DISTANCE BASED CONNECTIVITY
STATUS NEIGHBORHOOD INDICES OF CERTAIN GRAPHS

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ABSTRACT

Many distance based indices of a graph have been appeared in the literature. The status of a vertex \( u \) in a connected graph \( G \) is defined as the sum of the distance between \( u \) and all other vertices of \( G \). In this paper, we introduce the sum connectivity status neighborhood index, product connectivity status neighborhood index, reciprocal product connectivity status neighborhood index, general first and second status neighborhood indices of a graph and compute their values for some standard graphs, wheel and friendship graphs.

Keywords: distance, status, sum and product connectivity status neighborhood indices, general status neighborhood indices, wheel graph, friendship graph.

Mathematics Subject Classification: 05C05, 05C07, 05C12, 05C35.

1. INTRODUCTION

In this paper, we are concerned with simple graphs. Let \( G \) be a connected graph. Let \( V(G) \) and \( E(G) \) be its vertex and edge sets respectively. The edge between the vertices \( u \) and \( v \) is denoted by \( uv \). The degree of a vertex \( u \) is the number of vertices adjacent to \( u \) and is denoted by \( d_G(u) \). The distance between two vertices \( u \) and \( v \) is the length of shortest path connecting \( u \) and \( v \). The status \( \sigma(u) \) of a vertex \( u \) in \( G \) is the sum of distances of all other vertices from \( u \) in \( G \). Let \( N(v) = N_G(v) = \{u:uv \in E(G)\} \). Let \( \sigma_n(v) = \sum_{u \in N(v)} \sigma(u) \) be the status sum of neighbor vertices. We refer [1] for undefined terms and notations from graph theory.

A graph index is a numerical parameter mathematically derived from the graph structure. Graph indices [2] have applications in various disciplines of Science and Technology [3, 4]. Some of the graph indices may be found in [5, 6, 7, 8].

The first and second status neighborhood indices are introduced by Kulli in [9], and they are defined as

\[
SN_1(G) = \sum_{uv \in E(G)} \left[ \sigma_n(u) + \sigma_n(v) \right], \quad SN_2(G) = \sum_{uv \in E(G)} \sigma_n(u)\sigma_n(v).
\]

Recently some new status neighborhood indices were studied in [10, 11].

We introduce some connectivity status neighborhood indices as follows:

The sum connectivity status neighborhood index of a graph \( G \) is defined as

\[
SSN(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{\sigma_n(u) + \sigma_n(v)}}.
\]
The product connectivity status neighborhood index of a graph $G$ is defined as
\[ PSN(G) = \sum_{uv \in E(G)} \frac{1}{\sigma_n(u) \sigma_n(v)}. \]

The reciprocal product connectivity status neighborhood index of a graph $G$ is defined as
\[ RPSN(G) = \sum_{uv \in E(G)} \frac{1}{\sigma_n(u) \sigma_n(v)}. \]

The modified first status neighborhood index of a graph $G$ is defined as
\[ mSN_1(G) = \sum_{uv \in E(G)} \frac{1}{\sigma_n(u) + \sigma_n(v)}. \]

The modified second status neighborhood index of a graph $G$ is defined as
\[ mSN_2(G) = \sum_{uv \in E(G)} \frac{1}{\sigma_n(u) \sigma_n(v)}. \]

We continue these generalizations and introduce the general first and second status neighborhood indices of a graph and they are defined as
\[ SN^a_1(G) = \sum_{uv \in E(G)} \left[ \sigma_n(u) + \sigma_n(v) \right]^a, \]
\[ SN^a_2(G) = \sum_{uv \in E(G)} \left[ \sigma_n(u) \sigma_n(v) \right]^a, \]
where $a$ is a real number.

Recently, some variants of status indices were studied, for example, in [12, 13, 14, 15, 16, 17, 18, 19, 20, 21].

In this paper, the sum connectivity status neighborhood index, product connectivity status neighborhood index, reciprocal product connectivity status neighborhood index, modified first and second status neighborhood indices, general first and second status neighborhood indices of some standard graphs, wheel and friendship graphs are determined.

2. RESULTS FOR COMPLETE GRAPHS

**Theorem 1:** The general first status neighborhood index of a complete graph $K_n$ is
\[ SN^a_1(K_n) = \frac{n(n-1)}{2} \left[ 2(n-1)^2 \right]^a. \] (1)

**Proof:** Let $K_n$ be a complete graph with $n$ vertices and $\frac{n(n-1)}{2}$ edges. For any vertex $u$ of $K_n$, $\sigma(u) = n - 1$. Clearly $\sigma(u) = (n-1)^2$ for any vertex $u$ of $K_n$. Therefore
\[ SN^a_1(K_n) = \sum_{uv \in E(K_n)} \left[ \sigma_n(u) + \sigma_n(u) \right]^a = \left[ (n-1)^2 + (n-1)^2 \right]^a \frac{n(n-1)}{2}. \]
\[ = \frac{n(n-1)}{2} \left[ 2(n-1)^2 \right]^a. \]
From Theorem 1, we establish the following results.

**Corollary 1.1:** Let $K_n$ be a complete graph with $n$ vertices. Then
(i) $SSN(K_n) = \frac{n}{2\sqrt{2}}$.
(ii) $mSN_1(K_n) = \frac{n}{4(n-1)}$.

**Proof:** Put $a = -\frac{1}{2}$, $-1$ in equation (1), we obtain the desired results.
Theorem 2: The general second status neighborhood index of a complete graph $K_n$ is

$$SN^2_n(K_n) = \frac{n(n-1)}{2} (n-1)^{4a}. \quad (2)$$

Proof: Let $K_n$ be a complete graph with $n$ vertices and $\frac{n(n-1)}{2}$ edges. For any vertex $u$ of $K_n$, $\sigma_n(u) = (n-1)^2$. Thus

$$SN^2_n(K_n) = \sum_{u \in V(K_n)} [\sigma_n(u) \sigma_n(u)]^a = \left[ (n-1)^2 (n-1)^2 \right]^a \frac{n(n-1)}{2}.
= \frac{n(n-1)}{2} (n-1)^{4a}.$$

We obtain the following results by using Theorem 2.

Corollary 2.1: Let $K_n$ be a complete graph with $n$ vertices. Then

(i) $PSN(K_n) = \frac{n}{2(n-1)}$.
(ii) $RPSN(K_n) = \frac{1}{2} n(n-1)^3$.
(iii) $m SN^2(K_n) = \frac{n}{2(n-1)^3}$.

Proof: Put $a = -\frac{1}{2}$, $\frac{1}{2}$, $-1$ in equation (2), we get the desired results.

3. RESULTS FOR COMPLETE BIPARTITE GRAPHS

Theorem 3: The general first status neighborhood index of a complete bipartite graph $K_{p,q}$ is

$$SN^1_{i}(K_{p,q}) = pq \left[ 2 \left( p^2 + q^2 \right) - 2(p + q) + 2pq \right]^a. \quad (3)$$

Proof: Let $K_{p,q}$ be a complete graph with $p+q$ vertices and $pq$ edges. For vertex set of $K_{p,q}$ can be partitioned into two independent sets $V_1$ and $V_2$ such that $u \in V_1$ and $v \in V_2$ for every edge $uv$ in $K_{p,q}$. Therefore $d_u(u) = q$, $d_v(v) = p$, where $K_{p,q}$. Then $\sigma(u) = q + 2p - 2$ and $\sigma(v) = p + 2q - 2$. Thus by calculation, we have $\sigma_n(u) = p(q + 2p - 2)$ and $\sigma_n(v) = q(p + 2q - 2)$. Therefore

$$SN^1_{i}(K_{p,q}) = \sum_{u \in V(K)} [\sigma_n(u) + \sigma_n(u)]^a = pq \left[ p(q + 2p - 2) + q(p + 2q - 2) \right]^a.
= pq \left[ 2 \left( p^2 + q^2 \right) - 2(p + q) + 2pq \right]^a.$$

By using Theorem 3, we establish the following results.

Corollary 3.1: Let $K_{p,q}$ be a complete bipartite graph. Then

(i) $SSN(K_{p,q}) = pq \left[ \left( p^2 + q^2 \right) - 2(p + q) + 2pq \right]^{\frac{1}{2}}.
(ii) m SN^1(K_{p,q}) = \frac{pq}{2 \left( p^2 + q^2 \right) - 2(p + q) + 2pq}.$

Proof: Put $a = -\frac{1}{2}$, $-1$ in equation (3), we obtain the desired results.

Theorem 4: The general second status neighborhood index of a complete bipartite graph $K_{p,q}$ is

$$SN^2_{i}(K_{p,q}) = pq \left[ 2pq \left( p^2 + q^2 \right) - 6pq(p + q) + 5p^2q^2 + 4pq \right]^a. \quad (4)$$

Proof: Let $K_{p,q}$ be a complete bipartite graph with $p+q$ vertices and $pq$ edges. By calculation, we obtain $\sigma_n(u) = p(q + 2p - 2)$ and $\sigma_n(v) = q(p + 2q - 2)$. Thus

$$SN^2_{i}(K_{p,q}) = \sum_{u \in V(K)} \sigma_n(u) \sigma_n(v) = pq \left[ p(q + 2p - 2)q(p + 2q - 2) \right]^a
= pq \left[ 2pq \left( p^2 + q^2 \right) - 6pq(p + q) + 5p^2q^2 + 4pq \right]^a.$$
We obtain the following results by using Theorem 4.

**Corollary 4.1:** Let $K_{p,q}$ be a complete bipartite graph. Then

\begin{align*}
(i) \quad & \text{PSN}(K_{p,q}) = pq \left[ 2pq \left( p^2 + q^2 \right) - 6pq(p + q) + 5p^2q^2 + 4pq \right]^{\frac{1}{2}}. \\
(ii) \quad & \text{RPSN}(K_{p,q}) = pq \left[ 2pq \left( p^2 + q^2 \right) - 6pq(p + q) + 5p^2q^2 + 4pq \right]^{\frac{1}{2}}. \\
(iii) \quad & mSN_{2}(K_{p,q}) = pq \left[ 2pq \left( p^2 + q^2 \right) - 6pq(p + q) + 5p^2q^2 + 4pq \right]^{-1}.
\end{align*}

**Proof:** Put $a = -\frac{1}{2}, \frac{1}{2}, -1$ in equation (4), we obtain the desired results.

### 4. RESULTS FOR WHEEL GRAPHS

A wheel graph is the join of $C_n$ and $K_1$ and it is denoted by $W_n$. This graph has $n+1$ vertices and $2n$ edges. A graph $W_4$ is presented in Figure 1.

![Figure-1: Wheel graph $W_4$](image)

In a graph $W_n$, there are two types of edges as follows:

\[
E_1 = \{uv \in E(W_n) \mid d_{W_n}(u) = d_{W_n}(v) = 3\}, \quad |E_1| = n.
\]

\[
E_2 = \{uv \in E(W_n) \mid d_{W_n}(u) = 3, d_{W_n}(v) = n\}, \quad |E_2| = n.
\]

Therefore, in $W_n$, there are two types of status edges as follows.

<table>
<thead>
<tr>
<th>$\sigma(u), \sigma(v) \setminus uv \in E(W_n)$</th>
<th>$(2n - 3, 2n - 3)$</th>
<th>$(n, 2n - 3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of edges</td>
<td>$n$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

**Table-1:** Status edge partition of $W_n$

By calculation, we obtain that there are two types of status neighborhood edges as given in Table 2.

<table>
<thead>
<tr>
<th>$\sigma_s(u), \sigma_s(v) \setminus uv \in E(W_n)$</th>
<th>$(5n - 6, 5n - 6)$</th>
<th>$5n - 6, n(2n - 3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of vertices</td>
<td>$n$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

**Table-2:** Status neighborhood edge partition of $W_n$

**Theorem 5:** The general first status neighborhood index of a wheel graph $W_n$ is given by

\[
SN_{1}^{u}(W_n) = n(10n - 12)^u + n(2n^2 + 2n - 6)^u.
\]

**Proof:** By definition and by using Table 2, we deduce

\[
SN_{1}^{u}(W_n) = \sum_{uv \in E(W_n)} \left[ \sigma_s(u) + \sigma_s(v) \right]^u
\]

\[
= n(5n - 6 + 5n - 6)^u + n(5n - 6 + 2n^2 - 3n)^u
\]

\[
= n(10n - 12)^u + n(2n^2 + 2n - 6)^u.
\]
From Theorem 5, we establish the following results.

**Corollary 5.1:** Let \( W_n \) be a wheel graph with \( n+1 \) vertices and \( 2n \) edges. Then

\[
\begin{align*}
(i) & \quad \text{SSN} \left( W_n \right) = \frac{n}{\sqrt{10n - 12}} + \frac{n}{2\sqrt{2n^2 + 2n - 6}} \\
(ii) & \quad m\text{SN}_1 \left( W_n \right) = \frac{n}{10n - 12} + \frac{n}{2n^2 + 2n - 6}
\end{align*}
\]

**Proof:** Put \( a = -\frac{1}{2}, \frac{1}{2}, -1 \) in equation (5), we obtain the desired results.

**Theorem 6:** The general second status neighborhood index of a wheel graph \( W_n \) is given by

\[
\text{SN}_2 \left( W_n \right) = n(5n - 6)^2a + n\left(10n^3 - 27n^2 + 18n\right)^a. 
\]

**Proof:** Let \( W_n \) be a wheel graph \( n+1 \) vertices \( 2n \) edges. By definition and by using Table 2, we derive

\[
\text{SN}_2 \left( W_n \right) = \sum_{\sigma(u), \sigma(v) \in \text{E}(W_n)} \left[ \sigma(u) \sigma(v) \right]^a \\
= n[(5n - 6)(5n - 6)]^a + n[(5n - 6)(2n^2 - 3n)]^a \\
= n(5n - 6)^2a + n\left(10n^3 - 27n^2 + 18n\right)^a.
\]

By using Theorem 6, we obtain the following results.

**Corollary 6.1:** Let \( W_n \) be a wheel graph with \( n+1 \) vertices and \( 2n \) edges. Then

\[
\begin{align*}
(i) & \quad \text{PSN} \left( W_n \right) = \frac{n}{5n - 6} + \frac{n}{\sqrt{10n^3 - 27n^2 + 18n}}. \\
(ii) & \quad \text{RPSN} \left( W_n \right) = n(5n - 6) + n\sqrt{10n^3 - 27n^2 + 18n}. \\
(iii) & \quad m\text{SN}_2 \left( W_n \right) = \frac{n}{(5n - 6)^2} + \frac{n}{10n^3 - 27n^2 + 18n}.
\end{align*}
\]

**Proof:** Put \( a = -\frac{1}{2}, \frac{1}{2}, -1 \) in equation (6), we get the desired results.

5. RESULTS FOR FRIENDSHIP GRAPHS

A friendship graph, denoted by \( F_n \), is the graph obtained by taking \( n \geq 2 \) copies of \( C_3 \) with vertex in common. A graph \( F_n \) has \( 2n+1 \) vertices and \( 3n \) edges. A graph \( F_4 \) is presented in Figure 2.

5. RESULTS FOR FRIENDSHIP GRAPHS

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**Figure-2:** Friendship graph \( F_4 \)

Let \( F = F_n \). In a graph \( F_n \), we obtain two types of edges as follows:

\[
\begin{align*}
E_1 &= \{ uv \in \text{E}(F) \mid d_1(u) = d_1(v) = 2 \}, \quad |E_1| = n. \\
E_2 &= \{ uv \in \text{E}(F) \mid d_2(u) = 2, d_2(v) = 2n \}, \quad |E_2| = 2n.
\end{align*}
\]

Therefore in a graph \( F_n \), we find two types of status edges as given in Table 3.

<table>
<thead>
<tr>
<th>( \sigma(u), \sigma(v) \mid uv \in \text{E}(F_n) )</th>
<th>( (4n - 2, 4n - 2) )</th>
<th>( (2n, 4n - 2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of edges</td>
<td>( n )</td>
<td>( 2n )</td>
</tr>
</tbody>
</table>

Table-3: Status edge partition of \( F_n \)

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By calculation, we find that there are two types of status neighborhood edges as given in Table 4.

<table>
<thead>
<tr>
<th>$\sigma_n(u) \cdot \sigma_n(v)$ \ uv \in E(F_n)</th>
<th>(6n - 2, 6n - 2)</th>
<th>(6n - 2, 2n(4n - 2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of edges</td>
<td>$n$</td>
<td>$2n$</td>
</tr>
</tbody>
</table>

Table 4: Status neighborhood edge partition of $F_n$

Theorem 7: The general first status neighborhood index of a friendship graph $F_n$ is given by

$$SN_1^a(F_n) = n(12n - 4)^a + 2n(8n^2 + 2n - 2)^a.$$  \hfill (7)

Proof: By definition and by using Table 4, we obtain

$$SN_1^a(F_n) = \sum_{uv \in E(F_n)} [\sigma_n(u) + \sigma_n(v)]^a$$

$$= n(6n - 2 + 6n - 2)^a + 2n(6n - 2 + 8n^2 - 4n)^a$$

$$= n(12n - 4)^a + 2n(8n^2 + 2n - 2)^a.$$  

By using Theorem 7, we obtain the following results.

Corollary 7.1: Let $F_n$ be a friendship graph with $2n+1$ vertices and $3n$ edges. Then

(i) $SSN(F_n) = \frac{n}{2\sqrt{3n-1}} + \frac{2n}{\sqrt{8n^2 + 2n - 2}}.$

(ii) $mSN_1(F_n) = \frac{n}{12n - 4} + \frac{n}{4n^2 + n - 1}.$

Proof: Put $a = -\frac{1}{2}$, $-1$ in equation (7), we get the desired results.

Theorem 8: The general second status neighborhood index of a friendship graph $F_n$ is

$$SN_2^a(F_n) = n(6n - 2)^{2a} + 2n\left[4(12n^3 - 10n^2 + 2n)\right]^a.$$  \hfill (8)

Proof: From definition and by using Table 4, we deduce

$$SN_2^a(F_n) = \sum_{uv \in E(F_n)} [\sigma_n(u)\sigma_n(v)]^a$$

$$= n[(6n - 2)(6n - 2)]^a + 2n[(6n - 2)(8n^2 - 4n)]^a$$

$$= n(6n - 2)^{2a} + 2n\left[4(12n^3 - 10n^2 + 2n)\right]^a.$$  

We obtain the following results from Theorem 8.

Corollary 8.1: Let $F_n$ be a friendship graph with $2n+1$ vertices and $3n$ edges. Then

(i) $PSN(F_n) = \frac{n}{6n - 2} + \frac{n}{\sqrt{12n^3 - 10n^2 + 2n}}.$

(ii) $RPSN(F_n) = n(6n - 2) + 4n\sqrt{12n^3 - 10n^2 + 2n}.$

(iii) $mSN_2(F_n) = \frac{n}{(6n - 2)^2} + \frac{n}{2(12n^3 - 10n^2 + 2n)}.$

Proof: Put $a = -\frac{1}{2}, \frac{1}{2}, -1$ in equation (8), we obtain the desired results.

REFERENCES


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