ON A GENERALIZED COMMON FIXED POINT THEOREM FOR WEAK ** COMMUTING MAPS IN 2-METRIC SPACES

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ABSTRACT

In this present research article, we prove the existence of a common fixed point for four self mappings defined on a complete 2- metric space through weak ** commutativity. The results of kubaik [3] are generalized in this work.

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Key words: fixed point,2-metric space, weak** commutativity, weak* commutativity, weak commutativity.

INTRODUCTION

The notion of 2-metric space was introduced by Gahler [1] in 1963 as a generalization of area function for Euclidean triangles. Many fixed point theorems were established by various authors like Brouwer, Banach, Schauder etc. A point \( x \in X \) is said to be a fixed point of a self-map \( f : X \rightarrow X \) if \( f(x) = x \), where \( X \) is a non-empty set. Theorems concerning fixed points of self-maps are known as fixed point theorems. Most of the fixed point theorems were proved for contraction mappings. It is well known that every contraction on a metric space is continuous. The converse is not necessarily true. The identity mapping on \([0, 1]\) simply serves the counter example.

In this present work we consider commuting self-maps on a 2-metric space. Let \( T_1 \) and \( T_2 \) be two mappings from a metric space \((X, d)\) into itself. \( T_1 \) and \( T_2 \) are said to commute if \( T_1 T_2 x = T_2 T_1 x \), for all \( x \) in \( X \). Sessa [5] introduced the concept of weak commutativity in metric spaces. In subsequent years the condition of weak commutativity was again made weaker. Weak* commutativity was introduced in metric space. In recent years weak** commutativity has been introduced and some theorems have been established. The existence of fixed point for weak**commutative self maps in 2-metric space are studied.

In this research article we present the concepts of weak commutativity, weak* commutativity and weak** commutativity in 2-metric space. Our results generalize the result of kubaik [3]

1. PRELIMINARIES

In this section we define weak**commutativity, weak* commutativity and weak commutativity. We also present an example to establish the fact that weak** commutativity does not imply commutativity.

1.1 Definition: Two self-maps \( A \) and \( S \) of a 2-metric space \((X, d)\) are called weak** commutative

\( (1) \ A(X) \subset S(X) \) and

\( (1.1) \ d\left(A^2S^2x, A^2x, a\right) \leq d\left(A^2S, xS A^2x, a\right) \leq d\left(AS, xS A^2x, a\right) \leq d\left(S^2A, A^2x, a\right) \)

for all \( x, a \) in \( X \).

1.2 Definition: Two self-maps \( A \) and \( S \) define on a 2-metric space \((X, d)\) are said to be weak* commutative if

\( (1) \ A(X) \subset S(X) \)

\( (1.1) \ d\left(A^2S^2x, S^2A^2x, a\right) \leq d\left(S^2x, A^2x, a\right) \)

for all \( x, a \) in \( X \).
1.3 Definition: Two self-maps \( A \) and \( S \) define on a 2-metric space \((X, d)\) are said to be **weak commutative** if

\[
(1) \quad d(AX, SAx, a) \leq d(Ax, Sx, a)
\]

for all \( x, a \in X \).

1.4 Example: Let \( X = [0,1] \) with 2-metric \( d \)-defined as

\[
d(x, y, z) = \min \{ |x - y|, |y - z|, |z - x| \}
\]

Let \( A \) and \( S \) be defined as

\[
A(x) = x + 4 \quad \text{and} \quad S(x) = \frac{x}{2}
\]

for all \( x \in X \). Then \( A \) and \( S \) are weak commutative but not weak commutative.

2. GENERALIZED FIXED POINT THEOREM

2.1 Theorem: Let \( A, B, S \) and \( T \) be four self-mappings of a complete 2-metric space \((X, d)\) such that

\[
(A^2, B^2) : X \to S^2(X) \cap T^2(X)
\]

and satisfy

\[
(1) \quad d(A^2x, B^2y, a) \leq c \max \left\{ d(S^2x, T^2y, a), d(S^2x, A^2x, a), d(T^2y, B^2y, a) \right\}
\]

For all \( x, y, a \in X \), where \( 0 < c < 1 \). If one of \( A, B, S \) and \( T \) is continuous and if \( A \) and \( B \) weak commutative with \( S \) and \( T \) respectively, then \( A, B, S \) and \( T \) have a unique common fixed point.

Proof: Let \( x_0 \) be an arbitrary point of \( X \) and

Since \( A^2(X) \) and \( B^2(X) \) are contained in \( S^2(X) \cap T^2(X) \),

We can define sequence \( \{x_n\} \) in \( X \) such that

\[
S^2x_{2n-1} = B^2x_{2n-2} \quad \text{and} \quad T^2x_{2n} = A^2x_{2n-1} \quad \text{for} \quad n = 1, 2, 3, \ldots
\]

By (i) we have

\[
d(S^2x_{2n-1}, T^2x_{2n-1}, a) = d(B^2x_{2n-2}, A^2x_{2n-1}, a) = d(A^2x_{2n-1}, B^2x_{2n-2}, a)
\]

\[
\leq c \max \left\{ d(S^2x_{2n-1}, T^2x_{2n-2}, a), d(S^2x_{2n-1}, A^2x_{2n-1}, a), d(T^2x_{2n-2}, B^2x_{2n-2}, a) \right\}
\]

\[
\leq c \max \left\{ \frac{1}{2} \left[ d(S^2x_{2n-1}, B^2x_{2n-2}, a) + d(T^2x_{2n-2}, A^2x_{2n-1}, a) \right] \right\}
\]

Thus

\[
d(S^2x_{2n-1}, T^2x_{2n-2}, a) \leq cd(S^2x_{2n-1}, T^2x_{2n-2}, a)
\]

For \( n = 1, 2, 3, \ldots \) and all \( a \in X \).

By induction we obtain

\[
d(S^2x_{2n-1}, T^2x_{2n-2}, a) \leq c^{2n-1}d(S^2x_1, T^2x_0, a) \quad \text{...........(2)}
\]

\[
d(S^2x_{2n+1}, T^2x_{2n}, a) \leq c^{2n-1}d(S^2x_1, T^2x_2, a) \quad \text{...........(3)}
\]

For \( n = 1, 2, 3, \ldots \) and all \( a \in X \).

Thus

\[
d(S^2x_{2n-1}, S^2x_{2n+1}, a) \leq d(S^2x_{2n-1}, S^2x_{2n+1}, T^2x_{2n}) + d(S^2x_{2n-1}, T^2x_{2n}, a) + d(T^2x_{2n}, S^2x_{2n+1}, a)
\]

\[
\leq \cdots c^{2n-1}d(S^2x_1, T^2x_0, a) + c^{2n-1}d(S^2x_1, T^2x_2, a)
\]

\[
\leq 0 + c^{2n-1} \left[ d(S^2x_1, T^2x_0, a) + cd(S^2x_1, T^2x_0, a) \right]
\]

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Since $d\left(S^2x_{2n-1}, S^2x_{2n+1}, T^2x_{2n}\right) = 0$ and $d\left(S^2x_1, T^2x_2, a\right) < c d\left(S^2x_1, T^2x_0, a\right)$

$d\left(S^2x_{2n-1}, S^2x_{2n+1}, a\right) \leq c^{2n-1}(1+c) d\left(S^2x_1, T^2x_0, a\right)$

Similarly $d\left(S^2x_{2n+1}, S^2x_{2n+3}, a\right) \leq c^{2n+1}(1+c) d\left(S^2x_1, T^2x_0, a\right)$

$d\left(S^2x_{2n+3}, S^2x_{2n+5}, a\right) \leq c^{2n+3}(1+c) d\left(S^2x_1, T^2x_0, a\right)$ and so on

Since $0 < c < 1$

c^{2n-1} \to 0$ as $n \to \infty$

So that $\left\{S^2x_{2n-1}\right\}$ is a Cauchy sequence in $X$, thus converges to a point $u$ in $X$

Consider

$d\left(T^2x_{2n}, u, a\right) \leq d\left(T^2x_{2n}, u, S^2x_{2n-1}\right) + d\left(T^2x_{2n}, S^2x_{2n-1}, a\right) + d\left(S^2x_{2n-1}, u, a\right)$

$\leq d\left(T^2x_{2n}, u, u\right) + d\left(T^2x_{2n}, u, a\right) + d\left(u, u, a\right)$

$d\left(T^2x_{2n}, u, a\right) \leq d\left(T^2x_{2n}, u, a\right)$

Which is a contradiction

$d\left(T^2x_{2n}, u, a\right) = 0$ for every $a$ in $X$

Therefore $\left\{T^2x_{2n}\right\}$ converges to $u$

Thus

$\lim_{n \to \infty} S^2x_{2n-1} = \lim_{n \to \infty} B^2x_{2n-2} = \lim_{n \to \infty} T^2x_{2n} = \lim_{n \to \infty} A^2x_{2n-1} = u$

Now suppose that $S$ is continuous, we have the sequence $\left\{A^2Sx_{2n-1}\right\}$ converges to $Su$

I.e. $\lim_{n \to \infty} A^2Sx_{2n-1} = u$

Since $A$ and $S$ are weak** commute

We have $d\left(A^2Sx, SA^2x, a\right) \leq d\left(A^2x, S^2x, a\right)$ for all $a \in X$

Put $x = x_{2n-1}$

$d\left(A^2Sx_{2n-1}, SA^2x_{2n-1}, a\right) \leq d\left(A^2x_{2n-1}, S^2x_{2n-1}, a\right)$

$\lim_{n \to \infty} d\left(A^2Sx_{2n-1}, SA^2x_{2n-1}, a\right) \leq \lim_{n \to \infty} d\left(A^2x_{2n-1}, S^2x_{2n-1}, a\right)$

$\leq d(u, u, a) = 0$

$\lim_{n \to \infty} d\left(A^2Sx_{2n-1}, SA^2x_{2n-1}, a\right) = 0$

Also $\lim_{n \to \infty} A^2x_{2n-1} = u$

Since $S$ is continuous

$\lim_{n \to \infty} SA^2x_{2n-1} = Su$

$\lim_{n \to \infty} d\left(A^2Sx_{2n-1}, Su, a\right) = 0 \forall a \in X$

$\Rightarrow \left\{A^2Sx_{2n-1}\right\}$ is convergent to $Su$

Since $B^2x_{2n} = u$ and $S$ is continuous

$\lim_{n \to \infty} SB^2x_{2n} = Su$

$\lim_{n \to \infty} SS^2x_{2n+1} = Su$
Since \( S^2x_{2n-1} = B^2x_{2n-2} \Rightarrow S^2x_{2n+1} = B^2x_{2n} \)
\[
\lim_{n \to \infty} S^3x_{2n+1} = Su
\]

Now we have
\[
d \left( A^2Sx_{2n-1}, B^2x_{2n}, a \right) \leq c \max \left\{ \frac{1}{2} \left[ d \left( S^3x_{2n+1}, T^2x_{2n}, a \right) + d \left( S^3x_{2n+1}, A^2Sx_{2n+1}, a \right) , d \left( T^2x_{2n}, B^2x_{2n}, a \right) \right] \right\}
\]

Letting \( n \to \infty \) \( d(su, u, a) = 0 \forall a \in X \)
\( \Rightarrow Su = u \)

Hence \( u \) is a fixed point of \( S \)
\( \Rightarrow S^2u = Su = u \)

Consider
\[
d \left( A^2u, B^2x_{2n}, a \right) \leq c \max \left\{ \frac{1}{2} \left[ d \left( S^2u, B^2x_{2n}, a \right) + d \left( T^2x_{2n}, A^2u, a \right) \right] \right\}
\]

Letting \( n \to \infty \) \( d(A^2u, u, a) = 0 \forall a \in X \)
\( \Rightarrow A^2u = u \)

Since \( B^2(x) \subseteq T^2(x) \) and \( u \in X \)

We have \( B^2u \in B^2(x) \)
\( \Rightarrow u \in B^2(x) \)
\( \Rightarrow u \in T^2(x) \)

There exist \( u_1 \in X \) Such that \( u = T^2(u_1) \)

Then \( d \left( u, B^2u_1, a \right) = d \left( A^2u, B^2u_1, a \right) \leq c \max \left\{ \frac{1}{2} \left[ d \left( S^2u, B^2u_1, a \right) + d \left( T^2u_1, A^2u, a \right) \right] \right\} \)
\[
d \left( u, B^2u_1, a \right) = 0
\]
\( \Rightarrow B^2u_1 = u \)

Therefore \( T^2u_1 = B^2u_1 = u \)

Since \( B \) and \( T \) are Weak** commutative
\[
d(B^2T^2x, T^2B^2x, a) \leq d(B^2Tx, TB^2x, a) \leq d(BT^2x, T^2Bx, a) \leq d(BTx, TBx, a) \leq d(B^2x, T^2x, a) \forall x, a \in X
\]
Put $x = u_i$

\[ d(B^2 T^2 x, T^2 B^2 x, a) \leq d(B^2 Tu_i, TB^2 u_i, a) \leq d(BT^2 u_i, T^2 Bu_i, a) \leq d(BT u_i, TB u_i, a) \leq d(B^2 u_i, T^2 u_i, a) \forall a \in X \]

\[ d(u, T^2 u, a) = 0 \]

\[ \Rightarrow T^2 u = u \forall a \in T \]

Hence $A^2 u = B^2 u = S^2 u = T^2 u = u$

Since $A^2 u = u$

\[ A(A^2 u) = Au \]

\[ A^3 u = Au \]

Then we have

\[ d(u, Au, a) = d(Au, u, a) = d(A^3 u, B^2 u, a) = d(A^2 Au, B^2 u, a) \]

\[ \leq c \max \left\{ d(S^2 Au, T^2 u, a), d(S^2 Au, A^3 u, a), d(T^2 u, B^2 u, a) \right\} \]

\[ \leq \frac{1}{2} \left[ d(S^2 Au, B^2 u, a) + d(T^2 u, A^3 u, a) \right] \]

\[ \Rightarrow d(u, Au, a) = 0 \]

\[ \Rightarrow Au = u \]

\[ \therefore Su = Au = u \]

Since B and T are weak** commutative

\[ d(B^2 T^2 u, T^2 B^2 u, a) \leq d(B^2 Tu, TB^2 u, a) \leq d(BT^2 u, T^2 Bu, a) \leq d(BT u, TB u, a) \leq d(B^2 u, T^2 u, a) \]

\[ d(u, u, a) \leq d(B^2 Tu, Tu, a) \leq d(Bu, T^2 Bu, a) \leq d(BTu, TB u, a) \leq d(u, u, a) \]

\[ 0 \leq d(B^2 Tu, Tu, a) \leq d(Bu, T^2 Bu, a) \leq d(BTu, TB u, a) \leq 0 \forall a \in X \]

\[ d(B^2 Tu, Tu, a) = 0 \Rightarrow B^2 Tu = Tu \]

\[ d(Bu, T^2 Bu, a) = 0 \Rightarrow T^2 Bu = Bu \]

\[ d(BTu, TB u, a) = 0 \Rightarrow BT u = TB u \]

\[ d(u, Tu, a) = d(A^3 u, B^2 Tu, a) \]

\[ \leq \frac{1}{2} \left[ d(S^2 u, B^2 Tu, a) + d(T^2 Tu, A^3 u, a) \right] \]

\[ \Rightarrow d(u, Tu, a) = 0 \]

\[ \therefore Tu = u \]

Since $B^2 u = u$

\[ BB^2 u = Bu \]

\[ B^3 u = Bu \]

We have

\[ d(u, Bu, a) = d(A^3 u, B^3 u, a) = d(A^2 u, B^2 Bu, a) \]

\[ \leq \frac{1}{2} \left[ d(S^2 u, B^2 Bu, a) + d(T^2 Bu, A^2 u, a) \right] \]
\[ d(u, Bu, a) = 0 \]
\[ \Rightarrow Bu = u \]
\[ \therefore Au = Su = Tu = Bu = u \]

Hence \( u \) is a common fixed point of \( A, S, T \) and \( B \)

Now we prove that \( u \) is a Unique fixed point of \( A, S, T \) and \( B \)

Suppose that there is a point \( v \in X \) such that
\[ A v = S v = B v = T v = v \]
\[ A^2 v = S^2 v = B^2 v = T^2 v = v \]

Then
\[ d(u, v, a) = d(A^2 u, B^2 v, a) \leq c \max \left\{ \frac{d(S^2 u, T^2 v, a) + d(S^2 u, A^2 u, a) + d(T^2 v, B^2 v, a)}{2}, d(S^2 u, B^2 v, a) + d(T^2 v, A^2 u, a) \right\} \]
\[ d(u, v, a) = 0 \]
\[ \therefore u = v \]

So, we proved that \( u \) is the unique common fixed point of \( A, B, S \) and \( T \).

### 2.2 Corollary:
Let \( S, T : X \to X \) and either \( S \) or \( T \) be continuous. Then \( S \) and \( T \) have a common fixed point \( z \) if there exists two self mappings \( A, B \) of \( X \) and \( A \) (resp. \( B \)) weakly commute with \( S \) (resp. \( T \)). Further \( z \) is the unique common fixed point of \( A, B, S \) and \( T \).

**Proof:** As \( A \) (resp. \( B \)) weakly commutes with \( S \) (resp. \( T \)). But weakly commutativity implies weak **commutativity.** Thus the proof of theorem [2.1] work.

### Remark:
1. The corollary (2.2) generalizes theorem 1 of kubaik [3] where continuity of both \( S \) and \( T \) and commutative of both \( A \) and \( B \) with \( S \) and \( T \) are assumed. But assumptions in corollary (2.2) are much weaker than that of kubaik [3] and thus theorem (2.1) is more general than kubaik [3].

### 2.3 Theorem:
Let \( A, B, S \) and \( T \) be four self-mappings of a complete 2-metric space \((X, d)\) such that
\[ (1) \ A^2(X) \subset T^2(X) \text{ and } B^2(X) \subset S^2(X), \]
\[ (11) \ d(A^2 x, B^2 y, a) \leq c \max \{d(S^2 x, T^2 x, a), d(S^2 x, A^2 x, a), d(T^2 y, B^2 y, a), d(S^2 x, B^2 y, a) + d(T^2 x, A^2 y, a)\} \]

For all \( x, y, a \) in \( X \), where \( 0 < c < 1 \). if one of \( A, B, S \) and \( T \) is continuous and if \( A \) and \( B \) weak**commute with \( S \) and \( T \) respectively, then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

### REFERENCES

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