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ON A GENERALIZED COMMON FIXED POINT THEOREM FOR WEAK ** COMMUTING MAPS IN 2-METRIC SPACES

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ABSTRACT

*In this present research article, we prove the existence of a common fixed point for four self mappings defined on a complete 2- metric space through weak ** commutativity. The results of kubaik [3] are generalized in this work.*

AMS Subject Classification: *47H10, 54H25.*

Key words: *fixed point,2- metric space , weak** commutativity, weak* commutativity, weak commutativity.*

INTRODUCTION

The notion of 2-metric space was introduced by *Gahler* [1] in 1963 as a generalization of area function for Euclidean triangles. Many fixed point theorems were established by various authors like *Brouwer*, *Banach*, *Schauder*etc. A point $x \in X$ is said to be a *fixed point* of a self-map $f : X \to X$ if $f(x) = x$, where X is a non- empty set. Theorems concerning fixed points of self-maps are known as fixed point theorems. Most of the fixed point theorems were proved for contraction mappings. It is well known that every contraction on a metric space is continuous. The converse is not necessarily true. The identity mapping on [0, 1] simply serves the counter example.

In this present work we consider commuting self-maps on a 2-metric space. Let T_1 and T_2be two mappings from a metric space (X, d) into itself. T_1 and T_2 are said to commute if $T_1T_2x = T_2T_1x$, for all x in X. *Sessa* [5] introduced the concept of weak commutativity in metric spaces. In subsequent years the condition of weak commutativity was again made weaker. Weak* commutativity was introduced in metric space. In recent years weak** commutativity has been introduced and some theorems have been established. The existence of fixed point for weak**commutative self maps in 2-metric space are studied.

In this research article we present the concepts of weak commutativity, weak* commutativity and weak** commutativityin 2-metric space. Our results generalize the result of *kubaik* [3]

1. PRELIMINARIES

In this section we define weak**commutativity, weak* commutativity and weak commutativity. We also present an example to establish the fact that weak** commutativity does not imply commutativity.

1.1 Definition: Two self-maps A and S of a 2-metric space (X, d) are called *weak** commutative*

(1) $A(X) \subset S(X)$ and

$(11)\, d\left(A^2S^2x,\ S^2A^2x,\ a\right) \le d\left(A^2Sx,\ SA^2x,\ a\right) \le d\left(AS^2x,\ S^2Ax,\ a\right) \le d\left(ASx,\ SAx,\ a\right) \le d\left(S^2x,\ A^2x,\ a\right)$

for all x, a in X.

1.2 Definition : Two self-maps A and S define on a 2-metric space (X, d) are said to be *weak* commutative* if

$(1)\ A(X) \subset S(X)$

$(11)\ d\left(A^2S^2x,\ S^2A^2x,\ a\right) \le d\left(S^2x,\ A^2x, a\right)$

for all x, a in X.

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1.3 Definition: Two self-maps A and S define on a 2-metric space (X, d) are said to be *weak commutative* if

$(1)\ A(X) \subset S(X)$

$(11)\ d\ (ASx\ , SAx\ , a)\ \le\ d(Ax,\ Sx,\ a)$

for all x, a in X.

1.4 Example: let $X = [0,1]$ with 2-metric d-defined as

$d(x,\ y,\ z)\ =\ min\ \{|x-y|,\ |y-z|,\ |z-x|\}$

Let A and S be defined as

$Ax = \dfrac{x}{x+4}$ and $Sx = \dfrac{x}{2}$ for all x in X

Then A and S are weak** commutative but not weak commutative.

2. GENERALIZED FIXED POINT THEOREM

2.1 Theorem: Let A, B, S and T be four self-mappings of a complete 2-metric space (X, d) such that $A^2,\ B^2 : X \to\ S^2(X)\ \cap T^2(X)$ and satisfy

(1) $d\left(A^2x, B^2y, a)\right)$

$$\le c\ ma\ x\left\{d\left(S^2x,\ T^2y,\ a\right), d\left(S^2x, A^2x, a\right), d\left(T^2y, B^2y, a\right)\frac{1}{2}\left[d\left(S^2x,\ B^2y, a\right) + d\left(T^2y,\ A^2x, a\right)\right]\right\}$$

For all x, y, a in X, where $0 < c < 1$. If one of A, B, S and T is continuous and if A and B weak** commutative with S and T respectively, then A, B, S and T have a unique common fixed point.

Proof: Let x_0 be an arbitrary point of X and

Since $A^2(X)$ and $B^2(X)$ are contained in $S^2(X) \cap T^2(X)$,

We can define sequence $\{x_n\}$ in X such that

$S^2x_{2n-1} =\ B^2x_{2n-2}$ and $T^2x_{2n} =\ A^2x_{2n-1}$ for $n\ =\ 1, 2, 3, \ldots\ldots\ldots\ldots$

By (i) we have

$$d\left(S^2x_{2n-1,},\ T^2x_{2n}, a\right) = d\left(B^2x_{2n-2},\ A^2x_{2n-1}, a\right) = d\left(A^2x_{2n-1},\ \mathrm{B}^2x_{2n-2}, a\right)$$

$$\le c\ max\begin{cases} d\left(S^2x_{2n-1}, T^2x_{2n-2}, a\right), d\left(S^2x_{2n-1}, A^2x_{2n-1}, a\right), d\left(T^2x_{2n-2}, B^2x_{2n-2}, a\right) \\ \frac{1}{2}\left[d\left(S^2x_{2n-1}, B^2x_{2n-2}, a\right) + d\left(T^2x_{2n-2}, A^2x_{\,2n-1}, a\right)\right] \end{cases}$$

$$\le c\ ma\ x\left\{d\left(S^2x_{2n-1}, T^2x_{2n-2}, a\right), d\left(S^2x_{2n-1}, T^2x_{2n}, a\right), \frac{1}{2}\left[d\left(T^2x_{2n-2}, T^2x_{2n}, a\right)\right]\right\}$$

Thus $d\left(S^2x_{2n-1,},\ T^2x_{2n}, a\right) \le cd\left(S^2x_{2n-1,},\ T^2x_{2n-2}, a\right)$

For $n =\ 1,\ 2,\ 3, \ldots\ldots\ldots\ldots\ldots\ldots$ and all $a\ \varepsilon\ X$.

By induction we obtain

$d\left(S^2x_{2n-1,},\ T^2x_{2n}, a\right) \le c^{2n-1}d\left(S^2x_{1,},\ T^2x_0, a\right)$............(2)

$d\left(S^2x_{2n+1,},\ T^2x_{2n}, a\right) \le c^{2n-1}d\left(S^2x_{1,},\ T^2x_2, a\right)$................(3)

For $n =\ 1,\ 2,\ 3, \ldots\ldots\ldots\ldots\ldots\ldots$ and all $a\ \varepsilon\ X$

Thus $d\left(S^2x_{2n-1,}, \mathrm{S}^2x_{2n+1}, a\right) \le d\left(S^2x_{2n-1,},\ \mathrm{S}^2x_{2n+1}, T^2x_{2n}\right) + d\left(S^2x_{2n-1},\ T^2x_{2n}, a\right) + d\left(T^2x_{2n,},\ \mathrm{S}^2x_{2n+1}, a\right)$

$$\le d\left(S^2x_{2n-1,},\ \mathrm{S}^2x_{2n+1}, T^2x_{2n}\right) + c^{2n-1}d\left(S^2x_{1,},\ T^2x_0, a\right) + c^{2n-1}d\left(S^2x_{1,},\ \mathrm{T}^2x_2, a\right)$$

$$\le 0 + c^{2n-1}\left[d\left(S^2x_{1,},\ T^2x_0, a\right) + cd\left(S^2x_{1,},\ \mathrm{T}^2x_0, a\right)\right]$$

Since $d\left(S^2x_{2n-1,},\ S^2x_{2n+1},T^2x_{2n}\right)=0$ and $d\left(S^2x_{1,},\ T^2x_2,a\right)\prec cd\left(S^2x_{1,},\ T^2x_0,a\right)$
$d\left(S^2x_{2n-1,},\ S^2x_{2n+1},a\right)\le c^{2n-1}(1+c)d\left(S^2x_{1,},\ T^2x_0,a\right)$

Similarly $d\left(S^2x_{2n+1,},\ S^2x_{2n+3},a\right)\le c^{2n+1}(1+c)d\left(S^2x_{1,},\ T^2x_0,a\right)$
$d\left(S^2x_{2n+3,},\ S^2x_{2n+5},a\right)\le c^{2n+3}(1+c)d\left(S^2x_{1,},\ T^2x_0,a\right)$ and So on
Since $0<c<1$
$c^{2n-1}\to 0$ as $n\to\infty$

So that $\left\{s^2x_{2n-1}\right\}$ is a Cauchy sequence in X, thus converges to a point **u** in X
Consider $d\left(T^2x_{2n},\mathrm{u},a\right)\le d\left(T^2x_{2n},u,\ \mathrm{S}^2x_{2n-1}\right)+d\left(T^2x_{2n},S^2x_{2n-1},a\right)+d\left(\mathrm{S}^2x_{2n-1},u,a\right)$
$\le d\left(T^2x_{2n},u,\ \mathrm{u}\right)+d\left(T^2x_{2n},u,a\right)+d\left(u,u,a\right)$
$d\left(T^2x_{2n},\mathrm{u},a\right)\le d\left(T^2x_{2n},u,\ \mathrm{a}\right)$

Which is a contradiction
$d\left(T^2x_{2n},\mathrm{u},a\right)=0$ for every a in X
Therefore $\left\{T^2x_{2n}\right\}$ converges to **u**
Thus $\lim\limits_{n\to\infty}s^2x_{2n-1}=\lim\limits_{n\to\infty}B^2x_{2n-2}=\lim\limits_{n\to\infty}T^2x_{2n}=\lim\limits_{n\to\infty}A^2x_{2n-1}=u$

Now suppose that S is continuous, we have the sequence $\left\{A^2Sx_{2n-1}\right\}$ converges to su
I.e. $\lim\limits_{n\to\infty}A^2Sx_{2n-1}=u$

Since A and S are weak** commute

We have $d(A^2Sx,SA^2x,a)\le d(A^2x,S^2x,a)$ for all $a\in X$
Put $x=x_{2n-1}$
$d(A^2Sx_{2n-1},SA^2x_{2n-1},a)\le d(A^2x_{2n-1},S^2x_{2n-1},a)$
$\lim\limits_{n\to\infty}d(A^2Sx_{2n-1},SA^2x_{2n-1},a)\le\lim\limits_{n\to\infty}d(A^2x_{2n-1},S^2x_{2n-1},a)$
$\le d(u,u,a)=0$
$\lim\limits_{n\to\infty}d(A^2Sx_{2n-1},SA^2x_{2n-1},a)=0$

Also $\lim\limits_{n\to\infty}A^2x_{2n-1}=u$

Since S is continuous
$\lim\limits_{n\to\infty}SA^2x_{2n-1}=Su$
$\lim\limits_{n\to\infty}d(A^2Sx_{2n-1},Su,a)=0\forall a\in X$
$\Rightarrow\left\{A^2Sx_{2n-1}\right\}$ is convergent to Su
Since $\lim\limits_{n\to\infty}B^2x_{2n}=u$ and S is continuous
$\lim\limits_{n\to\infty}SB^2x_{2n}=Su$
$\lim\limits_{n\to\infty}SS^2x_{2n+1}=Su$

Since $S^2x_{2n-1} = B^2x_{2n-2} \Rightarrow S^2x_{2n+1} = B^2x_{2n}$

$\lim_{n\to\infty} S^3x_{2n+1} = Su$

Now we have

$$d\left(A^2Sx_{2n-1},\ \mathrm{B}^2x_{2n},a\right) \le c\ max\begin{Bmatrix} d\left(S^3x_{2n+1},T^2x_{2n},a\right), d\left(S^3x_{2n+1},A^2Sx_{2n+1},a\right), d\left(T^2x_{2n},B^2x_{2n},a\right) \\ \frac{1}{2}\left[d\left(S^3x_{2n+1},B^2x_{2n},a\right)+d\left(T^2x_{2n},A^2Sx_{2n+1},a\right)\right] \end{Bmatrix}$$

Letting $n\to\infty$ $d(su,u,a)=0 \forall a\in X$

$\Rightarrow Su = u$

Hence u is a fixed point of S

$\Rightarrow S^2u = Su = u$

Consider

$$d\left(A^2u,\ \mathrm{B}^2x_{2n},a\right)\le c\ max\begin{Bmatrix} d\left(S^2u,T^2x_{2n},a\right), d\left(S^2u,A^2u,a\right), d\left(T^2x_{2n},B^2x_{2n},a\right) \\ \frac{1}{2}\left[d\left(S^2u,B^2x_{2n},a\right)+d\left(T^2x_{2n},A^2u,a\right)\right] \end{Bmatrix}$$

Letting $n\to\infty$ $d(A^2u,u,a)=0 \forall a\in X$

$\Rightarrow A^2u = u$

Consider

$$d\left(u,\ \mathrm{B}^2u,a\right) = d\left(A^2u,\ \mathrm{B}^2u,a\right)\le c\ max\begin{Bmatrix} d\left(S^2u,T^2u,a\right), d\left(S^2u,A^2u,a\right), d\left(T^2u,B^2u,a\right) \\ \frac{1}{2}\left[d\left(S^2u,B^2u,a\right)+d\left(T^2u,A^2u,a\right)\right] \end{Bmatrix}$$

$d\left(u,\ \mathrm{B}^2u,a\right)=0$

$\Rightarrow B^2u = u$

Since $B^2\left(x\right)\subset T^2\left(x\right)$ and $u\in X$

We have $B^2u\in B^2\left(x\right)$

$\Rightarrow u\in B^2\left(x\right)$

$\Rightarrow u\in T^2\left(x\right)$

There exist $u_1\in X$ Such that $u = T^2\left(u_1\right)$

Then $d\left(u,\ \mathrm{B}^2u_1,a\right) = d\left(A^2u,\ \mathrm{B}^2u_1,a\right)\le c\ max\begin{Bmatrix} d\left(S^2u,T^2u_1,a\right), d\left(S^2u,A^2u,a\right), d\left(T^2u_1,B^2u_1,a\right) \\ \frac{1}{2}\left[d\left(S^2u,B^2u_1,a\right)+d\left(T^2u_1,A^2u,a\right)\right] \end{Bmatrix}$

$d\left(u,\ \mathrm{B}^2u_1,a\right)=0$

$\Rightarrow B^2u_1 = u$

There fore $T^2u_1 = B^2u_1 = u$

Since B and T are Weak** commutative

$$d(B^2T^2x,T^2B^2x,a)\le d(B^2Tx,TB^2x,a)\le d(BT^2x,T^2Bx,a)\le d(BTx,TBx,a)\le d(B^2x,T^2x,a)\forall x,a\in X$$

Put $x = u_1$

$$d(B^2T^2x, T^2B^2x, a) \le d(B^2Tu_1, TB^2u_1, a) \le d(BT^2u_1, T^2Bu_1, a) \le d(BTu_1, TBu_1, a) \le d(B^2u_1, T^2u_1, a) \forall a \in X$$

$$d\left(u, T^2u, a\right) = 0$$

$$\Rightarrow T^2u = u \forall a \in T$$

Hence $A^2u = B^2u = S^2u = T^2u = u$

Since $A^2u = u$

$$A\left(A^2u\right) = Au$$

$$A^3u = Au$$

Then we have

$$d\left(u, Au, a\right) = d\left(Au, u, a\right) = d\left(A^3u, B^2u, a\right) = d\left(A^2Au, B^2u, a\right)$$

$$\le c\ max\begin{Bmatrix} d\left(S^2Au, T^2u, a\right), d\left(S^2Au, A^3u, a\right), d\left(T^2u, B^2u, a\right) \\ \frac{1}{2}\left[d\left(S^2Au, B^2u, a\right) + d\left(T^2u, A^3u, a\right)\right] \end{Bmatrix}$$

$$\Rightarrow d\left(u, Au, a\right) = 0$$

$$\Rightarrow Au = u$$

$$\therefore Su = Au = u$$

Since B and T are weak** commutative

$$d(B^2T^2u, T^2B^2u, a) \le d(B^2Tu, TB^2u, a) \le d(BT^2u, T^2Bu, a) \le d(BTu, TBu, a) \le d(B^2u, T^2u, a)$$

$$d(u, u, a) \le d(B^2Tu, Tu, a) \le d(Bu, T^2Bu, a) \le d(BTu, TBu, a) \le d(u, u, a)$$

$$0 \le d(B^2Tu, Tu, a) \le d(Bu, T^2Bu, a) \le d(BTu, TBu, a) \le 0 \forall a \in X$$

$$d\left(B^2Tu, Tu, a\right) = 0 \Rightarrow B^2Tu = Tu$$

$$d\left(Bu, T^2Bu, a\right) = 0 \Rightarrow T^2Bu = Bu$$

$$d\left(BTu, TBu, a\right) = 0 \Rightarrow BTu = TBu$$

$$d\left(u, Tu, a\right) = d\left(A^2u, B^2Tu, a\right)$$

$$\le c\ max\begin{Bmatrix} d\left(S^2u, T^2Tu, a\right), d\left(S^2u, A^2u, a\right), d\left(T^2Tu, B^2Tu, a\right) \\ \frac{1}{2}\left[d\left(S^2u, B^2Tu, a\right) + d\left(T^2Tu, A^2u, a\right)\right] \end{Bmatrix}$$

$$d\left(u, Tu, a\right) = 0$$

$$\therefore Tu = u$$

Since $B^2u = u$

$$BB^2u = Bu$$

$$B^3u = Bu$$

We have

$$d\left(u, Bu, a\right) = d\left(A^2u, B^3u, a\right) = d\left(A^2u, B^2Bu, a\right)$$

$$\le c\ max\begin{Bmatrix} d\left(S^2u, T^2Bu, a\right), d\left(S^2u, A^2u, a\right), d\left(T^2Bu, B^2Bu, a\right) \\ \frac{1}{2}\left[d\left(S^2u, B^2Bu, a\right) + d\left(T^2Bu, A^2u, a\right)\right] \end{Bmatrix}$$

$d(u, Bu, a) = 0$

$\Rightarrow Bu = u$

$\therefore Au = Su = Tu = Bu = u$

Hence u is a common fixed point of A, S, T and B

Now we prove that u is a Unique fixed point of A, S, T and B

Suppose that there is a point $v \in X$ such that

$Av = sv = Bv = Tv = v$

$A^2v = s^2v = B^2v = T^2v = v$

Then $d(u, \mathrm{v}, a) = d(A^2u, \mathrm{B}^2v, a) \le c\ max \begin{Bmatrix} d(S^2u, T^2v, a), d(S^2u, A^2u, a), d(T^2v, B^2v, a) \\ \frac{1}{2}\left[d(S^2u, B^2v, a) + d(T^2v, A^2u, a)\right] \end{Bmatrix}$

$d(u, \mathrm{v}, a) = 0$

$\therefore u = v$

So, we proved that u is the unique common fixed point of A, B, S and T.

2.2 Corollary: Let S, T: X→X and either S or T be continuous. Then S and T have a common fixed point z if there exists two self mappings A,B of X and A (resp. B) weakly commute with S(resp. T) . Further z is the unique common fixed point of A, B, S and T.

Proof: As A (resp. B) weakly commutes with S (resp. T). But weakly commutativity implies weak **commutativity. Thus the proof of theorem [2.1] work.

Remark:

1. The corollary (2.2) generalizes theorem 1 of *kubaik* [3] where continuity of both S and T and commutative of both A and B with S and T are assumed. But assumptions in corollary (2.2) are much weaker than that of *kubaik* [3] and thus theorem (2.1) is more general than *kubaik* [3].

2.3 Theorem: Let A, B, S and T be four self-mappings of a complete 2-metric space (X, d) such that
(1) $A^2(X) \subset T^2(X)$ and $B^2(X) \subset S^2(X)$,
(11) $d(A^2x, B^2y, a) \le c$ max. $\{d(S^2x, T^2x, a), d(S^2x, A^2x, a), d(T^2y, B^2y, a), [d(S^2x, B^2y, a)+d(T^2x, A^2y, a)]\}$

For all x, y, a in X, where 0<c<1. if one of A, B, S and T is continuous and if A and B weak**commute with S and T respectively, then A, B, S and T have a unique common fixed point in X.

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