RIGHT ANGLE TRIANGLE
WITH APPLICATION OF TWO CIRCLES ON THE BASE OF TRIANGLE

RAJBALI YADAV*
Ordnance Factory Khamaria, Jabalpur (M.P,) - 482005, India.

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ABSTRACT
In this paper relation among diameter and radius of two circles drawn on the base of right angle triangle and height of the triangle is shown. By the application of two circles on the base of right angled triangle it is shown that this right angle triangle is an extension of an obtuse triangle in which difference of two angle is, 90°. Finally it will be proved that square difference of diameter of larger circle and radius of smaller circle, drawn on base of right angle triangle is equal to height square of right angle triangle.

Keywords: Circle, Maximum area, Obtuse triangle, Right angle triangle.

1. INTRODUCTION
This theory is, a deduction from maximum or minimum area of ΔADE (as shown in diagram1). For being maximum or minimum area of ΔADE difference of two angle in ΔABC must be 90°. In this condition ΔABC become an obtuse triangle. For constructing such triangle, in two circles of different diameter (as shown in diagram1) if larger diameter circle pass through centre of smaller diameter circle, line drawn from intersection of circles and initial point of center line of first circle and a perpendicular line drawn from end point of second circle center line, which cut diagonal at some point, through this point line drawn to last point of first circle, an obtuse triangle (ΔABC) with difference of two angle 90° is obtain. In right angle ΔADC, square difference of, diameter of larger circle and radius of smaller circle is equal to height square of right angle triangle.

Definition 1.1: Obtuse Triangle. An obtuse-angled triangle is a triangle in which one of the interior angles measures more than 90°. In an obtuse triangle, if one angle measures more than 90° than the sum of remaining two angles is less than 90°.

2. RIGHT ANGLED TRIANGLE WITH APPLICATION OF TWO CIRCLES ON THE BASE OF TRIANGLE
2.1. In right angle triangle if base of right angle triangle is equal to sum of diameter and radius of two circles in which larger diameter circle pass through center of smaller radius circle and Hypotenuse passes from intersection of these two circles then ‘Square difference of diameter of larger circle and radius of smaller circle is equal to height square of right angle triangle’. It can be mathematically written as: $d^2 - r^2 = h^2$

Corresponding Author: Rajbali Yadav*
Ordnance Factory Khamaria, Jabalpur (M.P,) - 482005, India.
Note:
C, initial point of center line CD 1st circle
D, End point of center line CD 2nd circle
B, Last point of 1st circle
E is mid of BC
AB = c, BC =a, AC = b
Angle ABC = β, Angle BAC = α, Angle ACB = C

Steps involved to prove above theorem:
For max. area of Δ ADE in ΔABC
1st \( a = \frac{b^2-c^2}{\sqrt{b^2+c^2}} \)
2nd \( \alpha +2\beta = 270^\circ \) Or \( \beta - C = 90^\circ \)
3rd \( BC \sin(\alpha +2\beta) +2DE \sin \alpha = 0 \)
4th \( AB^2 +AC^2 = (2DE)^2 \)

And finally
\( DE^2 - CE^2 = AD^2 \) Or \( d^2 - r^2 = h^2 \)

2.1 Proof.
in Δ ABC (Diagram 3)
E is mid of BC
D is perpendicular at at BC
BD = (a/2-DE), DC = (a/2+DE)

In Δ ADB
\( AD^2 = c^2 - (a/2-DE)^2 \) ……… 1

In Δ ADC
\( AD^2 = b^2 - (a/2+DE)^2 \) …… 2

After solving 1 & 2
\( c^2 - (a/2-DE)^2 = b^2 - (a/2+DE)^2 \)
\( c^2 - (a/2)^2 - DE^2 + DE \times a = b^2 - (a/2)^2 - DE^2 - DE \times a \)
\( c^2 + DE \times a = b^2 - DE \times a \)
\( DE = \frac{b^2 - c^2}{2a} \)

Put the value of DE in equation 2
\( AD^2 = b^2 - \left[ \frac{a}{2} - \frac{(b^2-c^2)}{2a} \right]^2 \) OR \( AD = \frac{\sqrt{(2ab)^2 - (a^2 + b^2 - c^2)^2}}{2a} \)

Area of Δ ADE (A1) = \( \frac{1}{2} \) AD \times DE
\( = \frac{1}{2} \times \frac{\sqrt{(2ab)^2 - (a^2 + b^2 - c^2)^2}}{2a} \times \frac{(b^2-c^2)}{2a} \)

\( A1 = \frac{(b^2 - c^2)}{8a^2} \times \sqrt{[(2ab)^2 - (a^2 + b^2 - c^2)^2]} \)
\( A1^2 = \frac{(b^2 - c^2)}{64a^4} \times \left[(2ab)^2 - (a^2 + b^2 - c^2)^2\right] \)

For being area of Δ ADE Max. Or Mini.
\( \frac{dA1}{da} = \frac{dA1}{db} = \frac{dA1}{dc} = 0 \)

Take \( \frac{dA1}{da} = 0 \)
\( 2A1 \frac{dA1}{da} = \frac{d}{da} \left[ \frac{(b^2-c^2)}{a^4} \times \left[(2ab)^2 - (a^2 + b^2 - c^2)^2\right] \right] \)
\( 128A1 \frac{dA1}{da} = (b^2 - c^2) \times \frac{d}{da} \left[ \frac{1}{a^4} \times \left[(2ab)^2 - (a^2 + b^2 - c^2)^2\right] \right] \)
\( = (b^2-c^2)^2 \times \frac{d}{da} \left[ 4b^2 / a^2 - (1 + b^4 / a^4 + 2b^2 / a^2 + c^4 / a^4 - 2c^2 / a^2 - 2b^2c^2 / a^4) \right] \)
\[
d/d[4b^4/a^2 - (1 + b4/a4 + 2b^2/a^2 + c4/a^2 - 2c^2/a^2 - 2b^2c^2/a^2) = 0 \quad \text{Put } dA1/da = 0
\]

\[
-8b^4/a3 + 4b^4/a5 + 4b^2/a3 + 4c^4/a5 - 4c^2/a3 - 8b^2c^2/a5 = 0
\]

\[
-b^4/a^2 + c^4/a^2 - 2b^2c^2/a^2 = b^2 + c^2
\]

\[
[(b^2 - c^2)/a]^2 = b^2 + c^2
\]

\[
a = b^2 - c^2/\sqrt{b^2 + c^2}
\]

\[
2.2 \text{ According to Cosine rule}
\]

\[
\cos \alpha = (b^2 + c^2 - a^2)/2bc
\]

\[
\text{Put the value of } a \text{ in equation (3)}
\]

\[
\cos \alpha = 2bc/(b^2 + c^2)
\]

\[
\text{Add 1 both side}
\]

\[
\cos \alpha + 1 = 1
\]

\[
\cos \alpha + 1 = (b + c)^2/(b^2 + c^2)
\]

Equation (2) can also be written as

\[
[a/(b - c)]^2 = (b + c)^2/(b^2 + c^2)
\]

\[
\text{Hence}
\]

\[
\cos \alpha + 1 = [a/(b - c)]^2
\]

\[
a/(b - c) = \sqrt{1 + \cos \alpha}
\]

\[
\text{C = 180° - (a + b)}
\]

\[
\sin \beta - \sin[180° - (a + b)] = \sin \alpha/\sqrt{2} \cos \alpha/2
\]

\[
\sin \beta - \sin(a + b) = \sqrt{2} \sin \alpha/2
\]

\[
2 \cos[(a + b)/2] \sin[(b - a)/2] = \sqrt{2} \sin \alpha/2
\]

\[
-2 \cos(\alpha + 2\beta) = \sqrt{2}
\]

\[
\cos(\alpha + 2\beta)/2 = -1/\sqrt{2} = \cos(180° - 45°)
\]

\[
\alpha + 2\beta = 270°
\]

\[
\alpha + \beta + \beta = 270°
\]

\[
180° - \beta = 270° \text{ or } \beta - C = 90°
\]

This shows that for maximum area of \( \Delta ADE \), difference of two angles in \( \Delta ABC \) will be 90° and for constructing such triangle when between two circles of different diameters (as shown in diagram1) larger diameter circle pass through centre of smaller dia. circle, line drawn from intersection of circles and initial point of center line of first circle and a perpendicular line drawn from end point of second circle center line, which cut diagonal at some point, through this point line drawn to last point of first circle, an obtuse triangle with difference of two angle 90° is obtain. This means for maximum area of \( \Delta ADE \) all relation found in \( \Delta ABC \) (diagram3), will also be true for Obtuse \( \Delta ABC \) with difference of two angle 90°, constructed using two circles as shown in diagram1.
In diagram (3)
E is mid of BC line
Area of Δ ADE (A1) = \( \frac{(b^2 - c^2)}{8a^2} \times \sqrt{[4(ab)^2 - (a^2 + b^2 - c^2)^2]} \)

And
Area of Δ ABC (A) = \( \frac{1}{4} \times \sqrt{[4(ab)^2 - (a^2 + b^2 - c^2)^2]} \)

Dividing A1/A
\[
\frac{A1}{A} = \frac{(b^2 - c^2)/8a^2}{1/4 \times \sqrt{[4(ab)^2 - (a^2 + b^2 - c^2)^2]}}
\]
\[
A1/A = (b^2 - c^2)/2a^2 \quad (4)
\]

According to diagram no.3
\[
\frac{A1}{A} = \frac{DE}{BC}
\]
Put the value of A1/A in equation (4)
\[
\frac{DE}{BC} = \frac{(b^2 - c^2)/2a^2}{2a^2} \quad (5)
\]

Using Sine rule
\[
\frac{2DE}{BC} = \frac{\sin^2\beta - \sin^2C}{\sin^2\alpha}
\]
\[
= \frac{\sin^2\alpha}{[\sin(180 - \alpha + \beta)\sin\beta - \sin(180 - \alpha + \beta)]} \quad C = 180 - (\alpha + \beta)
\]
\[
= \frac{\sin\beta + \sin(\alpha + \beta)\sin\beta - \sin(\alpha + \beta)}{\sin^2\alpha}
\]
\[
= \frac{2\sin[(\alpha + 2\beta)/2] \cos(\alpha/2) - 2\cos[(\alpha + 2\beta)/2] \sin(\alpha/2)}{\sin^2\alpha}
\]
\[
= -\frac{2\sin(\alpha + 2\beta)/2 \cos(\alpha + 2\beta)/2 \times 2\sin(\alpha/2)\cos(\alpha/2)}{\sin^2\alpha}
\]
\[
= -\sin[(\alpha + 2\beta) \times \sin\alpha]/\sin^2\alpha
\]
2DE/BC = -\sin[(\alpha + 2\beta)/\sin\alpha
2DE Sina = - BC Sin((\alpha + 2\beta)

BC Sin(\alpha + 2\beta) + 2DE Sin\alpha = 0

By previous relation it is proved that for maximum area of Δ ADE
\( \alpha + 2\beta = 270^\circ \)

Put this value in above equation
BC Sin 270° + 2DE Sina = 0, - BC +2DE Sina = 0
Sina = BC/2DE as BC = a and DE = a1 So, Sina = a/2a1

According to equation (5) 2DE/BC = (b^2 - c^2)/a^2
\[
\sin\alpha = BC/2DE
\]
\[
= 1/(b^2 - c^2)/a^2
\]
\[
\sin\alpha = a^2/(b^2 - c^2) \quad (7)
\]

According to equation no. 2
a^2 = \( \frac{(b^2 - c^2)^2}{b^2 + c^2} \)
put the value of a^2 in equation no. 7
\[
\sin \alpha = \frac{\left(b^2 - c^2\right)^2}{\left(b^2 + c^2\right)} \div \frac{\left(b^2 - c^2\right)}{1}
\]

\[
\sin \alpha = \frac{b^2 - c^2}{b^2 + c^2}
\]

According to equation (6) \(\sin \alpha = a/2a1\)

Put the value of \(a\) from equation no. 2 and \(\sin \alpha\) from equation (8) in equation (6)

\[
\left[\left(b^2 - c^2\right)^2 \div \left(b^2 + c^2\right)\right] = \frac{\left(b^2 - c^2\right)}{b^2 - c^2}
\]

\[
\frac{\left(b^2 + c^2\right)}{2a1 \times \sqrt{\left(b^2 + c^2\right)}}
\]

\[
2a1 = \frac{\left(b^2 + c^2\right)}{\sqrt{\left(b^2 + c^2\right)}}
\]

\[
(2a1)^2 = b^2 + c^2 \quad \text{OR} \quad (2DE)^2 = AC^2 + AB^2
\]

As \(\, AB = c \quad AC = b \quad DE = a1\)

\[
AC^2 + AB^2 = (2DE)^2
\]

For maximum area of \(\Delta ADE\)

Finally it will be proved that in diagram 1 & 2

\(DE^2 - CE^2 = AD^2\)

\(\text{Or} \quad d^2 - r^2 = h^2\)

In \(\Delta ABD\)

\[
AB^2 = AD^2 + BD^2
\]

\[
= AD^2 + (DE + BE)^2
\]

\[
= AD^2 + DE^2 + BE^2 + 2DE \times BE
\]

\[
AB^2 = AD^2 + DE^2 + BE^2 + DE \times BC
\]

In diagram 1 & 2 \(BE = BC/2\)

In \(\Delta ADC\)

\[
AC^2 = AD^2 + DC^2
\]

\[
= AD^2 + (CE - DE)^2
\]

\[
= AD^2 + CE^2 + DE^2 - 2CE \times DE
\]

\[
= AD^2 + CE^2 + DE^2 - BC \times DE
\]

In diagram 1 & 2 \(CE = BC/2\)

ADD eqn. (10) & (11)

\[
AB^2 + AC^2 = 2AD^2 + 2DE^2 + BC^2/2
\]

According to eqn. (9) \(AC^2 + AB^2 = (2DE)^2\)

Put the value of \(AC^2 + AB^2\) in eqn (12)

\[
4DE^2 = 2AD^2 + 2DE^2 + BC^2/2
\]

\[
2DE^2 = 2AD^2 + BC^2/2
\]

\[
4DE^2 = 4AD^2 + BC^2 \quad \text{BC} = 2CE
\]

\[
4DE^2 = 4AD^2 + 4CE^2
\]

\[
DE^2 = AD^2 + CE^2
\]

\(\text{Or} \quad DE^2 - CE^2 = AD^2\)

In diagram 1 & 2

\(DE = d\)

\(CE = r\)

\(AD = h\)

\(\text{hence} \quad d^2 - r^2 = h^2\)
3. CONCLUSION

This theory can help to understand right angled triangle after knowing one set of constraint i.e diameter of circles, in which one pass through centre of other. Height and Hypotenuse of right triangle can be find easily.

4. REFERENCE

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