MULTIPLICATIVE ABC, GA, AG, AUGMENTED AND HARMONIC STATUS INDICES 
OF GRAPHS

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ABSTRACT

In this study, we introduce the multiplicative atom bond connectivity status index, multiplicative geometric-arithmetic status index, multiplicative arithmetic-geometric status index, multiplicative augmented status index and multiplicative harmonic status index of a graph. We compute these multiplicative status indices for complete graphs, cycles, complete bipartite graphs, wheel graphs and friendship graphs.

Key words: Multiplicative ABC status index, multiplicative GA status index, multiplicative augmented status index, multiplicative harmonic status index, graph.

Mathematics Subjects Classification: 05C05, 05C07, 05C35, 05C90.

1. INTRODUCTION

Let $G= (V(G), E(G))$ be a simple, finite, connected graph. The degree $d_G(u)$ of a vertex $u$ is the number of vertices adjacent to $u$. The distance between two vertices $u$ and $v$, denoted by $d(u, v)$, is the length of the shortest $u$-$v$ path in a graph $G$. The status of a vertex $u$ in $G$ is defined as the sum of its distance from every other vertex in $G$ and is denoted by $\sigma(u)$. For graph theoretic terminology, we refer [1].

A graph index is a numerical parameter mathematically derived from the graph structure. Several graph indices have their applications in various disciplines of Science and Technology, see [2, 3]. Many status indices of a graph such as harmonic status index [4], first and second status connectivity indices [5], first and second hyper status indices [6], multiplicative first and second status indices [7], multiplicative $F$-status index [8], multiplicative $(a, b)$-status index [9], first and second status coincides [10], geometric-arithmetic status index [11] studied in the literate of graph indices.

We introduce the multiplicative atom bond connectivity status index, multiplicative geometric-arithemetic status index, multiplicative arithmetic-geometric status index, multiplicative augmented status index, multiplicative harmonic status index of a graph as follows:

The multiplicative atom bond connectivity status index of a graph $G$ is defined as

$$ABCSII(G) = \prod_{u \in E(G)} \frac{\sigma(u) + \sigma(v) - 2}{\sigma(u)\sigma(v)}.$$ 

The multiplicative geometric-arithmetic status index of a graph $G$ is defined as

$$GASII(G) = \prod_{u \in E(G)} \frac{2\sqrt{\sigma(u)\sigma(v)}}{\sigma(u) + \sigma(v)}.$$ 

The multiplicative arithmetic-geometric status index of a graph $G$ is defined as

$$AGSII(G) = \prod_{u \in E(G)} \frac{\sigma(u) + \sigma(v)}{2\sqrt{\sigma(u)\sigma(v)}}.$$ 

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The multiplicative augmented status index of a graph $G$ is defined as
\[
ASIII(G) = \prod_{uv \in E(G)} \left( \frac{\sigma(u)\sigma(v)}{\sigma(u) + \sigma(v) - 2} \right)^3.
\]

The multiplicative harmonic status index of a graph $G$ is defined as
\[
HSII(G) = \prod_{uv \in E(G)} \frac{2}{\sigma(u) + \sigma(v)}.
\]

Recently, some new multiplicative indices were studied, for example, in [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]. In this paper, some new multiplicative status indices of complete graphs, wheel graphs friendship graphs are determined.

2. RESULTS FOR COMPLETE GRAPHS

Theorem 1: Let $K_n$ be a complete graph with $n$ vertices and $\frac{n(n-1)}{2}$ edges. Then

1. $ABCSII(K_n) = \left[ \frac{2n - 4}{(n-1)^2} \right]^{\frac{n(n-1)}{4}}.$
2. $GASII(K_n) = 1.$
3. $AGSII(K_n) = 1.$
4. $ASIII(K_n) = \left( \frac{n - 1}{2n - 4} \right)^{\frac{3}{2}n(n-1)}.$
5. $HSII(K_n) = \left( \frac{1}{n-1} \right)^{\frac{(n-1)}{2}}.$

Proof: For any vertex $u$ in $K_n$, $\sigma(u) = n - 1$. Therefore

1. $ABCSII(K_n) = \prod_{uv \in E(K_n)} \left( \frac{\sigma(u) + \sigma(v) - 2}{\sigma(u) + \sigma(v)} \right)^{\frac{1}{2}n(n-1)} = \left[ \frac{n-1+n-1-2}{(n-1)(n-1)} \right]^{\frac{1}{2}n(n-1)} = \left[ \frac{2n-4}{(n-1)^2} \right]^{\frac{1}{4}n(n-1)}.$
2. $GASII(K_n) = \prod_{uv \in E(K_n)} \frac{\sigma(u) + \sigma(v)}{2\sqrt{\sigma(u)\sigma(v)}} = \left( \frac{n-1+n-1}{2\sqrt{(n-1)(n-1)}} \right)^{\frac{n(n-1)}{2}} = 1.$
3. $AGSII(K_n) = \prod_{uv \in E(K_n)} \frac{2\sqrt{\sigma(u)\sigma(v)}}{\sigma(u) + \sigma(v)} = \left( \frac{2\sqrt{(n-1)(n-1)}}{n-1+n-1} \right)^{\frac{n(n-1)}{2}} = 1.$
4. $ASIII(K_n) = \prod_{uv \in E(K_n)} \left( \frac{\sigma(u)\sigma(v)}{\sigma(u) + \sigma(v) - 2} \right)^{\frac{3}{2}n(n-1)} = \left( \frac{n-1(n-1)}{n-1+n-1-2} \right)^{\frac{3}{2}n(n-1)} = \left( \frac{(n-1)^2}{2n-4} \right)^{\frac{3}{2}n(n-1)}.$
5. $HSII(K_n) = \prod_{uv \in E(K_n)} \frac{2}{\sigma(u) + \sigma(v)} = \left( \frac{2}{n-1+n-1} \right)^{\frac{1}{2}n(n-1)} = \left( \frac{1}{n-1} \right)^{\frac{(n-1)}{2}}.$

3. RESULTS FOR COMPLETE BIPARTITE GRAPHS

Theorem 2: Let $K_{p,q}$ be a complete bipartite graph with $p+q$ vertices and $pq$ edges. Then

1. $ABCSII(K_{p,q}) = \left[ \frac{3(p+q)-6}{2(p^2+q^2)-6(p+q)+5pq+4} \right]^{\frac{1}{2pq}}.$
2. $GASII(K_{p,q}) = \left( \frac{2\sqrt{2(p^2+q^2)-6(p+q)+5pq+4}}{3(p+q)-4} \right)^{pq}.$
\[
AGHII(K_{p,q}) = \frac{3(p+q)-4}{2\sqrt{2(p^2+q^2)}-6(p+q)+5pq+4}.
\]

\[
ASHII(K_{p,q}) = \left(\frac{2(p^2+q^2)-6(p+q)+5pq+4}{3(p+q)}\right)^{pq}.
\]

\[
HSII(K_{p,q}) = \left(\frac{2}{3(p+q)-4}\right)^{pq}.
\]

**Proof:** The vertex set of \(K_{p,q}\) can be partitioned into two independent sets \(V_1\) and \(V_2\) such that \(u \in V_1\) and \(v \in V_2\) for every edge \(uv\) in \(K_{p,q}\). Thus \(d_G(u) = q\), \(d_G(v) = p\). Then \(\sigma(u) = q + 2(p - 1)\) and \(\sigma(v) = p + 2(q - 1)\). Therefore

\[
ABCII(K_{p,q}) = \sqrt{\frac{\sigma(u) + \sigma(v) - 2}{\sigma(u)\sigma(v)}} = \left[\frac{3(p+q)-6}{2\sqrt{(p^2+q^2)}-6(p+q)+5pq+4}\right]^{\frac{1}{2pq}}.
\]

\[
\prod_{uv \in E(K_{p,q})} \frac{2\sqrt{\sigma(u)\sigma(v)}}{\sigma(u) + \sigma(v)}^{pq}.
\]

\[
AGSII(K_{p,q}) = \left(\frac{q + 2p - 2 + p + 2q - 2}{2\sqrt{(q + 2p - 2)(p + 2q - 2)}}\right)^{pq}.
\]

\[
\prod_{uv \in E(K_{p,q})} \left(\frac{\sigma(u)\sigma(v)}{\sigma(u) + \sigma(v) - 2}\right)^3^{pq}.
\]

\[
ASHII(K_{p,q}) = \left(\frac{(q + 2p - 2)(q + 2p - 2)}{q + 2p - 2 + p + 2q - 2}\right)^{3pq}.
\]

\[
ASII(K_{p,q}) = \left(\frac{2(p^2+q^2)-6(p+q)+5pq+4}{3(p+q)-6}\right)^{3pq}.
\]

\[
\prod_{uv \in E(K_{p,q})} \frac{2}{\sigma(u) + \sigma(v)} = \left(\frac{2}{q + 2p - 2 + p + 2q - 2}\right)^{pq}.
\]

\[
\left(\frac{2}{3(p+q)-4}\right)^{pq}.
\]
4. RESULTS FOR CYCLES

Theorem 3: Let \( C_n \) be a cycle with \( n \) vertices and \( n \) edges. Then

(1) \( \text{ABC} \text{SII}(C_n) = \left(\frac{8(n^2 - 4)}{n^4}\right)^\frac{n}{2} \), if \( n \) is even,
    \[ = \left(\frac{8(n^2 - 5)}{(n^2 - 1)^2}\right)^\frac{n}{2} \], if \( n \) is odd.

(2) \( \text{GAS} \text{SII}(C_n) = 1 \), if \( n \) is even,
    \[ = 1 \], if \( n \) is odd.

(3) \( \text{AGS} \text{SII}(C_n) = 1 \), if \( n \) is even,
    \[ = 1 \], if \( n \) is odd.

(4) \( \text{ASIII} \text{SII}(C_n) = \left(\frac{n^4}{8(n^2 - 4)}\right)^{\frac{3n}{2}} \), if \( n \) is even,
    \[ = \left(\frac{(n^2 - 1)^2}{8(n^2 - 5)}\right)^{\frac{3n}{2}} \], if \( n \) is odd.

(5) \( \text{HSII} \text{SII}(C_n) = \left(\frac{4}{n^2}\right)^n \), if \( n \) is even,
    \[ = \left(\frac{4}{n^2 - 1}\right)^n \], if \( n \) is odd.

Proof:

Case-1: Suppose \( n \) is even. Then \( \sigma(u) = \frac{n^2}{4} \) for any vertex \( u \) in \( C_n \). Thus

(1) \( \text{ABC} \text{SII}(C_n) = \prod_{u \in V(C_n)} \sqrt{\frac{\sigma(u) + \sigma(v) - 2}{\sigma(u) \sigma(v)}} = \left[\frac{n^2}{4} + \frac{n^2}{4} - 2}{\frac{n^2 \times n^2}{4 \times 4}}\right]^{\frac{n}{2}} = \left(\frac{8(n^2 - 4)}{n^4}\right)^{\frac{n}{2}} . \)

(2) \( \text{GAS} \text{SII}(C_n) = \prod_{u \in V(C_n)} 2\sqrt{\frac{\sigma(u)}{\sigma(u) + \sigma(v)}} = \left[2 \sqrt{\frac{n^2 \times n^2}{4 \times 4}}\right]^{\frac{n}{2}} = 1 . \)

(3) \( \text{AGS} \text{SII}(C_n) = \prod_{u \in V(C_n)} \frac{\sigma(u) + \sigma(v)}{2\sqrt{\sigma(u) \sigma(v)}} = \left[\frac{n^2 + n^2}{4 + 4}\right]^{\frac{n}{2}} = 1 . \)

(4) \( \text{ASIII} \text{SII}(C_n) = \prod_{u \in V(C_n)} \left(\frac{\sigma(u) \sigma(v)}{\sigma(u) + \sigma(v) - 2}\right)^3 = \left[\frac{n^2 \times n^2}{4 + 4} - 2}{\frac{n^2 + n^2}{4 \times 4}}\right]^{3n} = \left(\frac{8(n^2 - 4)}{n^2 - 5}\right)^{3n} . \)

(5) \( \text{HSII} \text{SII}(C_n) = \prod_{u \in V(C_n)} \frac{2}{\sigma(u) + \sigma(v)} = \left[\frac{2}{\frac{n^2}{4} + \frac{n^2}{4}}\right]^{n} = \left(\frac{4}{n^2}\right)^n . \)
Case-2: Suppose $n$ is odd. Then $\sigma(u) = \frac{n^2 - 1}{4}$ for any vertex $u$ in $C_n$. Therefore

(1)  $ABCSII (C_n) = \prod_{u \in E(C_n)} \sqrt{\frac{\sigma(u) + \sigma(v) - 2}{\sigma(u)\sigma(v)}} = \frac{n^2 - 1 + n^2 - 1 - 2}{4} = \left(\frac{n^2 - 5}{(n^2 - 1)}\right)^{\frac{n}{2}}.$

(2)  $GASII (C_n) = \prod_{u \in E(C_n)} \frac{2\sqrt{\sigma(u)\sigma(v)}}{\sigma(u) + \sigma(v)} = \frac{2\sqrt{\frac{n^2 - 1 \times n^2 - 1}{4}}}{n^2 - 1 + n^2 - 1} = \lim_{n \to \infty} \frac{n}{4} = 1.$

(3)  $AGSII (C_n) = \prod_{u \in E(C_n)} \frac{\sigma(u) + \sigma(v)}{2\sqrt{\sigma(u)\sigma(v)}} = \frac{\frac{n^2 - 1 + n^2 - 1}{4}}{2\sqrt{\frac{n^2 - 1 \times n^2 - 1}{4}}} = 1.$

(4)  $ASIII (C_n) = \prod_{u \in E(C_n)} \left(\frac{\sigma(u)\sigma(v)}{\sigma(u) + \sigma(v) - 2}\right)^3 = \left[\frac{n^2 - 1 \times n^2 - 1}{4}\right]^{\frac{3n}{4}} = \left[\frac{n^2 - 1}{8(n^2 - 5)}\right]^n.$

(5)  $HSII (C_n) = \prod_{u \in E(C_n)} \frac{2}{\sigma(u) + \sigma(v)} = \left(\frac{2}{\frac{n^2 - 1 + n^2 - 1}{4}}\right)^n = \left(\frac{4}{n^2 - 1}\right)^n.$

5. RESULT FOR WHEEL GRAPHS

A wheel graph $W_n$ is the join of $C_n$ and $K_1$. A graph $W_n$ has $n + 1$ vertices and $2n$ edges. A graph $W_n$ is presented in Figure 1.

![Figure 1: Wheel graph $W_n$](image)

In $W_n$, there are two types of edges as given in Table 1.

<table>
<thead>
<tr>
<th>$d_{w_n}(u), d_{w_n}(v)$ \ $uv \in E(W_n)$</th>
<th>$d_{w_n}(u), d_{w_n}(v)$ \ $uv \in E(W_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of edges $n$</td>
<td>Number of edges $n$</td>
</tr>
</tbody>
</table>

Table-1: Edge partition of $W_n$

Therefore, there are two types of status edges in $W_n$ as given in Table 2.

<table>
<thead>
<tr>
<th>$\sigma(u), \sigma(v)$ \ $uv \in E(W_n)$</th>
<th>$\sigma(u), \sigma(v)$ \ $uv \in E(W_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of edges $n$</td>
<td>Number of edges $n$</td>
</tr>
</tbody>
</table>

Table-2: Status edge partition of $W_n$
Theorem 4: Let $W_n$ be a wheel graph with $n+1$ vertices and $2n$ edges. Then

1. \[ \text{ABCSII} (W_n) = \left( \frac{2 \sqrt{n^2 - 2}}{2n - 3} \right)^n \times \left( \frac{3n - 5}{n(2n - 3)} \right)^{\frac{n}{2}}. \]

2. \[ \text{GASII} (W_n) = \left( \frac{2 \sqrt{a^2 + b^2}}{3n - 3} \right)^n. \]

3. \[ \text{AGSII} (W_n) = \left( \frac{3n - 3}{2 \sqrt{n(2n - 3)}} \right)^n. \]

4. \[ \text{ASII} (W_n) = \left( \frac{(2n - 3)^2}{4(n - 2)} \right)^{\frac{3n}{2}} \times \left( \frac{n(2n - 3)}{3n - 5} \right)^{\frac{n}{2}}. \]

5. \[ \text{HSII} (W_n) = \left( \frac{1}{2n - 3} \right)^n \times \left( \frac{2}{3n - 3} \right)^n. \]

Proof: By definitions and by using Table 2, we obtain

1. \[ \text{ABCSII} (W_n) = \prod_{u \in E(W_n)} \frac{\sigma(u) + \sigma(v) - 2}{\sigma(u) \sigma(v)} = \left( \frac{2n - 3 + 2n - 3 - 2}{(2n - 3)(2n - 3)} \right)^n \times \left( \frac{n + 2n - 3 - 2}{n(2n - 3)} \right)^{\frac{n}{2}} \]

2. \[ \text{GASII} (W_n) = \prod_{u \in E(W_n)} \frac{2 \sqrt{\sigma(u) \sigma(v)}}{\sigma(u) \sigma(v)} = \left( \frac{2 \sqrt{(2n - 3)(2n - 3)}}{2n - 3 + 2n - 3} \right)^n \times \left( \frac{2 \sqrt{n(2n - 3)}}{n + 2n - 3} \right)^n \]

3. \[ \text{AGSII} (W_n) = \prod_{u \in E(W_n)} \frac{\sigma(u) + \sigma(v)}{2 \sqrt{\sigma(u) \sigma(v)}} = \left( \frac{2n - 3 + 2n - 3}{2 \sqrt{(2n - 3)(2n - 3)}} \right)^n \times \left( \frac{n + 2n - 3}{2 \sqrt{n(2n - 3)}} \right)^n \]

4. \[ \text{ASII} (W_n) = \prod_{u \in E(W_n)} \left( \frac{\sigma(u) \sigma(v)}{\sigma(u) + \sigma(v) - 2} \right)^{\frac{3n}{2}} \times \left( \frac{n(2n - 3)}{3n - 5} \right)^{\frac{n}{2}} \]

5. \[ \text{HSII} (W_n) = \prod_{u \in E(W_n)} \frac{2}{\sigma(u) + \sigma(v)} = \left( \frac{2}{2n - 3 + 2n - 3} \right)^n \times \left( \frac{2}{n + 2n - 3} \right)^n \]

4. RESULTS FOR FRIENDSHIP GRAPHS

A friendship graph $F_n$ is the graph obtained by taking $n \geq 2$ copies of $C_3$ with vertex in common. A graph $F_4$ is shown in Figure 2.
A graph $F_n$ has $2n+1$ vertices and $3n$ edges. In $F_n$, there are two types of edges as follows:

$$E_1 = \{uv \in (F_n) \mid d_{F_n}(u) = d_{F_n}(v) = 2\}, \quad |E_1| = n.$$  

$$E_2 = \{uv \in (F_n) \mid d_{F_n}(u) = 2, d_{F_n}(v) = 2n\}, \quad |E_1| = 2n.$$  

Therefore, there are two types of status edges in $F_n$ as given in Table 3.

<table>
<thead>
<tr>
<th>$\sigma(u), \sigma(v) \setminus uv \in E(F_n)$</th>
<th>$(4n - 2, 4n - 2)$</th>
<th>$(2n, 4n - 2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of edges</td>
<td>$n$</td>
<td>$2n$</td>
</tr>
</tbody>
</table>

**Theorem 5:** Let $F_n$ be a friendship graph with $2n+1$ vertices and $3n$ edges. Then

1. $ABCSII(F_n) = \left( \frac{8n - 6}{(4n - 2)^2} \right)^{\frac{n}{2}} \times \left( \frac{3n - 2}{n(4n - 2)} \right)^n.$
2. $GASII(F_n) = \left( \frac{\sqrt{2n(4n - 2)}}{3n - 1} \right)^{2n}.$
3. $AGSII(F_n) = \left( \frac{3n - 1}{\sqrt{2n(4n - 2)}} \right)^{2n}.$
4. $ASIII(F_n) = \left( \frac{(4n - 2)^2}{8n - 6} \right)^{3n} \times \left( \frac{n(4n - 2)}{3n - 2} \right)^{6n}.$
5. $HSII(F_n) = \left( \frac{1}{4n - 2} \right)^n \times \left( \frac{1}{3n - 1} \right)^{2n}.$

**Proof:** by using definitions and Table 3, we deduce

1. $ABCSII(F_n) = \prod_{uv \in E(F_n)} \sqrt{\frac{\sigma(u) + \sigma(v) - 2}{\sigma(u)\sigma(v)}} = \left( \frac{4n - 2 + 4n - 2 - 2}{(4n - 2)(4n - 2)} \right)^{\frac{1}{2^n}} \times \left( \frac{2n + 4n - 2 - 2}{2n(4n - 2)} \right)^{\frac{1}{2^{2n}}}$

$= \left( \frac{8n - 6}{(4n - 2)^2} \right)^{\frac{n}{2}} \times \left( \frac{3n - 2}{n(4n - 2)} \right)^n.$

2. $GASII(F_n) = \prod_{uv \in E(F_n)} \frac{2\sqrt{\sigma(u)\sigma(v)}}{\sigma(u) + \sigma(v)} = \left( \frac{2\sqrt{(4n - 2)(4n - 2)}}{4n - 2 + 4n - 2} \right)^n \times \left( \frac{2\sqrt{2n(4n - 2)}}{2n + 4n - 2} \right)^{2n}$

$= \left( \frac{\sqrt{2n(4n - 2)}}{3n - 1} \right)^{2n}.$

3. $AGSII(F_n) = \prod_{uv \in E(F_n)} \frac{\sigma(u) + \sigma(v)}{2\sqrt{\sigma(u)\sigma(v)}} = \left( \frac{4n - 2 + 4n - 2}{2\sqrt{(4n - 2)(4n - 2)}} \right)^n \times \left( \frac{2n + 4n - 2}{2\sqrt{2n(4n - 2)}} \right)^{2n}$

$= \left( \frac{3n - 1}{\sqrt{2n(4n - 2)}} \right)^{2n}.$
\[
S_{III}(F_a) = \prod_{uv \in E(F_a)} \left( \frac{\sigma(u) \sigma(v)}{\sigma(u) + \sigma(v) - 2} \right)^3 = \left( \frac{(4n-2)(4n-2)}{4n-2 + 4n-2} \right)^n \times \left( \frac{2n(4n-2)}{2n + 4n-2} \right)^6n
\]

\[
S_{II}(F_a) = \prod_{uv \in E(F_a)} \left( \frac{2}{\sigma(u) + \sigma(v)} \right) = \left( \frac{2}{4n-2 + 4n-2} \right)^n \times \left( \frac{2}{2n + 4n-2} \right)^2n
\]

\[
= \left( \frac{1}{3n-1} \right)^2n
\]

REFERENCES

9. V.R. Kulli, Computation of multiplicative \((a, b)\)-status index of certain graphs, submitted.


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