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Z-OPEN SETS AND Z-CONTINUITY IN TOPOLOGICAL SPACES

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ABSTRACT

The aim of this paper is to introduce and study the notion of Z-open sets and Z-continuity. Some characterizations of these notions are presented. Also, some topological operations such as: Z-boundary, Z-exterior, Z-limit… etc, are introduced.

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***Keywords**: Z-open sets , locally-Z closed , Z-boundary, Z-exterior, Z-limit, Z*-nbd and Z-continuity.*

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1. INTRODUCTION

In 1982, Mashhour, Abd EL-Monsef and EL-Deeb [17] introduced preopen sets and pre-continuous mappings in topological spaces. Also, in 1996 Andrijevi'c introduced the notion b-open sets [3]. In 1997, Park, Lee and Son [14] have introduced and studied δ-semiopen in topological spaces. Also, 2008, Ekici [6] introduced e-open sets and e-continuous map in topological spaces. The purpose of this paper is to introduce and study the notion of Z-open sets and Z-continuity. Some topological operations such as: Z-limit, Z-boundary and Z-exterior…atc are introduced. Also, some characterizations of these notions are presented.

2. PRELIMINARIES

A subset A of a topological space (X,τ) is called regular open (resp. regular closed) [16] if A= int(cl(A)) (resp. A= cl(int(A))).The delta interior [17] of a subset A of X is the union of all regular open sets of X contained in A is denoted by δ-int(A) . A subset A of a space X is called δ-open if it is the union of regular open sets . The complement of δ-open set is called δ-closed .Alternatively, a set A of (X,τ) is called δ-closed [17] if A=δ-cl(A), where δ-cl(A) = $\{x\in X: A \cap int(cl(U)) \neq \emptyset, U\in\tau \text{ and } x\in U\}$. Throughout this paper (X, τ) and (Y, σ)(simply X and Y) represent non-empty topological spaces on which no separation axioms are assumed ,unless otherwise mentioned . For a subset A of a space (X, τ), cl(A), int(A) and X \ A denote the closure of A , the interior of A and the complement of A respectively. A subset A of a space (X ,τ) is called α-open [7] (resp. α-open [13], δ-semiopen [14], semiopen [11], δ-preopen [15], preopen [12], b-open [3] or γ-open [4] or sp-open [5], e-open [6], β-open [1] or semi-preopen [2], e*-open [8] or δ-β-open [10]) if $A \subseteq int(cl(\delta\text{-}int(A)))$, (resp. $A \subseteq int(cl(int(A)))$, $A \subseteq cl(\delta\text{-}int(A))$, $A \subseteq cl(int(A))$, $A \subseteq int(\delta\text{-}cl(A))$, $A \subseteq int(cl(A))$, $A \subseteq int(cl(A)) \cup cl(int(A))$, $A \subseteq cl(\delta\text{-}int(A)) \cup int(\delta\text{-}cl(A))$, $A \subseteq cl(int(cl(A)))$, $A\subseteq cl(int(\delta\text{-}cl(A)))$. The complement of a δ-semiopen (resp. semiopen, δ-preopen, preopen) set is called δ-semi-closed (resp.semi-closed, δ-pre-closed, pre-closed). The intersection of all δ-semi-closed (resp. semi-closed, δ-pre-closed, pre-closed) sets containing A is called the δ-semi-closure(resp. semi-closure, δ-pre-closure, pre-closure) of A and is denoted by δ-scl(A) (resp. scl(A), δ-pcl(A) , pcl(A)). The union of all δ-semiopen (resp. semiopen, δ-preopen, preopen) sets contained in A is called the δ-semi-interior (resp. semi-interior, δ-pre-interior, pre-interior) of A and is denoted by δ-sint(A)(resp. sint(A), δ-pint(A), pint(A)).The family of all δ-open (resp. a-open, α-open, δ-semiopen, semiopen, δ-preopen, preopen, b-open, e-open, β-open, e*-open) is denoted by aO(X) (resp. **a**O(X), αO(X), δSO(X), SO(X), δPO(X), PO(X), BO(X), eO(X), βO(X), e*O(X)).

Lemma: 2.1[17]**.** Let A ,B be two subsets of (X, τ) .Then:

(1) A is δ-open if and only if A = δ-int(A),

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(2) $X \setminus (\delta\text{-int}(A)) = \delta\text{-cl}(X \setminus A)$ and $\delta\text{-int}(X \setminus A) = X \setminus (\delta\text{-cl}(A))$,
(3) $cl(A) \subseteq \delta\text{-cl}(A)$ (resp. $\delta\text{-int}(A) \subseteq int(A)$), for any subset A of X,
(4) $\delta\text{-cl}(A \cup B) = \delta\text{-cl}(A) \cup \delta\text{-cl}(B)$, $\delta\text{-int}(A \cap B) = \delta\text{-int}(A) \cap \delta\text{-int}(B)$.

Proposition: 2.1. Let A be a subset of a space (X, τ). Then:
(1) $scl(A) = A \cup int(cl(A))$, $sint(A) = A \cap cl(int(A))$ [11],
(2) $pcl(A) = A \cup cl(int(A))$, $pint(A) = A \cap int(cl(A))$ [12] ,
(3) $\delta\text{-scl}(X \setminus A) = X \setminus \delta\text{-sint}(A)$, $\delta\text{-scl}(A \cup B) \subset \delta\text{-scl}(A) \cup \delta\text{-scl}(B)$[14] ,
(4) $\delta\text{-pcl}(X \setminus A) = X \setminus \delta\text{-pint}(A)$, $\delta\text{-pcl}(A \cup B) \subset \delta\text{-pcl}(A) \cup \delta\text{-pcl}(B)$[15].

Lemma: 2.2[14]**.** The following hold for a subset H of a space (X, τ).
(1) $\delta\text{-pcl}(H) = H \cap cl(\delta\text{-int}(H))$ and $\delta\text{-pint}(H) = H \cap int(\delta\text{-cl}(H))$,
(2) $\delta\text{-sint}(H) = H \cap cl(\delta\text{-int}(H))$ and $\delta\text{-scl}(H) = H \cup int(\delta\text{-cl}(H))$.

Lemma: 2.3. [6] The following hold for a subset H of a space (X, τ). $cl(\delta\text{-int}(H)) = \delta\text{-cl}(\delta\text{-int}(H))$ and $int(\delta\text{-cl}(H)) = \delta\text{-int}(\delta\text{-cl}(H)$,

Definition: 2.1. A function $f:(X, \tau) \rightarrow (Y, \sigma)$ is called precontinuous [12](resp. δ-semicontinuous [9], γ-continuous[4], e-continuous [6]) if $f^{-1}(V)$ is preopen (resp. δ-semiopen, γ-open, e-open) for each $V \in \sigma$.

3. Z-OPEN SETS

Definition: 3.1 A subset A of a topological space (X, τ) is said to be:
(1) a Z-open set if $A \subseteq cl(\delta\text{-int}(A)) \cup int(cl(A))$,
(2) a Z-closed set if $int(\delta\text{-cl}(A)) \cap cl(int(A)) \subseteq A$.

The family of all Z-open (resp. Z-closed) subsets of a space (X, τ) will be as always denoted by ZO(X) (resp. ZC(X)).

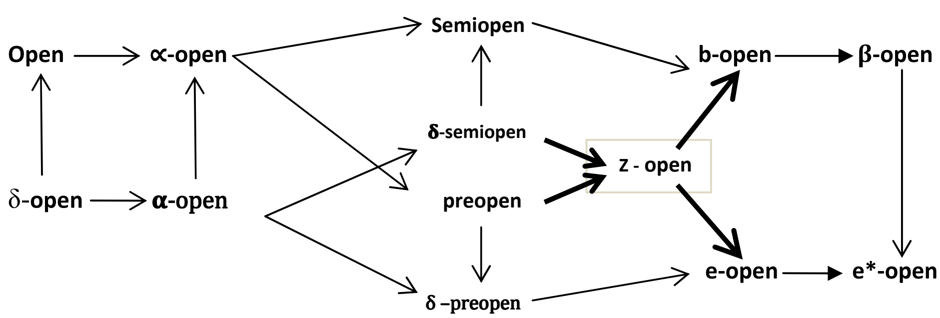
Remark: 3.1 One may notice that
(1) Every δ-semiopen (resp. preopen) set is Z-open,
(2) Every Z-open set is b-open (resp. e-open).

But the converse of the a bovine are not necessarily true in general as shown by the following examples.

Example: 3.1 Let $X = \{a, b, c, d\}$, with topology $\tau = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}$. Then:
(1) A subset $\{a\}$ of X is Z-open but not δ-semiopen,
(2) A subset $\{a, d\}$ of X is b-open but not Z-open,
(3) A subset $\{b, c\}$ of X is e-open but not Z-open.

Example: 3.2 Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$.Then$\{b ,c\}$ is a Z-open set but not preopen.

Remark: 3.2 According to Definition 3.1 and Remark 3.1, the following diagram holds for a subset A of a space X:



Theorem: 3.1 Let (X, τ) be a topological space .Then:
(1) If $A \in \delta O(X)$ and $B \in ZO(X)$, then $A \cap B$ is Z-open,
(2) If $A \in \tau$ and $B \in ZO(X)$, then $A \cap B$ is b-open,
(3) If $A \in aO(X, \tau)$ and $B \in ZO(X, \tau)$, then $A \cap B \in ZO(X, \tau_A)$.

Proof: (1) Suppose that $A \in \delta O(X)$. Then $A = \delta\text{-int}(A)$. Since $B \in ZO(X)$, then
$B \subseteq cl(\delta\text{-int}(B)) \cup int(cl(B))$ and hence $A \cap B \subseteq \delta\text{-int}(A) \cap (cl(\delta\text{-int}(B)) \cup int(cl(B)))$

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$= (\delta\text{-int}(A) \cap cl(\delta\text{-int}(B))) \cup (\delta\text{-int}(A) \cap int(cl(B)))$
$\subseteq cl(\delta\text{-int}(A) \cap (\delta\text{-int}(B))) \cup int(int(A) \cap cl(B))) \subseteq cl(\delta\text{-int}(A \cap B)) \cup int(cl(A \cap B))$.
Thus $A \cap B \subseteq cl(\delta\text{-int}(A \cap B)) \cup int(cl(A \cap B))$.Therefore, $A \cap B$ is Z-open,

(2) It is similar to that of (1),

(3) Since $A \cap B \subset int(cl(\delta\text{-int}(A))) \cap (cl(\delta\text{-int}(B)) \cup int(cl(B)))$
$= (int(cl(\delta\text{-int}(A))) \cap cl(\delta\text{-int}(B))) \cup (int(cl(\delta\text{-int}(A))) \cap int(cl(B)))$
$\subset cl\ (cl(\delta\text{-int}(A)) \cap \delta\text{-int}(B)) \cup int\ (cl(\delta\text{-int}(A)) \cap int(cl(B)))$
$\subset cl\ (cl(\delta\text{-int}(A) \cap \delta\text{-int}(B))) \cup int\ (cl(\delta\text{-int}(A)) \cap int(cl(B)))$ and hence
$A \cap B \subset (A \cap cl(\delta\text{-int}(A) \cap \delta\text{-int}(B))) \cup (A \cap int(cl(\delta\text{-int}(A)) \cap int(cl(B))))$
$\subset cl_A(\delta\text{-int}(A) \cap \delta\text{-int}(B)) \cup int_A(cl(\delta\text{-int}(A) \cap int(cl(B))))$
$\subset cl_A(\delta\text{-int}(A) \cap \delta\text{-int}(B)) \cup int_A(cl(\delta\text{-int}(A) \cap cl(B))) \subset cl_A(\delta\text{-int}(A) \cap \delta\text{-int}(B)) \cup int_A(cl(cl(\delta\text{-int}(A) \cap B)))$. Since $\delta\text{-int}(A) \cap \delta\text{-int}(B) \subset \delta\text{-int}(A) \subset A$ which is δ-open in A , then $A \cap B \subset cl_A\ \delta\text{-int}_A\ (\delta\text{-int}(A) \cap \delta\text{-int}(B)) \cup int_A(A \cap cl(\delta\text{-int}(A) \cap B)) \subset cl_A\ \delta\text{-int}_A\ (A \cap B) \cup int_A\ cl_A(\delta\text{-int}(A) \cap B)) \subset cl_A\ \delta\text{-int}_A\ (A \cap B) \cup int_A\ cl_A(A \cap B)$.

Therefore $A \cap B \in ZO(X, \tau_A)$.

Proposition: 3.1 Let (X, τ) be a topological space .Then the closure of a Z-open subset of X is semiopen.

Proof: Let $A \in ZO(X)$.Then $cl(A) \subseteq cl(cl(\delta\text{-int}(A)) \cup int(cl(A))) \subseteq cl(\delta\text{-int}(A)) \cup cl(int(cl(A))) = cl(int(cl(A)))$. Therefore, cl(A) is semiopen.

Proposition: 3.2 Let A be a Z-open subset of a topological space $(X, \tau)$and $\delta\text{-int}(A) = \emptyset$. Then A is preopen.

Proof: obvious.

Lemma: 3.1 Let (X, τ) be a topological space .Then the following statements are hold .
(1) The union of arbitrary Z-open sets is Z-open,
(2)The intersection of arbitrary Z-closed sets is Z-closed.

Proof: (1) Let $\{A_i, i \in I\}$be a family of Z-open sets. Then $A_i \subseteq cl(\delta\text{-int}(A_i)) \cup int(cl(A_i))$ and hence $\cup_i A_i \subseteq \cup_i (cl(\delta\text{-int}(A_i)) \cup int(cl(A_i))) \subset cl(\delta\text{-int}(\cup_i A_i)) \cup int(cl(\cup_i A_i))$, for all $i \in I$. Thus $\cup_i A_i$ is Z-open.
(2) It follows from (1)

Remark: 3.3 By the following we show that the intersection of any two Z-open sets is not Z-open.

Example: 3.3 Let X = {a, b, c} with topology τ = {∅, {b}, {c}, {b, c}, X}.Then A = {a, c} and B = {a, b} are Z-open sets, but A ∩ B = {a} is not Z-open.

Definition: 3.2 Let (X, τ) be a topological space .Then:
(1) The union of all Z-open sets of X contained in A is called the Z-interior of A and is denoted by Z-int(A),
(2) The intersection of all Z-closed sets of X containing A is called the Z-closure of A and is denoted by Z-cl(A).

Theorem: 3.2 Let A, B be two subsets of a topological space (X, τ). Then the following are hold:
(1) $Z\text{-cl}(X \setminus A) = X \setminus Z\text{-int}(A)$,
(2) $Z\text{-int}(X \setminus A) = X \setminus Z\text{-cl}(A)$,
(3) If $A \subseteq B$, then $Z\text{-cl}(A) \subseteq Z\text{-cl}(B)$ and $Z\text{-int}(A) \subseteq Z\text{-int}(B)$,
(4) $x \in Z\text{-cl}(A)$ if and only if for each a Z-open set U contains x, $U \cap A \neq \emptyset$,
(5) $x \in Z\text{-int}(A)$ if and only if there exist a Z-open set W such that $x \in W \subset A$.
(6) $Z\text{-cl}(Z\text{-cl}(A)) = Z\text{-cl}(A)$ and $Z\text{-int}(Z\text{-int}(A)) = Z\text{-int}(A)$,
(7) $Z\text{-cl}(A) \cup Z\text{-cl}(B) \subset Z\text{-cl}(A \cup B)$ and $Z\text{-int}(A) \cup Z\text{-int}(B) \subset Z\text{-int}(A \cup B)$,
(8) $Z\text{-int}(A \cap B) \subset Z\text{-int}(A) \cap Z\text{-int}(B)$ and $Z\text{-cl}(A \cap B) \subset Z\text{-cl}(A) \cap Z\text{-cl}(B)$.

Proof: (1) It follows from Definition 3.2.

Remark: 3.4 By the following example we show that the inclusion relation in parts (7) and (8) of the above theorem cannot be replaced by equality.

Example 3.4 Let X = {a, b, c, d}, with topology τ = {∅, {a}, {b}, {a, b}, X}.Then
(1) If A ={a, d}, B ={b, d}, then A∪B ={a, b, d} and hence Z-cl(A) = A ,Z-cl(B) = B and Z-cl(A∪B) =X . Thus $Z\text{-cl}(A \cup B) \not\subset Z\text{-cl}(A) \cup Z\text{-cl}(B)$,

(2) If E ={a, b}, F ={a, c}, then E∩F ={a} and hence Z-cl(E) =X , Z-cl(F) = F and Z-cl(E∩F) ={a}.Thus Z-cl(E) ∩ Z-cl(F) ⊄ Z-cl(E∩F).

(3) If M ={c, d} , N ={b, d }, then M∪N ={b, c, d} and hence Z-int(M) =∅ ,Z-int(N) = N and Z-int(M∪N) ={b, c, d}. Thus Z-int(M∪N) ⊄ Z- int(M) ∪ Z- int(N).

Theorem: 3.3 Let (X, τ) be a topological space and A⊂ X .Then A is a Z-open set if and only if A= δ-sint(A)∪ pint(A).

Proof: Let A be a Z-open set. Then A ⊆ cl(δ-int(A)) ∪ int(cl(A)) and hence by Proposition 2.1 and Lemma 2.2, δ-sint(A) ∪ pint(A) = (A ∩ cl(δ-int(A))) ∪ (A ∩ int(cl(A))) = A ∩ (cl(δ-int(A)) ∪int(cl(A))) = A. The Converse it follows from Proposition 2.1 and Lemma 2.2.

Proposition: 3.3. Let (X, τ) be a topological space and A⊂ X . Then A is a Z-closed set if and only if A= δ-scl(A) ∩ pcl(A).

Proof: It follows from Theorem 3.3.

Theorem: 3.4 Let A be a subset of a space(X, τ). Then:
(1) Z-cl(A) = δ-scl(A) ∩ pcl(A),
(2) Z-int(A) = δ-sint(A) ∪ pint(A).

Proof: (1) It is easy to see that Z-cl(A) ⊆ δ-scl(A) ∩pcl(A).Also, δ-scl(A) ∩ pcl(A) = (A∪ int(δ-cl(A)) ∩(A□ cl(int(A)) = A∪ (int(δ-cl(A)) ∩cl(int(A))).Since Z-cl(A) is Z-closed, then Z-cl(A) ⊃ int(δ-cl(Z-cl(A))) ∩ cl(int(Z-cl(A))) ⊃ int(δ-cl(A)) ∩ cl(int(A)).

Thus A ∪ (int(δ-cl(A)) ∩ cl(int(A))) ⊂ A ∪ Z-cl(A) = Z-cl(A) and hence, δ-scl(A) ∩pcl(A) ⊂ Z-cl(A). So, Z-cl(A) = δ-scl(A) ∩pcl(A).

(2) It follows from (1).

Theorem: 3.5 Let A be a subset of a space (X, τ).Then
(1)A is a Z-open set if and only if A =Z-int(A),
(2) A is a Z-closed set if and only if A= Z-cl(A).

Proof: (1) It follows from Theorems 3.3, 3.4.

Lemma: 3.2 Let A be a subset of a topological space (X, τ). Then the following statement are hold :
(1) δ-pint(pcl(A)) = pcl(A) ∩ int(δ-cl(A)),
(2) δ-pcl(pint(A)) = pint(A) ∪ cl(δ-int(A)).

Proof: (1) By Lemma 1.3, δ-pint(pcl(A)) = pcl(A)∩ int(δ-cl(pcl(A))) = pcl(A)∩int(δ-cl(A ∪ int(cl(A)))) = pcl(A) ∩ int(δ-cl(A)).
(2) It follows from (1).

Proposition: 3.4 Let A be a subset of a topological space (X, τ).Then:
(1) Z-cl(A) = A ∪ δ-pint(pcl(A)),
(2) Z-int(A) = A ∩ δ-pcl(pint(A)).

Proof : (1) By Lemma 3.2, A ∪ δ-pint(pcl(A)) = A∪ (pcl(A) ∩ int(δ-cl(A)))
= (A ∪pcl(A)) ∩ (A ∪int(δ-cl(A))) = pcl(A) ∩ δ-scl(A) = Z-cl(A).

(2) It follows from (1).

Theorem: 3.6 Let A be a subset of a topological space (X, τ).Then the following are equivalent :
(1) A is a Z-open set,
(2) A ⊆ δ-pcl(pint(A)),
(3) there exists U ∈ PO(X) such that U⊂ A ⊂ δ-pcl(U),
(4) δ-pcl(A) = δ-pcl(pint(A)).

Proof: (1) → (2) Let A be a Z-open set. Then by Theorem 3.5, A = Z-int(A) and By Proposition 3.4, A = A ∩ δ-pcl(pint(A)) and hence , A ⊆ δ-pcl(pint(A)).

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(2) → (1) Let A$\subseteq$ δ-pcl(pint(A)) .Then by Proposition 3.4, A $\subseteq$ A∩ δ-pcl(pint(A)) = Z-int(A),and hence A = Z-int(A). Thus A is Z-open.

(2) → (3).It follows from putting U= pint(A),

(3) → (2). Let there exists U ∈ PO(X) such that U⊂ A ⊂ δ-pcl(U). Since U ⊂A, then δ-pcl(U) ⊂ δ-pcl(pint(A)), therefore A ⊂ δ-pcl(U) ⊂ δ-pcl(pint(A)),

(2) ↔ (4). It is clear.

Theorem: 3.7 Let A be a subset of a topological space X .Then the following are equivalent:
(1) A is a Z-closed set,
(2) δ-pint(pcl(A)) $\subseteq$ A,
(3) there exists U ∈ PC(X) such that δ-pint(U) ⊂ A ⊂ U,
(4) δ-pint(A) = δ-pint(pcl(A)).

Proof: It follows from Theorem 3.6.

Proposition: 3.5 If A is a Z-open subset of a topological space (X, τ) such that A⊂ B⊂ δ-pcl(A), then B is Z-open .

Proof: It is clear.

Definition: 3.3 A subset A of a topological space (X, τ)is said to be locally Z-closed if A = U ∩ F, where U∈ τ and F ∈ZC(X) .

Theorem: 3.8 Let H be a subset of a space X. Then H is locally Z-closed if and only if H = U ∩ Z-cl(H).

Proof: Since H is a locally Z-closed set, then H = U ∩ F, where U ∈ τ and F ∈ ZC(X) and hence

H$\subseteq$ Z-cl(H) $\subseteq$ Z-cl(F) = F. Thus H $\subseteq$ U ∩ Z-cl(H) $\subseteq$ U ∩ Z-cl(F) =H.

Therefore H = U ∩ Z-cl(H). The Converse is clear.

Theorem: 3.9 Let A be a locally Z-closed subset of a space (X, τ).Then the following statement are hold:
(1) Z-cl(A) \ A is a Z-closed set,
(2)(A∪ (X \ Z-cl(A))) is a Z-open,
(3) A $\subseteq$ Z-int(A ∪ (X \ Z-cl(A))) .

Proof.(1) If A is a locally Z-closed set, then there exists an open set U such that
A=U∩Z-cl(A). Hence, Z-cl(A) \ A = Z-cl(A) \ (U ∩ Z-cl(A)) = Z-cl(A) ∩(X \ (U∩ Z-cl(A)))
= Z-cl(A) ∩((X \ U) ∪ (X \ Z-cl(A))) = Z-cl(A) ∩ (X \ U) which is Z-closed .

(2) Since Z-cl(A) \ A is Z-closed , then X \ (Z-cl(A) \ A)) is a Z-open set. Since
X \ (Z-cl(A) \ A)) = ((X \ Z-cl(A)) ∪ (X ∩ A)) =(A ∪(X \ Z-cl(A))), then A∪ (X \ Z-cl(A))) is Z-open.

(3) It follows from (2).

Definition: 3.4 A subset A of a space (X, τ) is said to be D(c, z) iff int(A) = Z-int(A).

Remark: 3.5 One may notice that the concepts of Z-open and D(c, z) are independent and by we show this the following example.

Example: 3.5 Let X ={a, b, c, d}, with τ ={∅,{a},{c},{a, c},{a, c ,d},{a, b, c}, X} and σ = {∅,{a},{c},{d},{a, c}, {a, d}, {c, d}, {a, c, d}, {a, b, c}, X}.

Then a subset {a, b} is Z-open but not D(c, z) in (X, τ). Also a subset {b, d} is D (c, z) but not Z-open in (X, σ) .

Theorem: 3.10 Let A be a subset of topological space X .Then the following are equivalent:
(1) A is an open set,
(2) A is Z-open and D(c, z).

Proof: Obvious.

4. SOME TOPOLOGICAL OPERATIONS

Definition: 4.1 Let (X, τ) be a topological space and $A \subset X$. Then the Z-boundary of A (briefly, Z-b(A)) is defined by Z-b(A) = Z-cl(A) ∩ Z-cl(X \ A) .

Theorem: 4.1 If A is a subsets of a topological space (X, τ), then the following statement are hold:
(1) Z-b(A) = Z-b(X \ A),
(2) Z-b(A) = Z-cl(A) \ Z-int(A),
(3) Z-b(A) ∩ Z-int(A) = ∅,
(4) Z-b(A) ∪ Z-int(A) = Z-cl(A).

Proof: (1) It is clear.

Theorem: 4.2 If A is a subset of a space X, then the following statement are hold:
(1) A is a Z-open set if and only if A ∩ Z-b(A) = ∅,
(2) A is a Z-closed set if and only if Z-b(A) ⊂ A ,
(3) A is a Z-clopen set if and only if Z-b(A) = ∅ .

Proof: (1) It follows from Theorem 4.1.

Definition: 4.2 Let (X, τ) be a topological space and $A \subset X$. Then the set X \ (Z-cl(A))is called the Z-exterior of A and is denoted by Z-ext(A). A point $p \in X$ is called a Z- exterior point of A, if it is a Z-interior point of X \ A.

Theorem: 4.3 If A and B are two subsets of a space (X, τ), then the following statement are hold:
(1) Z-ext(A) = Z-int (X \ A),
(2) Z-ext(A) ∩ Z-b(A) = ∅,
(3) Z-ext(A) ∪ Z-b(A) = Z-cl(X \ A),
(4){Z-int(A) , Z-b(A) and Z-ext(A)} form a partition of X.
(5) If A ⊂ B , then Z-ext(B) ⊂ Z-ext(A),
(6) Z-ext(A∪ B)⊂ Z-ext(A) ∪ Z-ext(B),
(7) Z-ext(A∩B) ⊃Z-ext(A) ∩ Z-ext(B),
(8) Z-ext(∅) = X and Z-ext(X)= ∅ .

Proof: It follows from Theorems 3.5 and 4.1.

Remark: 4.1 The inclusion relation in parts (6) and (7) of the above theorem cannot be replaced by equality as is shown by the following example.

Example: 4.1 Let X = {a, b, c, d} with topology τ = {∅, {a}, {b, c}, {a, b, c}, X}.

If A= {b, c} and B= {a, c}, then Z-ext(A) ={a, d}, Z-ext(B) = {b}.But Z-ext(A∪ B) =∅,

Therefore Z-ext(A) ∪ Z-ext(B)⊄Z-ext(A∪B) . Also, Z-ext (A∩B) = {a, b, d}, hence, Z-ext (A∩B) ⊄ Z-ext (A) ∩ Z-ext(B).

Definition: 4.3 Let A is a subset of a topological space (X, τ), Then a point $P \in X$ is called a Z-limit point of a set $A \subset X$ if every Z-open set $G \subset X$ containing p contains a point of A other than p. The set of all Z-limit points of A is called a Z-derived set of A and is denoted by Z-d(A) .

Theorem: 4.4 If A and B are two subsets of a space X, then the following statement are hold:
(1) If A⊂ B, then Z-d(A) ⊂ Z-d(B) ,
(2) Z-d(A) ∪ Z-d(B) ⊂ Z-d(A ∪ B),
(3) Z-d(A ∩ B) ⊂ Z-d(A) ∩ Z-d(B),
(4) A is a Z-closed set if and only if it contains each of its Z-limit points,
(5) Z-cl(A) = A ∪ Z-d(A) .

Proof: It is clear.

Definition: 4.4 A subset N of a topological space $(X, \tau)$is called a Z-neighbourhood (briefly, Z-nbd) of a point $P \in X$ if there exists a Z-open set W such that $P \in W \subseteq N$. The class of all Z-nbds of $P \in X$ is called the Z-neighbourhood system of P and denoted by $Z\text{-}N_p$.

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Theorem: 4.5 A subset G of a space X is Z-open if and only if it is Z-nbd, for every point $P \in G$.

Proof: It is clear**.**

Theorem: 4.6 In a topological space (X, τ), let $Z\text{-}N_p$ be the Z-nbd. System of a point $P \in X$, then the following statement are hold:
(1)$Z\text{-}N_p$ is not empty and p belongs to each member of $Z\text{-}N_p$,
(2) Each superset of members of N_p belongs to $Z\text{-}N_p$,
(3) Each member $N \in Z\text{-}N_p$ is a superset of a member $W \in Z\text{-}N_p$, where W is Z-nbd of each point $P \in W$.
(4) The intersection of δ-nbd of a point p and Z-nbd of p is Z-nbd of p.

Proof: (4) It follows from Theorem 3.1.

5. Z-CONTINUOUS MAPPINGS

Definition: 5.1 A mapping $f: (X, \tau) \to (Y, \sigma)$ is called Z-continuous if the inverse image of each member of (Y, σ) is Z-open in (X, τ).

Remark: 5.1 Let $f: (X, \tau) \to (Y, \sigma)$ be mapping from a space (X, τ) into a space (Y, σ), The following diagram hold:

$$\begin{tikzcd}
\delta\text{-Semi continuity} \arrow[dr] & & \gamma\text{-continuity} \\
& Z\text{-continuity} \arrow[ur] \arrow[dr] & \\
\text{pre continuity} \arrow[ur] & & e\text{-continuity}
\end{tikzcd}$$

Now , the following examples show that these implication are not reversible .

Example: 5.1 Let $X = Y = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}$, $\sigma_1 = \{\emptyset, \{b, c\}, Y\}$, $\sigma_2 = \{\emptyset, \{a\}, Y\}$, $\sigma_3 = \{\emptyset, \{a, d\}, Y\}$

(1) The identity $f: (X, \tau) \to (Y, \sigma_1)$ is e-continuous but not Z-continuous,
(2)The identity $f: (X, \tau) \to (Y, \sigma_2)$ is Z-continuous but not δ-semi continuous.
(3)The identity $f: (X, \tau) \to (Y, \sigma_3)$ is γ-continuous but not Z-continuous.

Example: 5.2 Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\sigma = \{\emptyset, \{b, c\}, Y\}$, then the identity $f: (X, \tau) \to (Y, \sigma)$ is Z-continuous but not precontinuous.

Theorem: 5.1 Let $f: (X, \tau) \to (Y, \sigma)$ be a mapping ,then the following statements are equivalent:
(1) f is Z-continuous.
(2) For each $x \in X$ and $V \in \sigma$ containing $f(X)$, there exists $U \in ZO(X)$ containing x such that $f(U) \subset V$,
(3) The inverse image of each closed set in Y is Z-closed in X,
(4) $\mathrm{int}(\delta\text{-cl}(f^{-1}(B))) \cap \mathrm{cl}(\mathrm{int}(f^{-1}(B))) \subset f^{-1}(\mathrm{cl}(B))$, for each $B \subset Y$,
(5) $f^{-1}(\mathrm{int}(B)) \subset \mathrm{cl}(\delta\text{-int}(f^{-1}(B))) \cup \mathrm{int}(\mathrm{cl}(f^{-1}(B)))$, for each $B \subset Y$,
(6) If f is bijective, then $\mathrm{int}(f(A)) \subset f(\mathrm{cl}(\delta\text{-int}(A))) \cup f(\mathrm{int}(\mathrm{cl}(A)))$, for each $A \subset X$,
(7) If f is bijective, then $f(\mathrm{int}(\delta\text{-cl}(A))) \cap f(\mathrm{cl}(\mathrm{int}(A))) \subset \mathrm{cl}(f(A))$, for each $A \subset X$.

Proof: (1) ↔ (2) and (1) ↔ (3) are obvious.

(3) → (4). Let $B \subset Y$, then by (3) $f^{-1}(\mathrm{cl}(B))$ is Z-closed . This means
$f^{-1}(\mathrm{cl}(B)) \supset \mathrm{int}(\delta\text{-cl}(f^{-1}(\mathrm{cl}(B)))) \cap \mathrm{cl}(\mathrm{int}(f^{-1}(\mathrm{cl}(B)))) \supset \mathrm{int}(\delta\text{-cl}(f^{-1}(B))) \cap \mathrm{cl}(\mathrm{int}(f^{-1}(B)))$.

(4) → (5). By replacing $Y \setminus B$ instead of B in (4) ,we have
$\mathrm{int}(\delta\text{-cl}(f^{-1}(Y \setminus B))) \cap \mathrm{cl}(\mathrm{int}(f^{-1}(Y \setminus B))) \subset f^{-1}(\mathrm{cl}(Y \setminus B))$ and therefore $f^{-1}(\mathrm{int}(B)) \subset \mathrm{cl}(\delta\text{-int}(f^{-1}(B))) \cup \mathrm{int}(\mathrm{cl}(f^{-1}(B)))$,

(5) → (6). Follows directly by replacing A instead of $f^{-1}(B)$ in (5) and applying the bijection of f.

(6) → (7). By complementation of (6) and applying the bijective of f , we have
$f(\mathrm{int}(\delta\text{-cl}(X \setminus A))) \cap f(\mathrm{cl}(\mathrm{int}(X \setminus A))) \subset \mathrm{cl}(f(X \setminus A))$ .We obtain the required by replacing A instead of $X \setminus A$.

(7) → (1). Let $V \in \sigma$. Set $W = Y \setminus V$, by (7), we have $f(\mathrm{int}(\delta\text{-cl}(f^{-1}(W)))) \cap f(\mathrm{cl}(\mathrm{int}(f^{-1}(W)))) \subset \mathrm{cl}(ff^{-1}(W)) \subset \mathrm{cl}(W) = W$. So $\mathrm{int}(\delta\text{-cl}(f^{-1}(W))) \cap \mathrm{cl}(\mathrm{int}(f^{-1}(W))) \subset f^{-1}(W)$ implies $f^{-1}(W)$ is Z-closed and therefore $f^{-1}(V) \in ZO(X)$.

Theorem: 5.2 Let $f:(X, \tau) \rightarrow (Y, \sigma)$ be a mapping ,then the following statements are equivalent:
(1) f is Z-continuous,
(2) $Z\text{-}cl(f^{-1}(B)) \subset f^{-1}(cl(B))$, for each $B \subset Y$,
(3) $f(Z\text{-}cl(A)) \subset cl(f(A))$, for each $A \subset X$,
(4) If f is bijective, then $int(f(A)) \subset f(Z\text{-}int(A))$,for each $A \subset X$,
(5) If f is bijective, then $f^{-1}(int(B)) \subset Z\text{-}int(f^{-1}(B))$, for each $A \subset X$.

Proof: (1) → (2). Let $B \subset Y$, $f^{-1}(cl(B))$ is Z-closed in X , then $Z\text{-}cl(f^{-1}(B)) \subset Z\text{-}cl(f^{-1}(cl(B))) = f^{-1}(cl(B))$.

(2) → (3). Let $A \subset X$, then $f(A) \subset Y$, by (2), $f^{-1}(cl(f(A))) \supset Z\text{-}cl(f^{-1}(f(A))) \supset Z\text{-}cl(A)$,

Therefore, $cl(f(A)) \supset f f^{-1}(cl(f(A))) \supset f(Z\text{-}cl(A))$,

(3) → (4). Follows directly by complementation of (3) and applying the bijection of f,

(4) → (5). By replacing $f^{-1}(B)$ instead of A in (4) and using the bijection, we have
$int(B) = int(f f^{-1}(B)) \subset f(Z\text{-}int(f^{-1}(B)))$, therefore $f^{-1}(int(B)) \subset Z\text{-}int(f^{-1}(B))$,

(5) → (1). Let $V \in \sigma$, by (5), $f^{-1}(V) = f^{-1}(int(V)) \subset Z\text{-}int(f^{-1}(V))$, therefore $f^{-1}(V) \in ZO(X)$.

Definition: 5.2 Let X and Y be spaces .A mapping $f : X \rightarrow Y$ is called Z-continuous at a point $P \in X$ if the inverse image of each Z-neighbourhood of f(P) is Z-neighbourhood of P.

Theorem: 5.3 Let X and Y be spaces .Then the mapping $f: X \rightarrow Y$ is Z-continuous if and only if it is Z-continuous at every point $x \in X$.

Proof: Let $H \subseteq Y$ be an open set containing f (p).Then $p \in f^{-1}(H)$, but f is Z-continuous, hence $f^{-1}(H)$ is an Z-open of X containing p, therefore, f is Z-continuous at every point $p \in X$,

On the other hand Suppose that $G \subseteq Y$ is open set for every $p \in f^{-1}(G)$ and f is Z-continuous at every point $p \in X$. Then there exists an Z-open set H containing p such that $p \in G \subseteq f^{-1}(G)$, i.e, $f^{-1}(G) = \cup\{H : p \in f^{-1}(G), H \text{ is Z-open}\}$, then $f^{-1}(G) \subseteq X$ is Z-open. SO, f is Z-continuous.

Remark: 5.2 The composition of two Z-continuous mappings need not be Z-continuous as show by the following example.

Example: 5.3 Let $X = Z = \{a, b, c\}$, $Y = \{a, b, c, d\}$ with topologies $\tau_x = \{\emptyset, \{a\}, X\}$, $\tau_Y = \{\emptyset, \{a, c\}, Y\}$, $\tau_Z = \{\emptyset, \{c\}, \{a, b\}, Z\}$.Let the identity mapping f and $g:Y \rightarrow Z$ defined as g(a)= a, g(b)=g(d)=b and g(c)=c. It is clear that f and g is Z-continuous but $g \circ f$ is not Z-continuous.

Theorem: 5.4 The restriction mapping $f/A :(A, \tau_A) \rightarrow (Y, \sigma)$ of a Z-continuous mapping $f: (X, \tau) \rightarrow (Y, \sigma)$ is Z-continuous if $A \in aO(X, \tau)$.

Proof: Let $U \in \sigma$ and f be a Z-continuous mapping. Then $f^{-1}(U) \in ZO(X, \tau)$.
Since $A \in aO(X, \tau)$, then by Theorem 3.1, $(f/A)^{-1}(U) = A \cap f^{-1}(U) \in ZO(X, \tau)$, therefore f /A is Z-continuous.

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