$V(W(x, y, \lambda_1), W(x, y, \lambda)) < \varepsilon.$$

Thus $W(x, y, \lambda_1) \in B(W(x, y, \lambda), \varepsilon) \cap (X - K)$.

This completes the proof.

Theorem: 3.7 Let (X, V) be a complete convex vector metric space with convex structure W which is continuous on third variable, C be a non-empty closed subset of X and $T : C \to X$ be a non-self mapping satisfying the contractive type condition

$$V(T(x), T(y)) \leq q \max \{V(x, y), V(x, T(x)), V(y, T(y)), V(x, T(y)), V(y, T(x))\}$$

for all $x, y \in C$ and $0 < q < 1$. If T has the additional property $T(\partial C) \subset C$ where ∂C is the boundary of C, then T has a unique fixed point in C.

Proof: *For* $x \in \partial C$, *we put* $x_0 = x$. Then $T(\partial C) \subset C$ implies that $T(x_0) \in C$. Let us put $x_1 = T(x_0)$. We define $y_2 = T(x_1)$. If $y_2 \in C$, let us take $x_2 = y_2$. If $y_2 \notin C$, then by Proposition 3.6, there exists $x_2 \in \partial C \cap Seg\,[x_1, y_2]$.

Continuing in this way we can obtain a sequence $\{x_n\}$ such that

$$x_n = T(x_{n-1}),\ if\ T(x_{n-1}) \in C,$$
$$x_n \in \partial C \cap Seg\,[x_{n-1}, y_n],\ if\ y_n = T(x_{n-1}) \notin C.$$

We show that the sequences $\{x_n\}$ *and* $\{T(x_n)\}$ are bounded.

For $n \in N$, *we define*

$$A_n = \{x_i\}_{i=0}^{n-1} \cup \{T(x_i)\}_{i=0}^{n-1}\ and\quad \alpha_n = diam\ A_n.$$

We shall show that $\alpha_n = \max \{V(x_0, T(x_i)) : 0 \leq i \leq n-1\}$.

We now discuss the following possible cases.

Case I: Suppose $\alpha_n = \max\{V(x_i, T(x_j)) : 0 \leq i, j \leq n-1\}$. The case $i = 0$ is trivial. So we suppose that $i \geq 1$.

(1) *If* $x_i = T(x_{i-1}) \in C$, then

$$V(T(x_{i-1}), T(x_j)) \quad \leq q \max \begin{Bmatrix} V(x_{i-1}, x_j), V(x_{i-1}, T(x_{i-1})), V(x_j, T(x_j)), \\ V(x_{i-1}, T(x_j)), V(x_j, T(x_{i-1})) \end{Bmatrix}$$

$$\leq q\,\alpha_n$$

$$So,\ \alpha_n = \max\{V(x_i, T(x_j)) : 0 \leq i,\ j \leq n-1 \text{ and } i \geq 1\} \leq q\alpha_n < \alpha_n, \quad \text{a contradiction}.$$

(2) *If* $x_i \neq T(x_{i-1})$ *i.e.,* $T(x_{i-1}) \notin C$, *then*

$$x_i \in \partial C \cap Seg\,[x_{i-1}, T(x_{i-1})].$$

$$So\ x_i = W(x_{i-1}, T(x_{i-1}), \lambda) \text{ for some } \lambda \in [0,1] \text{ and } x_{i-1} = T(x_{i-2}).$$

Now

$$\begin{aligned} V(x_i, T(x_j)) &= V(W(x_{i-1}, T(x_{i-1}), \lambda), T(x_j)) \\ &\leq \lambda\, V(x_{i-1}, T(x_j)) + (1-\lambda)\, V(T(x_{i-1}), T(x_j)) \\ &\leq \lambda\, V(T(x_{i-2}), T(x_j)) + (1-\lambda)\, V(T(x_{i-1}), T(x_j)) \\ &\leq \lambda\, q\, \alpha_n + (1-\lambda)\, q\, \alpha_n \\ &= q\, \alpha_n. \end{aligned}$$

$$So,\ \alpha_n = \max\{V(x_i, T(x_j)) : 0 \leq i,\ j \leq n-1 \text{ and } i \geq 1\} \leq q\alpha_n < \alpha_n, \text{ a contradiction}.$$

Case II: *Suppose* $\alpha_n = \max\ \{V(x_i, x_j) : 0 \leq i,\ j \leq n-1\}$.

If $x_j = T(x_{j-1})$, we have Case I again.

If $x_j \neq T(x_{j-1})$, *then* $x_j \in \partial C \cap Seg\,[x_{j-1}, T(x_{j-1})]$, $j \geq 2$ *and* $x_{j-1} = T(x_{j-2})$, so this is similar to Case I(2).

By an argument similar to that used above, the case

$$\alpha_n = \max\ \{V(T(x_i), T(x_j)) : 0 \leq i,\ j \leq n-1\}$$

is also impossible. Thus, it must be the case that

$$\alpha_n = \max\ \{V(x_0, T(x_i)) : 0 \leq i \leq n-1\}.$$

If $\alpha_n = \theta$, then x_0 is the fixed point of T, so we can suppose that $\alpha_n \neq \theta$ *for any* $n \in N$.

Further, for $0 \leq i \leq n-1$,

$$\begin{aligned} V(x_0, T(x_i)) &\leq V(x_0, T(x_0)) + V(T(x_0), T(x_i)) \\ &\leq V(x_0, T(x_0)) + q\, \alpha_n. \end{aligned}$$

$$So,\ \alpha_n = \max\ \{V(x_0, T(x_i)) : 0 \leq i \leq n-1\} \leq V(x_0, T(x_0)) + q\, \alpha_n$$

$$\text{i.e.,} \qquad \alpha_n \leq \frac{1}{1-q} V(x_0, T(x_0)) \tag{3.1}$$

If $\alpha_n = \{\alpha_n^k\}_k$ *and* $V(x_0, T(x_0)) = \{d_k(x_0, T(x_0))\}_k$, *then it follows from* (3.1) *that*

$$\alpha_n^k \leq \frac{1}{1-q}\, d_k(x_0, T(x_0)) \text{ for all } k.$$

For fixed k, $\{\alpha_n^k\}_n$ is non-decreasing, so there exists $c_k \in R$ such that

$$c_k = \lim_n \alpha_n^k \text{ and } c_k \leq \frac{1}{1-q}\, d_k(x_0, T(x_0)).$$

So there exists $c = (c_1, c_2, \cdots, c_k, \cdots) \in S$ *such that*

$$c \leq \frac{1}{1-q} V(x_0, T(x_0)).$$

We define

$B_n = \{x_i\}_{i\geq n} \cup \{T(x_i)\}_{i\geq n}$, and $\beta_n = diam\ B_n$ for integer $n \geq 2$. Then by the same technique as given above, one can show that

$\beta_n = \sup\{V(x_n, T(x_j)) : j \geq n\}$ and that if $\beta_n = \{\beta_n^k\}_k$, then for fixed k, $\{\beta_n^k\}_n$ is non-increasing and bounded. So there exists a limit and it must be zero(see [2]). Therefore, $\lim_n \beta_n = \theta$. So $\{x_n\}$ and $\{T(x_n)\}$ are vector Cauchy sequences. Since (X, V) is a complete vector metric space and C is closed, there exists $z \in C$ such that $z = \lim_{n\to\infty} x_n$.

Again, $V(x_n, T(x_n)) \leq \beta_n$, $\beta_n \to \theta$, $n \to \infty$ implies that

$$V(x_n, T(x_n)) \to \theta \ as\ n \to \infty.$$

Now

$$V(T(x_n), z) \leq V(T(x_n), x_n) + V(x_n, z) \to \theta \ as\ n \to \infty,$$

gives that $\lim_{n\to\infty} T(x_n) = z.$

Assume that $z \neq T(z)$. Then

$$V(T(x_n), T(z)) \leq q \max\{V(x_n, z), V(x_n, T(x_n)), V(z, T(z)), V(x_n, T(z)), V(z, T(x_n))\}.$$

For $n \to \infty$, we have

$$V(z, T(z)) \leq q\, V(z, T(z)) < V(z, T(z)),$$

which is a contradiction. Therefore $z = T(z)$. The uniqueness follows from contractive condition.

The following example shows that the contractive type condition in Theorem 3.7 can not be relaxed in order that Theorem 3.7 is true.

Example: 3.8 Let $X = [0, \infty)$ with usual metric d. Then (X, d) is a complete metric space. So (X, V) is a complete vector metric space where $V(x, y) = \{d(x, y), d(x, y), \cdots\}$. We define $W : X \times X \times I \to X$ by

$$W(x, y, \lambda) = \lambda x + (1-\lambda) y$$

for all $x, y \in X$ and $\lambda \in I = [0,1]$.

For $x, y, u \in X$ and $\lambda \in I$, *it is easy to check that*

$$V(u, W(x, y, \lambda)) \leq \lambda\, V(u, x) + (1-\lambda)\, V(u, y).$$

Thus (X, V) is a complete convex vector metric space with convex structure W which is continuous on the third variable.

Let $C = [0,1]$ be the closed unit interval of reals. Then $C \subset X$ and let $T : C \to X$ be a non-self mapping defined by

$$T(x) = \frac{1}{2} \text{ for } 0 \leq x \leq 1 \text{ except } x = \frac{1}{70}, \frac{1}{2^i} (i = 1, 2, 3, \cdots)$$

$$= 0 \text{ for } x = \frac{1}{70},$$

$$\text{and} \quad T\left(\frac{1}{2^i}\right) = 2^{i+1} \quad \text{for } i \geq 1.$$

Then $T(\partial C) \subset C$, but T has no fixed point in C.

We now verify that for $x = \frac{1}{3}$, $y = \frac{1}{70}$, T does not satisfy the contractive condition

$$V(T(x), T(y)) \leq q \max\{V(x, y), V(x, T(x)), V(y, T(y)), V(x, T(y)), V(y, T(x))\} \ for\ 0 < q < 1.$$

Now, $V(T(x),T(y)) = \{d(T(x),T(y)),\, d(T(x),T(y)),\, \cdots\} = \left\{d\left(\frac{1}{2},0\right),\, d\left(\frac{1}{2},0\right),\, \cdots\right\}$

$$= \left\{\frac{1}{2},\ \frac{1}{2},\ \cdots\right\}.$$

Similarly,

$$V(x,T(x)) = \left\{\frac{1}{6},\ \frac{1}{6},\ \cdots\right\};$$

$$V(y,T(y)) = \left\{\frac{1}{70},\ \frac{1}{70},\cdots\right\};$$

$$V(x,y) = \left\{\frac{67}{210},\ \frac{67}{210},\ \cdots\right\};$$

$$V(x,T(y)) = \left\{\frac{1}{3},\ \frac{1}{3},\ \cdots\right\};$$

$$V(y,T(x)) = \left\{\frac{17}{35},\ \frac{17}{35},\ \cdots\right\}.$$

$$\begin{aligned} \textit{Therefore},\quad & q \max\{V(x,y), V(x,T(x)), V(y,T(y)), V(x,T(y)), V(y,T(x))\} \\ & = q\left\{\frac{17}{35},\ \frac{17}{35},\ \cdots\right\} \\ & < \left\{\frac{17}{35},\ \frac{17}{35},\ \cdots\right\} \\ & = \left\{\frac{1}{2}.\frac{34}{35},\frac{1}{2}.\frac{34}{35},\ \cdots\right\} \\ & < \left\{\frac{1}{2},\ \frac{1}{2},\ \cdots\right\} \\ & = V(T(x),T(y)),\ \textit{for } 0 < q < 1. \end{aligned}$$

Theorem: 3.9 Let (X,V) be a complete convex vector metric space with convex structure W which is continuous on third variable, C be a non- empty closed subset of X and $T_i : C \to X$ be a non-self mapping satisfying the condition

$$V(T_i(x), T_i(y)) \le q \max\{V(x,y), V(x,T_i(x)), V(y,T_i(y)), V(x,T_i(y)), V(y,T_i(x))\}$$

for all $x, y \in C$ and $0 < q < 1$ with the additional property $T_i(\partial C) \subset C$. If u_i is the fixed point of T_i for $i = 1, 2, 3, \cdots$ *and* $T(x) = \lim_{i\to\infty} T_i(x)$ *for all* $x \in C$ *with* $T(\partial C) \subset C$, then T has a unique fixed point u in C *iff* $u = \lim_{i\to\infty} u_i$.

Proof: We have $T(x) = \lim_{i\to\infty} T_i(x)$ for $x \in C$. For $x, y \in C$, we have

$V(T_i(x), T_i(y)) \le q \max\{V(x,y), V(x,T_i(x)), V(y,T_i(y)), V(x,T_i(y)), V(y,T_i(x))\}$ where $0 < q < 1$.

As $i \to \infty$,

$V(T(x), T(y)) \le q \max\{V(x,y), V(x,T(x)), V(y,T(y)), V(x,T(y)), V(y,T(x))\}$ where $0 < q < 1$.

Hence by Theorem 3.7, T has a unique fixed point, say $u \in C$. By hypothesis $u_i = T_i(u_i)$ for $i = 1,2,3,\cdots$.

Now for $0 < q < 1$, we have

$$\begin{aligned} V(u,u_i) &= V(T(u),T_i(u_i)) \\ &\le V(T(u),T_i(u)) + V(T_i(u),T_i(u_i)) \\ &\le V(T(u),T_i(u)) + q \max\begin{Bmatrix} V(u,u_i), V(u,T_i(u)), V(u_i,T_i(u_i)), \\ V(u,T_i(u_i)), V(u_i,T_i(u)) \end{Bmatrix} \\ &\le V(T(u),T_i(u)) + q \max\{V(u,u_i), V(u,T_i(u)), V(u,u_i)+V(u,T_i(u))\} \\ &\le V(T(u),T_i(u)) + q\,\{V(u,u_i)+V(T(u),T_i(u))\}, \end{aligned}$$

which implies that,

$$V(u,u_i) \le \left(\frac{1+q}{1-q}\right) V(T(u),T_i(u)) \to 0 \text{ as } i \to \infty .$$

$Therefore,\ u = \lim_{i\to\infty} u_i$.

Conversely, let $u = \lim_{i\to\infty} u_i$, then

$$V(T_i(u),T_i(u_i)) \le q \max\begin{Bmatrix} V(u,u_i), V(u,T_i(u)), V(u_i,T_i(u_i)), \\ V(u,T_i(u_i)), V(u_i,T_i(u)) \end{Bmatrix}$$

which implies that,

$$V(T_i(u),u_i) \le q \max\{V(u,u_i), V(u,T_i(u)), V(u_i,T_i(u))\}.$$

Taking limit as $i \to \infty$, we have

$$V(T(u),u) \le q\, V(u,T(u)).$$

So, $(1-q)\, V(u,T(u)) \le \theta$ *which gives that* $T(u) = u.$

We now prove a fixed point theorem for multivalued mappings. For this purpose we need the following notations.

Notations: Let (X,V) be a vector metric space and let A,B be any two subsets of X. We denote

$$D(A,B) = \inf_{x\in A,y\in B} V(x,y) = \left(\inf_{x\in A,y\in B} a_1(x,y),\ \inf_{x\in A,y\in B} a_2(x,y), \cdots, \inf_{x\in A,y\in B} a_n(x,y), \cdots\right),$$

$$\rho(A,B) = \sup_{x\in A,y\in B} V(x,y) = \left(\sup_{x\in A,y\in B} a_1(x,y),\ \sup_{x\in A,y\in B} a_2(x,y), \cdots, \sup_{x\in A,y\in B} a_n(x,y), \cdots\right)$$

where $V(x,y) = \{a_i(x,y)\}$.

$$BN(X) = \{A : \Phi \ne A \subset X \text{ and } \delta(A) < +\infty\}.$$

Theorem: 3.10 Let (X,V) be a complete convex vector metric space with convex structure W which is continuous on third variable, C be a non-empty closed subset of X and $F: C \to BN(X)$ be a multivalued mapping satisfying the condition

$$\rho(F(x),F(y)) \le q \max\{V(x,y), \rho(x,F(x)), \rho(y,F(y)), D(x,F(y)), D(y,F(x))\} \tag{3.2}$$

for all $x, y \in C$ and $0 < q < 1$. If F has the additional property $F(\partial C) \subset C$ where $F(\partial C) = \cup\{F(x) : x \in \partial C\}$, then F has a unique fixed point u in C and $F(u) = \{u\}$.

Proof: Take $0 < a < 1$ and define a single valued mapping $T: C \to X$ as follows: For each $x \in C$, let $T(x)$ be a point of $F(x)$, which satisfies the condition

$$V(x,T(x)) \ge q^a\ \rho(x,F(x)).$$

Then T has the property $T(\partial C) \subset C$ since $F(\partial C) \subset C$.

Now for every $x, y \in C$, we have

$$\begin{aligned} V(T(x), T(y)) &\le \rho(F(x), F(y)) \\ &\le q \max\{V(x,y), \rho(x, F(x)), \rho(y, F(y)), D(x, F(y)), D(y, F(x))\} \\ &= q\, q^{-a} \max\begin{Bmatrix} q^a V(x,y),\, q^a \rho(x, F(x)),\, q^a \rho(y, F(y)), \\ q^a D(x, F(y)),\, q^a D(y, F(x)) \end{Bmatrix} \\ &\le q^{1-a} \max\begin{Bmatrix} V(x,y), V(x, T(x)), V(y, T(y)), \\ V(x, T(y)), V(y, T(x)) \end{Bmatrix} \end{aligned}$$

for $0 < q < 1$.

Hence by Theorem 3.7, T has a unique fixed point in C. Let $u \in C$ be such that $u = T(u)$.

Clearly $u = T(u)$ implies $u \in F(u)$. Since F satisfies (3.2), $u \in F(u)$ implies

$$\rho(F(u), F(u)) \le q\ \rho(u, F(u)).$$

This may happen only if $F(u) = \{u\}$. Therefore, $u \in C$ is a fixed point of T iff u is a fixed point of F. Thus F has a unique fixed point u in C and $F(u) = \{u\}$.

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