International Journal of Mathematical Archive-2(10), 2011, Page: 1854-1864
Available online through www.ijma.info ISSN 2229 – 5046

NEW SUB CLASSES OF THE CLASS OF CLOSE TO CONVEX FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

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(Received on: 23-09-11; Accepted on: 10-10-11)

ABSTRACT

We introduce some subclasses of the class of close to convex functions w.r.t. symmetric points, derive inclusion relations, establish integral representation formulas and determine coefficient estimates for the functions of such classes. The results are sharp.

Mathematical Subject Classification: *30C45.*

Keywords: *Subordination, Bounded functions; univalent starlike, univalent convex, close to convex and α - close to convex functions w.r.t. symmetric points.*

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1. INTRODUCTION AND DEFINITIONS

Principle of subordination ([5], [7])

Let *f(z)* and *F(z)* be two analytic functions in the unit disc $E=\{z;\ |z|<1\}$. Then *f(z)* is called subordinate to *F(z)* if there exists a function $w(z)$ analytic in E and satisfying the conditions $w(z)=0, |w(z)|<1$ such that $f(z)=F(w(z))$ and we write as $f(z)\prec F(z)$. If $F(z)$ is univalent in E, the above definition is equivalent to $f(0)=F(0)$ and $f(E)\subset F(E)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form

(1.1) $$w(z)=\sum_{k=1}^{\infty} d_k z^k,\quad z\in E$$

with the conditions $w(0)=0$ and $|w(z)|<1$.

Let $\mathcal{A}$ be the class of functions of the form

(1.2) $$f(z)=z+\sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disc *E*.

Sakaguchi [8] introduced the concept of starlike univalent functions w.r.t. symmetric points. A function $f(z)\in\mathcal{A}$ is called univalent star-like w.r.t. symmetric points if and only if

(1.3) $$Re\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\}>0,\ z\in E,$$

and the class of functions satisfying (1.3) may be denoted by $S_s{}^*$. Das and Singh [2] extended the idea of symmetric points to convex and close to convex functions. A function $f(z)\in\mathcal{A}$ is said to be univalent convex w. r. t. symmetric points if and only if

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(1.4) $$Re\left\{\frac{(zf'(z))'}{(f(z)-f(-z))'}\right\} > 0, \quad z \in E,$$

and the class of such functions is denoted by K_s. A function $f(z) \in \mathcal{A}$ is said to belong to the class C_s of close to convex functions w.r.t. symmetric points if there exists a function

(1.5) $$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in S_s^*$$

for which

(1.6) $$Re\left\{\frac{zf'(z)}{g(z)-g(-z)}\right\} > 0, \; z \in E.$$

If

(1.7) $$h(z) = z + \sum_{k=2}^{\infty} c_k z^k \in K_s$$

for which

(1.8) $$Re\left\{\frac{zf'(z)}{h(z)-h(-z)}\right\} > 0, \; z \in E.$$

the class of functions satisfying (1.8) is denoted by $C_{1(s)}$, $C_{1(s)} \subset C_s$.

Let C_s' represent the class of functions $f(z) \in \mathcal{A}$ which satisfy the condition

(1.9) $$Re\left\{\frac{(zf'(z))'}{(g(z)-g(-z))'}\right\} > 0, \quad g \in S_s^*, \; z \in E.$$

If

(1.10) $$Re\left\{\frac{(zf'(z))'}{(h(z)-h(-z))'}\right\} > 0, \quad h \in K_s, \; z \in E,$$

the corresponding class is denoted by $C'_{1(s)}$. Goel and the first author [4] introduced and studied the subclass $S_s^*(A,B)$ of S_s^*. $f(z) \in S_s^*(A,B)$ if

(1.11) $$\left[\frac{2zf'(z)}{f(z)-f(-z)}\right] \prec \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1, \; z \in E.$$

If

(1.12) $$\left[\frac{2(zf'(z))'}{(f(z)-f(-z))'}\right] \prec \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1, \; z \in E,$$

the class of functions satisfying (1.12) is denoted by $K_s(A,B)$ which is a subclass of K_s. It is obvious that

$$f(z) \in S_s^*(A,B) \text{ implies that } zf'(z) \in K_s(A,B).$$

The author etal. [6] introduced the subclass $C_s(A,B;C,D)$ of C_s and obtained its coefficient estimates.

A function $f(z)$ in $\mathcal{A}$ belongs to $C_s(A,B;C,D)$ if

(1.13) $$\left[\frac{2zf'(z)}{g(z)-g(-z)}\right] \prec \frac{1+Cz}{1+Dz}, \; g \in S_s^*(A,B), \; -1 \le D \le B < A \le C \le 1, \; z \in E.$$

If

(1.14) $$\left[\frac{2zf'(z)}{h(z)-h(-z)}\right] \prec \frac{1+Cz}{1+Dz}, \; h \in K_s(A,B), \; -1 \le D \le B < A \le C \le 1, \; z \in E.$$

the corresponding class of functions satisfying (1.14) is denoted by $C_{1(s)}(A,B;C,D)$, $C_s'(A,B;C,D)$ is the class of functions $f(z)$ in $\mathcal{A}$ for which

(1.15) $$\left[\frac{2(zf'(z))'}{(g(z)-g(-z))'}\right] \prec \frac{1+Cz}{1+Dz}, \; g \in S_s^*(A,B), \; -1 \le D \le B < A \le C \le 1, \; z \in E.$$

Moreover $\mathrm{C}'_{1(\mathrm{s})}(A,B;C,D)$is the class of functions $f(z)$ in $\mathcal{A}$ such that

(1.16) $$\left[\frac{2\left(zf'(z)\right)'}{\left(h(z)-h(-z)\right)'}\right] \prec \frac{1+\mathrm{C}z}{1+\mathrm{D}z},\ h\in K_s(A,B),\ -1\le \mathrm{D}\le \mathrm{B}< A\le \mathrm{C}\le 1,\ z\in \mathrm{E}.$$

Let $\alpha\ge 0$ and $\frac{f(z)f'(z)}{z}\neq 0$. Then $\mathrm{C}_s(\alpha)$ is the class of α-close to convex functions $f(z)$ in $\mathcal{A}$ w.r.t. symmetric points if there exists $g(z)\in S_s^*$ such that

(1.17) $$Re\left\{\frac{2(1-\alpha)zf'(z)}{g(z)-g(-z)}+\frac{2\alpha\left(zf'(z)\right)'}{\left(g(z)-g(-z)\right)'}\right\}>0,\ z\in \mathrm{E}.$$

If

(1.18) $$Re\left\{\frac{2(1-\alpha)zf'(z)}{h(z)-h(-z)}+\frac{2\alpha\left(zf'(z)\right)'}{\left(h(z)-h(-z)\right)'}\right\}>0,\ h\in K_s(A,B),$$

the class of functions satisfying (1.18) and is denoted by $\mathrm{C}_{1(\mathrm{s})}(\alpha)$.

Also $C_s(\alpha;\ A,B;C,D)$ and $C_{1(s)}(\alpha;\ A,B;C,D)$ represent the classes of functions $f(z)$ in $\mathcal{A}$ which satisfy, respectively, the conditions

(1.19) $$\left[\frac{2(1-\alpha)zf'(z)}{g(z)-g(-z)}+\frac{2\alpha\left(zf'(z)\right)'}{\left(g(z)-g(-z)\right)'}\right]\prec\frac{1+\mathrm{C}z}{1+\mathrm{D}z},\ g\in S_s^*(A,B),\ -1\le \mathrm{D}\le \mathrm{B}<A\le \mathrm{C}\le 1, z\in \mathrm{E}.$$

and

(1.20) $$\left[\frac{2(1-\alpha)zf'(z)}{h(z)-h(-z)}+\frac{2\alpha\left(zf'(z)\right)'}{\left(h(z)-h(-z)\right)'}\right]\prec\frac{1+\mathrm{C}z}{1+\mathrm{D}z},\ h\in K_s(A,B),\ -1\le \mathrm{D}\le \mathrm{B}<A\le \mathrm{C}\le 1, z\in \mathrm{E}.$$

Throughout this paper we assume that

(1.21) $$\left\{\begin{array}{c}\alpha\ge 0, g\in S_s^*(A,B), h\in K_s(A,B), G(z)=\frac{g(z)-g(-z)}{2}, H(z)=\frac{h(z)-h(-z)}{2},\\ \wp(z)=\frac{1+Cw(z)}{1+Dw(z)}, w\in \mathrm{u},\ -1\le \mathrm{D}\le \mathrm{B}<A\le \mathrm{C}\le 1,\ z\in \mathrm{E}.\end{array}\right\}$$

2. PRELIMINARY LEMMAS

Lemma: 2.1[1] Let $\alpha\ge 0$ and $\mathcal{D}(z)$ be starlike in *E*. Let $\mathcal{N}(z)$ be analytic in *E* such that $\mathcal{N}(0)=\mathcal{D}(0)=0=\mathcal{N}'(0)-1$ $=\mathcal{D}'(0)-1$, then $Re\left\{\frac{\mathcal{N}(z)}{\mathcal{D}(z)}\right\}>0$ whenever

$$Re\left\{(1-\alpha)\frac{\mathcal{N}(z)}{\mathcal{D}(z)}+\alpha\frac{\mathcal{N}'(z)}{\mathcal{D}'(z)}\right\}>0.$$

Lemma: 2.2 Under the same conditions of lemma 2.1, $\frac{\mathcal{N}(z)}{\mathcal{D}(z)}\prec\frac{1+\mathrm{C}z}{1+\mathrm{D}z}$ whenever

$$\left\{(1-\alpha)\frac{\mathcal{N}(z)}{\mathcal{D}(z)}+\alpha\frac{\mathcal{N}'(z)}{\mathcal{D}'(z)}\right\}\prec\frac{1+\mathrm{C}z}{1+\mathrm{D}z}.$$

Proof: By definition of subordinates, $\left\{(1-\alpha)\frac{\mathcal{N}(z)}{\mathcal{D}(z)}+\alpha\frac{\mathcal{N}'(z)}{\mathcal{D}'(z)}\right\}=\frac{1+\mathrm{C}w(z)}{1+\mathrm{D}w(z)}$.

This implies that

$$\mathrm{Re}\left\{(1-\alpha)\frac{\mathcal{N}(z)}{\mathcal{D}(z)}+\alpha\frac{\mathcal{N}'(z)}{\mathcal{D}'(z)}\right\}=\mathrm{Re}\left\{\frac{1+\mathrm{C}w(z)}{1+\mathrm{D}w(z)}\right\}\ge\frac{1-\mathrm{C}r}{1-\mathrm{D}r}>\frac{1-\mathrm{C}}{1-\mathrm{D}}=\beta\ \ (0\le\beta<1)$$

which can be put into the form

(2.1) $$\frac{1}{(1-\beta)}\mathrm{Re}\left\{(1-\alpha)\left(\frac{\mathcal{N}(z)}{\mathcal{D}(z)}-\beta\right)+\alpha\left(\frac{\mathcal{N}'(z)}{\mathcal{D}'(z)}-\beta\right)\right\}>0.$$

Setting

(2.2) $\qquad \mathcal{M}(z) = \frac{\mathcal{N}(z)-\beta\mathcal{D}(z)}{(1-\beta)}$,

(2.1) takes the form $\mathrm{Re}\left\{(1-\alpha)\frac{\mathcal{M}(z)}{\mathcal{D}(z)} + \alpha\frac{\mathcal{M}'(z)}{\mathcal{D}'(z)}\right\} > 0$.

By lemma 2.1, $\mathrm{Re}\left\{\frac{\mathcal{M}(z)}{\mathcal{D}(z)}\right\} > 0$ whenever $\mathrm{Re}\left\{(1-\alpha)\frac{\mathcal{M}(z)}{\mathcal{D}(z)} + \alpha\frac{\mathcal{M}'(z)}{\mathcal{D}'(z)}\right\} > 0$.

This means that $Re\left\{\frac{\mathcal{N}(z)}{\mathcal{D}(z)}\right\} > \beta$ whenever $Re\left\{(1-\alpha)\frac{\mathcal{N}(z)}{\mathcal{D}(z)} + \alpha\frac{\mathcal{N}'(z)}{\mathcal{D}'(z)}\right\} > \beta$.

That is $\frac{\mathcal{N}(z)}{\mathcal{D}(z)} \prec \frac{1+Cz}{1+Dz}$ whenever $\left\{(1-\alpha)\frac{\mathcal{N}(z)}{\mathcal{D}(z)} + \alpha\frac{\mathcal{N}'(z)}{\mathcal{D}'(z)}\right\} \prec \frac{1+Cz}{1+Dz}$.

Lemma: 2.3[3] Let $\wp(z) = \frac{1+Cw(z)}{1+Dw(z)} = 1 + \sum_{k=1}^{\infty} p_k z^k$. Then

$$|p_n| \leq (C-D).$$

The estimates are sharp for the functions $\wp_n(z) = \frac{1+C\delta z^n}{1+D\delta z^n}$, $|\delta| = 1$ and $n \geq 1$.

Lemma: 2.4 [**4**] Let $g \in S_s^*(A,\ B)$. Then for $n \geq 1$,

$$|b_{2n}| \leq \frac{(A-B)}{2^n n!}\prod_{j=1}^{n-1}(A-B+2j),$$

$$|b_{2n+1}| \leq \frac{(A-B)}{2^n n!}\prod_{j=1}^{n-1}(A-B+2j).$$

The bounds are sharp being attained for the function $g_0(z)$ defined by

$$(2.3) \qquad g_0(z) = \begin{cases} \log\left(\frac{(1+z)^{(1-A)/2(1-B)}(1+Bz)^{(A-B)/(1-B^2)}}{(1-z)^{(1+A)/2(1+B)}}\right), & B \neq -1; \\ \log\left(\frac{1+z}{1-z}\right)^{(1-A)/4} + \left(\frac{1+A}{2}\right)\left(\frac{z}{1-z}\right), & B = -1; \\ \log\left(\frac{(1+z)^{(1-A)/2}}{(1-z)^{(1+A)/2}}\right), & B = 0. \end{cases}$$

Since $f(z) \in S_s^*(A,\ B)$ implies that $zf'(z) \in K_s(A,\ B)$, we have the following

Lemma: 2.5 Let $f \in K_s(A,\ B)$, then

$$|c_{2n}| \leq \frac{1}{2n}\left[\frac{(A-B)}{2^n n!}\prod_{j=1}^{n-1}(A-B+2j)\right],$$

$$|c_{2n+1}| \leq \frac{1}{2n+1}\left[\frac{(A-B)}{2^n n!}\prod_{j=1}^{n-1}(A-B+2j)\right].$$

The extremal function is given by

(2.4) $\qquad h_0(z) = \int_0^z \frac{g_0(t)}{t}dt$, $g_0(z)$ is defined by (2.3).

3. INCLUSION RELATIONS AND INTEGRAL REPRESENTATION FORMULAS

Theorem: 3.1 $C_s(\alpha;\ A,B;C,D) \subset C_s(A,B;C,D) \subset C_s$.

Proof: Putting $\mathcal{N}(z) = zf'(z)$ and $\mathcal{D}(z) = G(z)$ (which is odd star-like) in lemma 2.2 so that

$$\frac{zf'(z)}{G(z)} \prec \frac{1+Cz}{1+Dz}, \text{ whenever } \left\{(1-\alpha)\frac{zf'(z)}{G(z)} + \alpha\frac{\left(zf'(z)\right)'}{G'(z)}\right\} \prec \frac{1+Cz}{1+Dz}$$

and the desired result follows.

From the above theorem it follows that all α-close to convex functions w. r. t. symmetric points are close to convex w. r. t. symmetric points. Similarly we can prove

Theorem: 3.2 $C_{1(s)}(\alpha;\ A,B;C,D) \subset C_{1(s)}(A,B;C,D) \subset C_{1(s)}$.

Theorem: 3.3 Let $f \in C_s(\alpha;\ A,B;C,D)$, then

(i) for $\alpha = 0, f(z) = \int_0^z \frac{G(t)\wp(t)}{t} dt.$
(ii) for $\alpha > 0, f(z) = (1+c)\left[\int_0^z \frac{1}{t\left(G(t)\right)^c}\left\{\int_0^t \left(G(u)\right)^c G'(u)\wp(u)du\right\} dt\right], \left(c = \frac{1}{\alpha} - 1\right).$

Proof: We have

(3.1) $$(1-\alpha)\frac{zf'(z)}{G(z)} + \alpha\frac{\left(zf'(z)\right)'}{G'(z)} = \wp(z)$$

For $\alpha = 0$, there is nothing to prove.

Consider the case when $\alpha > 0$.

Dividing (3.1) by α and putting $c = \frac{1}{\alpha} - 1$, (3.1) takes the form

(3.2) $$\frac{czf'(z)}{G(z)} + \frac{\left(zf'(z)\right)'}{G'(z)} = (1+c)\wp(z)$$

Multiplying (3.2) by $\left(G(z)\right)^c G'(z)$, we get

$$czf'(z)(G(z))^{c-1}G'(z) + \left(zf'(z)\right)'\left(G(z)\right)^c = (1+c)[(G(z))^c G'(z)\wp(z)]$$

which reduces to

(3.3) $$\frac{d}{dz}[zf'(z)(G(z))^c] = (1+c)[(G(z))^c G'(z)\wp(z)]$$

Integrating (3.3) from 0 to z,

$$f'(z) = \frac{(1+c)}{z(G(z))^c}\left[\int_0^z \left(G(u)\right)^c G'(u)\wp(u)du\right]$$

which on integration gives the desired result.

On the same lines we can prove

Theorem: 3.4 Let $f \in C_{1(s)}(\alpha;\ A,B;C,D)$, then

(i) for $\alpha = 0, f(z) = \int_0^z \frac{H(t)\wp(t)}{t} dt.$
(ii) for $\alpha > 0, f(z) = (1+c)\left[\int_0^z \frac{1}{t\left(H(t)\right)^c}\left\{\int_0^t \left(H(u)\right)^c H'(u)\wp(u)du\right\} dt\right], \left(c = \frac{1}{\alpha} - 1\right).$

4. COEFFICIENT ESTIMATES FOR THE CLASSES $C_s'(A,B;C,D)$ and $C_s(\alpha;A,B;C,D)$

Theorem: 4.1 Let $f \in C_s'(A,B;C,D)$, then

(4.1) $$|a_{2n}| \leq \frac{(C-D)}{(2n)^2}\left\{1+(A-B)\left(\sum_{k=2}^{n}\frac{(2k-1)}{2^{k-1}(k-1)!}\prod_{j=1}^{k-2}(A-B+2j)\right)\right\},$$

(4.2) $$|a_{2n+1}|\frac{1}{(2n+1)^2}\left[\frac{(2n+1)(A-B)}{2^n n!}\prod_{j=1}^{n-1}(A-B+2j)+(C-D)\left\{1+(A-B)\sum_{k=2}^{n}\frac{(2k-1)}{2^{k-1}(k-1)!}\prod_{j=1}^{k-2}(A-B+2j)\right\}\right].$$

Proof: We have

$\left(zf'(z)\right)' = G'(z)\wp(z)$ which on expansion yields

(4.3) $$1+2^2a_2z+3^2a_3z^2+\cdots+(2n)^2a_{2n}z^{2n-1}+(2n+1)^2a_{2n+1}z^{2n}+\cdots$$
$$=[1+p_1z+p_2z^2+\cdots+p_{2n-1}z^{2n-1}+p_{2n}z^{2n}+\cdots]\,[1+3b_3z^2+5b_5z^4+\cdots+(2n-1)b_{2n-1}z^{2n-2}+(2n+1)b_{2n+1}z^{2n}+\cdots]$$

Equating coefficients of z^{2n-1} and z^{2n} in (4.3), we get

(4.4) $$(2n)^2a_{2n} = (2n-1)p_1b_{2n-1}+(2n-3)p_3b_{2n-3}+\cdots+3p_{2n-3}b_3+p_{2n-1}.$$

(4.5) $$(2n+1)^2a_{2n+1} = (2n+1)b_{2n+1}+(2n-1)p_2b_{2n-1}+\cdots+3p_{2n-2}b_3+p_{2n}.$$

Using lemmas 2.3 and 2.4 in (4.4), we obtain

$$(2n)^2|a_{2n}| \leq (C-D)\left(1+\sum_{k=2}^{n}(2k-1)|b_{2k-1}|\right) \leq (C-D)\left[1+(A-B)\left\{\sum_{k=2}^{n}\frac{(2k-1)}{2^{k-1}(k-1)!}\prod_{j=1}^{k-2}(A-B+2j)\right\}\right]$$

from which (4.1) follows.

Again applying lemmas 2.3 and 2.4 in (4.5), we have

$$(2n+1)^2|a_{2n+1}| \leq (2n+1)|b_{2n+1}|+(C-D)\left(1+\sum_{k=2}^{n}(2k-1)|b_{2k-1}|\right)$$
$$\leq \frac{(2n+1)(A-B)}{2^n n!}\prod_{j=1}^{n-1}(A-B+2j)+(Cb-D)\left[1+(A-B)\left\{\sum_{k=2}^{n}\frac{(2k-1)}{2^{k-1}(k-1)!}\prod_{j=1}^{k-2}(A-B+2j)\right\}\right]$$

which gives the desired result.

The extremal function is obtained by choosing $G(z)=\frac{g_0(z)-g_0(-z)}{2}$, $g_0(z)$ is defined by (2.3), and $\wp(z)=\frac{1+Cz}{1+Dz}$ in the integral representation formula

$$f(z)=\left[\int_0^z \frac{1}{t}\left\{\int_0^t G'(u)\wp(u)du\right\}dt\right].$$

Taking $A = C = 1$ and $B = D = -1$ in the theorem, we have the following:

Corollary: 4.1 If $f \in C_s$, then $|a_{2n}| \leq \frac{1}{2}$ and $|a_{2n+1}| \leq \frac{1+(2n+1)^2}{2(2n+1)^2}$.

Theorem 4.2 Let $f \in C_s(\alpha; A, B; C, D)$. Then

(4.6) $\quad |a_2| \leq \frac{(C-D)}{2(1+\alpha)}$,

(4.7) $\quad |a_3| \leq \frac{(1+2\alpha)(A-B)+2(C-D)}{6(1+2\alpha)}$,

(4.8) $\quad |a_4| \leq \frac{(C-D)}{8(1+\alpha)(1+3\alpha)}[(1+5\alpha)(A-B)+2(1+\alpha)]$,

(4.9) $\quad |a_5| \leq \frac{1}{5(1+2\alpha)(1+4\alpha)}\begin{bmatrix} \frac{1}{8}(1+2\alpha)(1+4\alpha)(A-B)(A-B+2) \\ +4(C-D)\{(1+8\alpha)(A-B)+2(1+2\alpha)\} \end{bmatrix}$,

(4.10) $\quad |a_6| \leq \frac{(C-D)}{48(1+\alpha)(1+3\alpha)(1+5\alpha)}\begin{bmatrix} 4\alpha|\alpha-1|\left(\frac{A-B}{4}\right)^2 + (1+3\alpha)(1+9\alpha)(A-B)(A-B+2) \\ +4(1+\alpha)(1+11\alpha)(A-B)+8(1+\alpha)(1+3\alpha) \end{bmatrix}$

The bounds are sharp.

Proof: Since $f \in C_s(\alpha; A, B; C, D)$, therefore

$$(1-\alpha)zf'(z)G'(z) + \alpha\left(zf'(z)\right)' G(z) = \wp(z)G(z)G'(z).$$

Expanding the series, we get

$$\begin{aligned}(1-\alpha)[z + 2a_2z^2 + 3a_3z^3 + 4a_4z^4 + 5a_5z^5 + 6a_6z^6 + \cdots][1 + 3b_3z^2 + 5b_5z^4 + \cdots] \\ + \alpha[1 + 4a_2z + 9a_3z^2 + 16a_4z^3 + 25a_5z^4 + 36a_6z^5 + \cdots][z + b_3z^3 + b_5z^5 + \ldots] \\ = [1 + p_1z + p_2z^2 + p_3z^3 + p_4z^4 + p_5z^5 + \cdots][z + b_3z^3 + b_5z^5 + \cdots][1 + 3b_3z^2 + 5b_5z^4 + \cdots]\end{aligned}$$

which on simplification yields

(4.11) $\quad 1 + 2(1+\alpha)a_2z + \{3(1+2\alpha)a_3 + (3-2\alpha)b_3\}z^2 + \{4(1+3\alpha)a_4 + (6-2\alpha)a_2b_3\}z^3 + \{5(1+4\alpha)a_5 +$
$5-4\alpha b5+9a3b3z4+6 1+5\alpha a6+4 3+\alpha a4b3+2 1+\alpha a2b5z5+\ldots$

$$\begin{aligned}= 1 + p_1z + (4b_3 + p_2)z^2 + (4p_1b_3 + p_3)z^3 + \{(6b_5 + 3b_3^2) + 4p_2b_3 + p_4\}z^4 \\ + \{p_1(6b_5 + 3b_3^2) + 4p_3b_3 + p_5\}z^5 + \cdots.\end{aligned}$$

Identifying terms in (4.11), we get

(4.12) $\quad 2(1+\alpha)a_2 = p_1$,

(4.13) $\quad 3(1+2\alpha)a_3 = (1+2\alpha)b_3 + p_2$,

(4.14) $\quad 4(1+3\alpha)a_4 = (2\alpha-6)a_2b_3 + 4p_1b_3 + p_3$,

(4.15) $\quad 5(1+4\alpha)a_5 = (1+4\alpha)b_5 - 9a_3b_3 + 3b_3^2 + 4p_2b_3 + p_4$,

(4.16) $\quad 6(1+5\alpha)a_6 = 4(3+\alpha)a_4b_3 - 2(5-3\alpha)a_2b_5 + p_1(6b_5 + 3b_3^2) + 4p_3b_3 + p_5$.

Applying lemma 2.3 to (4.12), we get (4.6). Using lemma 2.3 and 2.4 in (4.13), (4.7) follows. (4.14) in conjunction with (4.12) leads us to

(4.17) $\quad 4(1+\alpha)(1+3\alpha)a_4 = (1+5\alpha)p_1b_3 + (1+\alpha)p_3$.

With the application of lemmas 2.3 and 2.4 in (4.17), (4.8) follows.

Eliminating a_3 from (4.13) and (4.15), we arrive at

(4.18) $\quad 5(1+2\alpha)(1+4\alpha)a_5 = (1+2\alpha)(1+4\alpha)b_5 + (1+8\alpha)p_2b_3 + (1+2\alpha)p_4$.

Using lemmas 2.3 and 2.4 in (2.18), (4.9) follows.

From (4.12), (4.14) and (4.16), we obtain

(4.19) $$6(1+\alpha)(1+3\alpha)(1+5\alpha)a_6 = 4\alpha(\alpha-1)p_1{b_3}^2 + (1+3\alpha)(1+9\alpha)p_1b_5 + (1+\alpha)(1+11\alpha)p_3b_3 + (1+\alpha)(1+3\alpha)p_5.$$

With the application of lemma 2.3 and 2.4 in (4.19), (4.10) follows.

The extremal function is obtained by choosing $G(z) = \frac{g_0(z)-g_0(-z)}{2}$, $g_0(z)$ is defined by (2.3), and $\wp(z) = \frac{1+Cz}{1+Dz}$ in the integral representation formula proved in theorem 3.3.

Letting $A = C = 1$ and $B = D = -1$ in the theorem, we have the following:

Corollary: 4.2 If $f \in C_s(\alpha)$, then

$$|a_2| \le \frac{1}{(1+\alpha)},$$

$$|a_3| \le \frac{3+2\alpha}{3(1+2\alpha)},$$

$$|a_4| \le \frac{1}{(1+\alpha)},$$

$$|a_5| \le \frac{5+26\alpha+8\alpha^2}{5(1+2\alpha)(1+4\alpha)},$$

$$|a_6| \le \begin{cases} \frac{3+32\alpha+37\alpha^2}{3(1+\alpha)(1+3\alpha)(1+5\alpha)}, & 0 \le \alpha \le 1, \\ \frac{1}{(1+\alpha)}, & \alpha \ge 1. \end{cases}$$

5. COEFFICIENT ESTIMATES FOR THE CLASSES $C <>_{1(s)} (A, B; C, D)$, $C'_{1(s)}(A, B; C, D)$ and $C_{1(s)}(\alpha; A, B; C, D)$

Theorem: 5.1 Let $f \in C_{1(s)}(A, B; C, D)$, then

(5.1) $$|a_{2n}| \le \frac{(C-D)}{2n}\left\{1 + \left(\sum_{k=2}^{n} \frac{(A-B)}{(2k-1)2^{k-1}(k-1)!}\prod_{j=1}^{k-2}(A-B+2j)\right)\right\},$$

(5.2) $$|a_{2n+1}| \le \frac{1}{2n+1}\left[\frac{(A-B)}{(2n+1)2^n n!}\prod_{j=1}^{n-1}(A-B+2j) + (C-D)\left\{1 + \sum_{k=2}^{n}\frac{(A-B)}{(2k-1)2^{k-1}(k-1)!}\prod_{j=1}^{k-2}(A-B+2j)\right\}\right].$$

Proof: We have $zf'(z) = H(z)\wp(z)$ which on expansion gives

(5.3) $$z + 2a_2z^2 + 3a_3z^3 + \cdots + 2na_{2n}z^{2n} + (2n+1)a_{2n+1}z^{2n+1} + \cdots = [1 + p_1z + p_2z^2 + \cdots + p_{2n-1}z^{2n-1} + p_{2n}z^{2n} + \cdots][z + c_3z^3 + \cdots + c_{2n-1}z^{2n-1} + c_{2n+1}z^{2n+1} + \cdots]$$

Equating coefficients of z^{2n} and z^{2n+1} in (5.3), we have

(5.4) $$2na_{2n} = p_1c_{2n-1} + p_3c_{2n-3} + \cdots + p_{2n-3}c_3 + p_{2n-1}.$$

(5.5) $$(2n+1)a_{2n+1} = c_{2n+1} + p_2c_{2n-1} + \cdots + p_{2n-2}c_3 + p_{2n}.$$

Using lemmas 2.3 and 2.5 in (5.4), we have

$$2n|a_{2n}| \leq (C-D)\left(1+\sum_{k=2}^{n}|c_{2k-1}|\right) \leq (C-D)\left[1+\left\{\sum_{k=2}^{n}\frac{(A-B)}{(2k-1)2^{k-1}(k-1)!}\prod_{j=1}^{k-2}(A-B+2j)\right\}\right]$$

from which (5.1) follows.

By the application of lemmas 2.3 and 2.5 in (5.5), we get

$$(2n+1)|a_{2n+1}| \leq |c_{2n+1}| + (C-D)\left(1+\sum_{k=2}^{n}|c_{2k-1}|\right)$$

$$\leq \frac{(A-B)}{(2n+1)2^n n!}\prod_{j=1}^{n-1}(A-B+2j) + (C-D)\left[1+\left\{\sum_{k=2}^{n}\frac{(A-B)}{(2k-1)2^{k-1}(k-1)!}\prod_{j=1}^{k-2}(A-B+2j)\right\}\right]$$

which gives (5.2).

Choosing$H(\mathrm{z}) = \frac{\mathrm{h}_0(z)-\mathrm{h}_0(-z)}{2}$, $\mathrm{h}_0(z)$ is defined by (2.4), and $\wp(z) = \frac{1+Cz}{1+Dz}$, the extremal function is

$$(z) = \int_0^z \frac{H(t)\wp(t)}{t}\,dt.$$

If $\mathrm{A} = \mathrm{C} = 1$ and $\mathrm{B} = \mathrm{D} = -1$, we have following

Corollary: 5.1 If $f \in C_{1(s)}$, then

$$|a_{2n}| \leq \frac{1}{n}\left[1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n+1}+\cdots\right],$$

$$|a_{2n+1}| \leq \frac{1}{2n+1}\left[\frac{1}{2n+1}+2\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}\right)\right].$$

Theorem: 5.2 Let $f \in \mathrm{C}'_{1(\mathrm{s})}(A,B;C,D)$, then

$$(5.6)\qquad |a_{2n}| \leq \frac{(C-D)}{(2n)^2}\left\{1+\left(\sum_{k=2}^{n}\frac{(A-B)}{2^{k-1}(k-1)!}\prod_{j=1}^{k-2}(A-B+2j)\right)\right\},$$

$$(5.7)\qquad |a_{2n+1}| \leq \frac{1}{(2n+1)^2}\left[\frac{(A-B)}{2^n n!}\prod_{j=1}^{n-1}(A-B+2j) + (C-D)\left\{1+\sum_{k=2}^{n}\frac{(A-B)}{2^{k-1}(k-1)!}\prod_{j=1}^{k-2}(A-B+2j)\right\}\right].$$

The results are sharp.

Proof: We have

$$(5.8)\qquad \left(zf'(z)\right)' = \wp(z)H'(z)$$

Equating coefficients of $\mathrm{z}^{2\mathrm{n}-1}$ and $\mathrm{z}^{2\mathrm{n}}$ in (5.8), we have

$$(5.9)\qquad (2n)^2 a_{2n} = (2n-1)p_1c_{2n-1} + (2n-3)p_3c_{2n-3} + \cdots + 3p_{2n-3}c_3 + p_{2n-1},$$

$$(5.10)\qquad (2n+1)^2 a_{2n+1} = (2n+1)c_{2n+1} + (2n-1)p_2c_{2n-1} + \cdots + 3p_{2n-2}c_3 + p_{2n}.$$

Using lemmas 2.3 and 2.5 in (5.9), we get

$$(2n)^2|a_{2n}| \leq (C-D)\left(1+\sum_{k=2}^{n}(2k-1)|c_{2k-1}|\right) \leq (C-D)\left(1+\left\{\sum_{k=2}^{n}\frac{(A-B)}{2^{k-1}(k-1)!}\prod_{j=1}^{k-2}(A-B+2j)\right\}\right)$$

which is the result (5.6).

From (5.10) by lemmas 2.3 and 2.5, we obtain

$$(2n+1)^2|a_{2n+1}| \le (2n+1)|c_{2n+1}| + (C-D)\left(1+\sum_{k=2}^{n}(2k-1)|c_{2k-1}|\right)$$
$$\le \frac{(A-B)}{2^n n!}\prod_{j=1}^{n-1}(A-B+2j) + (C-D)\left(1+\left\{\sum_{k=2}^{n}\frac{(A-B)}{2^{k-1}(k-1)!}\prod_{j=1}^{k-2}(A-B+2j)\right\}\right)$$

from which (5.7) follows.

Choosing $H(z) = \frac{h_0(z)-h_0(-z)}{2}$, $h_0(z)$ is defined by (2.4), and $\wp(z) = \frac{1+Cz}{1+Dz}$, the extremal function is

$$f(z) = \left[\int_0^z \frac{1}{t}\left\{\int_0^t H'(u)\wp(u)du\right\}dt\right].$$

Taking $A = C = 1$ and $B = D = -1$ in the theorem, we get

Corollary 5.2 If $f \in C_{1(s)}$, then $|a_{2n}| \le \frac{1}{2n}$ and $|a_{2n+1}| \le \frac{1}{2n+1}$. That is $|a_n| \le \frac{1}{n}$.

Proceeding as in theorem 4.2 and using lemmas 2.3 and 2.5, we can prove

Theorem 5.3 Let $f \in C_{1(s)}(\alpha; A, B; C, D)$. Then

(5.11) $|a_2| \le \frac{(C-D)}{2(1+\alpha)}$,

(5.12) $|a_3| \le \frac{(1+2\alpha)(A-B)+6(C-D)}{18(1+2\alpha)}$,

(5.13) $|a_4| \le \frac{(C-D)}{24(1+\alpha)(1+3\alpha)}[(1+5\alpha)(A-B)+6(1+\alpha)]$,

(5.14) $|a_5| \le \frac{1}{600(1+2\alpha)(1+4\alpha)}\begin{bmatrix}3(1+2\alpha)(1+4\alpha)(A-B)(A-B+2)\\+20(C-D)\{(1+8\alpha)(A-B)+6(1+2\alpha)\}\end{bmatrix}$,

(5.15) $|a_6| \le \frac{(C-D)}{2160(1+\alpha)(1+3\alpha)(1+5\alpha)}\begin{bmatrix}40\alpha|\alpha-1|(A-B)^2+9(1+3\alpha)(1+9\alpha)(A-B)(A-B+2)\\+60(1+\alpha)(1+11\alpha)(A-B)+360(1+\alpha)(1+3\alpha)\end{bmatrix}$.

The extremal function is obtained by choosing $H(z) = \frac{h_0(z)-h_0(-z)}{2}$ and $\wp(z) = \frac{1+Cz}{1+Dz}$ in the integral representation formula

(i) for $\alpha = 0$, $f(z) = \int_0^z \frac{H(t)\wp(t)}{t}dt$,

(ii) for $\alpha > 0$, $f(z) = (1+c)\left[\int_0^z \frac{1}{t(H(t))^c}\left\{\int_0^t (H(u))^c H'(u)\wp(u)du\right\}dt\right], \left(c = \frac{1}{\alpha}-1\right)$.

Letting $A = C = 1$ and $B = D = -1$, we have

Corollary 5.3 If $f \in C_{1(s)}(\alpha)$, then

$$|a_2| \le \frac{1}{(1+\alpha)},$$

$$|a_3| \le \frac{7+2\alpha}{9(1+2\alpha)},$$

$$|a_4| \le \frac{2(1+2\alpha)}{3(1+\alpha)(1+3\alpha)},$$

$$|a_5| \leq \frac{43+158\alpha+24\alpha^2}{75(1+2\alpha)(1+4\alpha)},$$

$$|a_6| \leq \begin{cases} \frac{69+488\alpha+523\alpha^2}{135(1+\alpha)(1+3\alpha)(1+5\alpha)}, & 0 \leq \alpha \leq 1, \\ \frac{69+448\alpha+563\alpha^2}{135(1+\alpha)(1+3\alpha)(1+5\alpha)}, & \alpha \geq 1. \end{cases}$$

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ABBREVIATED TITLE: *New sub-classes of the class of close to convex functions with respect to symmetric points.*

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