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BEST SIMULTANEOUS APPROXIMATION IN FUZZY ANTI-*n*-NORMED LINEAR SPACES

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ABSTRACT

The main aim of this paper is to consider the t-best simultaneous approximation in fuzzy anti-n-normed linear spaces. We develop the theory of t-best simultaneous approximation in its quotient spaces. Then we discuss the relationship in t-proximinality and t-Chebyshevity of a given space and its quotient space.

Key Words: *Fuzzy anti-n-norms, simultaneous approximation, simultaneous t-proximinality, simultaneous t-chebyshevity, Quotient space.*

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1. INTRODUCTION:

Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. The theory of fuzzy sets was introduced by Zadeh [24] in 965. The idea of fuzzy norm was initiated by Katsaras in [13]. Felbin [6] defined a fuzzy norm on a linear space whose associated fuzzy metric is of Kaleva and Seikkala type [12]. Cheng and Mordeson [4] introduced an idea of a fuzzy norm on a linear space whose associated metric is Kramosil and Michalek type [14].

Bag and Samanta in [1] gave a definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [14]. They also studied some properties of the fuzzy norm in [2] and [3]. Bag and Samanta discussed the notion of convergent sequence and Cauchy sequence in fuzzy normed linear space in [1]. They also made in [3] a comparative study of the fuzzy norms defined by Katsaras [13], Felbin [6], and Bag and Samanta [1]. The concept of *n*-norm on a linear space has been introduced and developed by Gahler in [7,8]. Following Misiak [16], Malceski [15] and Gunawan and Mashadi [10] developed the theory of *n*-normed space. Narayana and Vijayabalaji [17] introduced the concept of fuzzy *n*-normed linear space. Vijayabalaji and Thillaigovindan [23] introduced the notion of and convergent sequence and Cauchy sequence in fuzzy *n*-normed linear space and studied the completeness of the fuzzy *n*-normed linear space. Many authors studied on fuzzy *n*-normed linear space [5]. Vaezpour and Karimi [22], studied on the set of all *t*-best approximations on fuzzy normed linear spaces. Goudarzi and Vaezpour [9] considered the set of all *t*-best simultaneous approximation in fuzzy normed linear spaces.

In [11] Iqbal H. Jebril and Samanta introduced fuzzy anti-norm on a linear space depending on the idea of fuzzy anti-norm was introduced by Bag and Samanta [3] and investigated their important properties. In [18,19] Surender Reddy introduced the notion of convergent sequence and Cauchy sequence in fuzzy anti-2-normed linear space and fuzzy anti-*n*-normed linear space. In [20] Surender Reddy studied on the set of all *t*-best approximations on fuzzy anti-*n*-normed linear spaces. In [21] Surender Reddy considered the set of all *t*-best simultaneous approximation in fuzzy anti-2-normed linear spaces.

In the present paper, we consider the set of all *t*-best simultaneous approximation in fuzzy anti-*n*-normed linear spaces and use the concept of simultaneous *t*-proximinality and simultaneous *t*-Chebyshevity to introduce the theory of *t*-best simultaneous approximation in quotient spaces.

2. PRELIMINARIES:

Definition 2.1: Let $n \in N$ and let X be a real linear space of dimension $\geq n$. A real valued function $\|\bullet,\bullet,\ldots,\bullet\|$ on $\underbrace{X \times X \times \ldots \times X}_{n} = X^n$ satisfying the following conditions

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nN_1: $\|x_1, x_2, \dots, x_n\| = 0$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent,

nN_2: $\|x_1, x_2, \dots, x_n\|$ is invariant under any permutation of $x_1, x_2, \dots, x_n$,

nN_3: $\|x_1, x_2, \dots, x_{n-1}, \alpha x_n\| = |\alpha| \, \|x_1, x_2, \dots, x_{n-1}, x_n\|$, for every $\alpha \in R$,

nN_4: $\|x_1, x_2, \dots, x_{n-1}, y+z\| \le \|x_1, x_2, \dots, x_{n-1}, y\| + \|x_1, x_2, \dots, x_{n-1}, z\|$ for all $y, z, x_1, x_2, \dots, x_{n-1} \in X$,

then the function $\|\bullet, \bullet, \dots, \bullet\|$ is called an *n*-norm on *X* and the pair $(X, \|\bullet, \bullet, \dots, \bullet\|)$ is called *n*-normed linear space.

Example 2.2: A trivial example of an *n*-normed linear space is $X = R^n$ equipped with the following Euclidean *n*-norm.

$$\|x_1, x_2, \dots, x_n\|_E = |\det(x_{ij})| = abs\left(\begin{vmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \dots & \dots & \dots & \dots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{vmatrix} \right),$$

where $x_i = (x_{i1}, x_{i2}, \dots, x_{in}) \in R^n$ for each $i = 1, 2, \dots, n$.

Definition 2.3: Let *X* be a linear space over a real field *F*. A fuzzy subset *N* of $\underbrace{X \times X \times \dots \times X}_{n} \times R$ is called a fuzzy *n*-norm on *X* if the following conditions are satisfied for all $x_1, x_2, \dots, x_n, y \in X$.

$(n - N_1)$ For all $t \in R$ with $t \le 0$, $N(x_1, x_2, \dots, x_n, t) = 0$,

$(n - N_2)$: For all $t \in R$ with $t > 0$, $N(x_1, x_2, \dots, x_n, t) = 1$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent,

$(n - N_3)$: $N(x_1, x_2, \dots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \dots, x_n$,

$(n - N_4)$: For all $t \in R$ with $t > 0$, $N(x_1, x_2, \dots, x_{n-1}, cx_n, t) = N(x_1, x_2, \dots, x_{n-1}, x_n, \frac{t}{|c|})$, if $c \ne 0$, $c \in F$,

$(n - N_5)$: For all $s, t \in R$,
$N(x_1, x_2, \dots, x_{n-1}, x_n + y, s + t) \ge \min\{N(x_1, x_2, \dots, x_{n-1}, x_n, s), N(x_1, x_2, \dots, x_{n-1}, y, t)\}$,

$(n - N_6)$: $N(x_1, x_2, \dots, x_n, t)$ is a non-decreasing function of $t \in R$ and $\lim_{t \to \infty} N(x_1, x_2, \dots, x_n, t) = 1$.

Then the pair (X, N) is called a fuzzy *n*-normed linear space (briefly F-n-NLS).

Example 2.4: Let $(X, \|\bullet, \bullet, \dots, \bullet\|)$ be a *n*-normed linear space. Define

$$N(x_1, x_2, \dots, x_n, t) = \frac{t}{t + \|x_1, x_2, \dots, x_n\|}, \text{ if } t > 0, \ t \in R, \ x_1, x_2, \dots, x_n \in X$$
$$= 0, \text{ if } t \le 0, \ t \in R, \ x_1, x_2, \dots, x_n \in X.$$

Then (X, N) is a fuzzy *n*-normed linear space.

Definition 2.5: Let *X* be a linear space over a real field *F*. A fuzzy subset *N* of $\underbrace{X \times X \times \dots \times X}_{n} \times R$ is called a fuzzy anti-*n*-norm on *X* if the following conditions are satisfied for all $x_1, x_2, \dots, x_n, y \in X$.

$(a - n - N_1)$ For all $t \in R$ with $t \le 0$, $N(x_1, x_2, \dots, x_n, t) = 1$,

$(a-n-N_2)$: For all $t \in R$ with $t>0$, $N(x_1, x_2, ..., x_n, t) = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent,

$(a-n-N_3)$: $N(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$,

$(a-n-N_4)$: For all $t \in R$ with $t>0$, $N(x_1, x_2, ..., x_{n-1}, cx_n, t) = N(x_1, x_2, ..., x_{n-1}, x_n, \frac{t}{|c|})$, if $c \neq 0$, $c \in F$,

$(a-n-N_5)$: For all $s, t \in R$,
$N(x_1, x_2, ..., x_{n-1}, x_n + y, s+t) \leq \max\{N(x_1, x_2, ..., x_{n-1}, x_n, s), N(x_1, x_2, ..., x_{n-1}, y, t)\}$,

$(a-n-N_6)$: $N(x_1, x_2, ..., x_n, t)$ is a non-increasing function of $t \in R$ and $\lim_{t\to\infty} N(x_1, x_2, ..., x_n, t) = 0$.

Then the pair (X, N) is called a fuzzy anti-*n*-normed linear space (briefly Fa-*n*-NLS).

Remark 2.6: From $(a-n-N_3)$, it follows that in Fa-*n*-NLS,

$(a-n-N_4)$: For all $t \in R$ with $t>0$, $N(x_1, x_2, ..., cx_i, ..., x_n, t) = N(x_1, x_2, ..., x_i, ..., x_n, \frac{t}{|c|})$, if $c \neq 0$, $c \in F$,

$(a-n-N_5)$: For all $s, t \in R$,
$N(x_1, x_2, ..., x_i + x_i', ..., x_n, s+t) \leq \max\{N(x_1, x_2, ..., x_i, ..., x_n, s), N(x_1, x_2, ..., x_i', ..., x_n, t)\}$.

Example 2.7: Let $(X, \|\bullet, \bullet, ..., \bullet\|)$ be a *n*-normed linear space. Define

$$N(x_1, x_2, ..., x_n, t) = \frac{k\|x_1, x_2, ..., x_n\|}{kt^n + m\|x_1, x_2, ..., x_n\|}, \text{ if } t>0,\ t \in R,\ k, m, n \in R^+,\ x_1, x_2, ..., x_n \in X$$
$$= 1, \text{ if } t \leq 0,\ t \in R,\ x_1, x_2, ..., x_n \in X.$$

Then (X, N) is a fuzzy anti-*n*-normed linear space. In particular if $k = m = n = 1$ we have

$$N(x_1, x_2, ..., x_n, t) = \frac{\|x_1, x_2, ..., x_n\|}{t + \|x_1, x_2, ..., x_n\|}, \text{ if } t>0,\ t \in R,\ x_1, x_2, ..., x_n \in X$$
$$= 1, \text{ if } t \leq 0,\ t \in R,\ x_1, x_2, ..., x_n \in X,$$

which is called the standard fuzzy anti-*n*-norm induced by the *n*-norm $\|\bullet, \bullet, ..., \bullet\|$.

Definition 2.8: A sequence $\{x_k\}$ in a fuzzy anti-*n*-normed linear space (X, N) is said to be converges to $x \in X$ if given $t>0$, $0<r<1$, there exists an integer $n_0 \in N$ such that

$$N(x_1, x_2, ..., x_{n-1}, x_k - x, t) < r, \ \forall\ k \geq n_0.$$

Theorem 2.9: In a fuzzy anti-*n*-normed linear space (X, N), a sequence $\{x_k\}$ converges to $x \in X$ if and only

$$\lim_{k\to\infty} N(x_1, x_2, ..., x_{n-1}, x_k - x, t) = 0, \ \forall\ t>0.$$

Definition 2.10: Let (X, N) be a fuzzy anti-*n*-normed linear space. Let $\{x_k\}$ be a sequence in *X* then $\{x_k\}$ is said to be a Cauchy sequence if $\lim_{k\to\infty} N(x_1, x_2, ..., x_{n-1}, x_{k+p} - x_k, t) = 0$, $\forall\ t>0$ and $p = 1, 2, 3, \ldots$.

Definition 2.11: A fuzzy anti-*n*-normed linear space (X, N) is said to be complete if every Cauchy sequence in X is convergent.

Definition 2.12: A complete fuzzy anti-*n*-normed linear space (X, N) is called a fuzzy anti-*n*-Banach space.

Definition 2.13: Let (X, N) be a fuzzy anti-*n*-normed linear space. The open ball $B(x,r,t)$ and the closed ball $B[x,r,t]$ with the center $x \in X$ and radius $0 < r < 1$, $t > 0$ are defined as follows:

$$B(x,r,t) = \{ y \in X : N(x_1, x_2, ..., x_{n-1}, x - y, t) < r\}$$

$$B[x,r,t] = \{ y \in X : N(x_1, x_2, ..., x_{n-1}, x - y, t) \leq r\}$$

Definition 2.14: Let (X, N) be a fuzzy anti-*n*-normed linear space. A subset *A* of *X* is said to be open if there exists $r \in (0,1)$ such that $B(x,r,t) \subset A$ for all $x \in A$ and $t > 0$.

Definition 2.15: Let (X, N) be a fuzzy anti-*n*-normed linear space. A subset *A* of X is said to be closed if for any sequence $\{x_k\}$ in *A* converges to $x \in A$.

i.e., $\lim_{k \to \infty} N(x_1, x_2, ..., x_{n-1}, x_k - x, t) = 0$, for all $t > 0$ implies that $x \in A$.

Corollary 2.16: Let (X, N) be a fuzzy anti-*n*-normed linear space. Then *N* is a continuous function on

$$\underbrace{X \times X \times ... \times X}_{n} \times R.$$

3. *t*-BEST SIMULTANEOUS APPROXIMATION:

Definition 3.1: Let (X, N) be a fuzzy anti-*n*-normed linear space. A subset *A* of X is called *F*-bounded if there exists $t > 0$ and $0 < r < 1$ such that $N(x_1, x_2, ..., x_{n-1}, x, t) < r$, $\forall \; x \in A$.

Definition 3.2: Let (X, N) be a fuzzy anti-*n*-normed linear space, *W* be a subset of X and *M* be a *F*-bounded subset in *X*. For $t > 0$, we define

$$d(M, W, t) = \inf_{w \in W} \sup_{m \in M} N(x_1, x_2, ..., x_{n-1}, m - w, t).$$

An element $w_0 \in W$ is called a *t*-best simultaneous approximation to M from W if for $t > 0$,

$$d(M, W, t) = \sup_{m \in M} N(x_1, x_2, ..., x_{n-1}, m - w_0, t).$$

The set of all *t*-best simultaneous approximations to M from W will be denoted by $S_W^t(M)$ and we have,

$$S_W^t(M) = \{ w \in W : \sup_{m \in M} N(x_1, x_2, ..., x_{n-1}, m - w, t) = d(M, W, t)\}$$

Definition 3.3: Let W be a subset of a fuzzy anti-*n*-normed linear space (X, N) then *W* is called a simultaneous *t*-proximinal subset of *X* if for each *F*-bounded set *M* in *X*, there exists at least one *t*-best simultaneous approximation from *W* to *M*. Also W is called a simultaneous *t*-Chebyshev subset of X if for each *F*-bounded set *M* in *X*, there exists a unique *t*-best simultaneous approximation from *W* to *M*.

Definition 3.4: Let (X, N) be a fuzzy anti-*n*-normed linear space. A subset E of X is said to be convex if $(1-\lambda)x + \lambda y \in E$ whenever $x, y \in E$ and $0 < \lambda < 1$.

Lemma 3.5: Every open ball in a fuzzy anti-*n*-normed linear space (X, N) is convex.

Theorem 3.6: Suppose that W is a subset of a fuzzy anti-*n*-normed linear space (X, N) and *M* is *F*-bounded in *X*. Then $S_W^t(M)$ is a *F*-bounded subset of X and if *W* is convex and is a closed subset of *X* then $S_W^t(M)$ is closed and is convex for each *F*-bounded subspace *M* of *X*.

Proof: Since *M* is *F*-bounded, there exists $t > 0$ and $0 < r < 1$ such that $N(x_1, x_2, ..., x_{n-1}, x, t) < r$, for all $x \in M$. If $w \in S_W^t(M)$, then

$$\sup_{m\in M} N(x_1, x_2, ..., x_{n-1}, m - w, t) = d(M, W, t).$$

Now, for all $m \in M$ and $w \in S_W^t(M)$,

$$\begin{aligned} N(x_1, x_2, ..., x_{n-1}, w, 2t) &= N(x_1, x_2, ..., x_{n-1}, w - m + m, 2t) \\ &\le \max\{N(x_1, x_2, ..., x_{n-1}, w - m, t), N(x_1, x_2, ..., x_{n-1}, m, t)\} \\ &\le \sup_{m\in M} \max\{N(x_1, x_2, ..., x_{n-1}, w - m, t), N(x_1, x_2, ..., x_{n-1}, m, t)\} \\ &\le \max\{\sup_{m\in M} N(x_1, x_2, ..., x_{n-1}, w - m, t), \sup_{m\in M} N(x_1, x_2, ..., x_{n-1}, m, t)\} \\ &\le \max\{d(M, W, t), r\} \le r_0, \text{ for some } 0 < r_0 < 1. \end{aligned}$$

Then $S_W^t(M)$ is *F*-bounded. Suppose that *W* is convex and is a closed subset of *X*. We show that $S_W^t(M)$ is convex and closed. Let $x, y \in S_W^t(M)$ and $0 < \lambda < 1$. Since *W* is convex, there exists $z_\lambda \in W$ such that $z_\lambda = \lambda x + (1 - \lambda) y$, for each $0 < \lambda < 1$. Now for $t > 0$ we have,

$$\begin{aligned} \sup_{m\in M} N(x_1, x_2, ..., x_{n-1}, (\lambda x + (1 - \lambda) y) - m, t) &= \sup_{m\in M} N(x_1, x_2, ..., x_{n-1}, z_\lambda - m, t) \\ &\ge d(M, W, t). \end{aligned}$$

On the other hand, for a given $t > 0$, take the natural number *n* such that $t > \frac{1}{n}$, we have

$$\begin{aligned} \sup_{m\in M} N(x_1, x_2, ..., x_{n-1}, (\lambda x + (1 - \lambda) y) - m, t) &= \sup_{m\in M} N(x_1, x_2, ..., x_{n-1}, \lambda(x - y) + y - m, t) \\ &\le \sup_{m\in M} \max\{N(x_1, x_2, ..., x_{n-1}, x - y, \frac{1}{\lambda n}), N(x_1, x_2, ..., x_{n-1}, y - m, t - \frac{1}{n})\} \\ &\le \max\{N(x_1, x_2, ..., x_{n-1}, x - y, \frac{1}{\lambda n}), \sup_{m\in M} N(x_1, x_2, ..., x_{n-1}, y - m, t - \frac{1}{n})\} \\ &\le \lim_{n\to\infty} \left(\sup_{m\in M} N(x_1, x_2, ..., x_{n-1}, y - m, t - \frac{1}{n}) \right) = d(M, W, t). \end{aligned}$$

So $S_W^t(M)$ is convex. Finally let $\{w_n\} \subset S_W^t(M)$ and suppose $\{w_n\}$ converges to some *w* in X. Since $\{w_n\} \subset W$ and W is closed so $w \in W$. Therefore by Corollary 2.16, for $t > 0$ we have

$$\begin{aligned} \sup_{m\in M} N(x_1, x_2, ..., x_{n-1}, m - w, t) &= \sup_{m\in M} N(x_1, x_2, ..., x_{n-1}, \lim_{n\to\infty} w_n - m, t) \\ &= \lim_{n\to\infty} \sup_{m\in M} N(x_1, x_2, ..., x_{n-1}, w_n - m, t) = d(M, W, t). \end{aligned}$$

Theorem 3.7: The following assertions are hold for $t > 0$,

(i) $d(M + x, W + x, t) = d(M, W, t), \quad \forall \ x \in X$,

(ii) $d(\lambda M, \lambda W, t) = d(M, W, \frac{t}{|\lambda|}), \quad \forall \ \lambda \in C$,

(iii) $S^t_{W+x}(M+x) = S^t_W(M)+x$, $\forall\ x \in X$,

(iv) $S^{|\lambda|t}_{\lambda W}(\lambda M) = \lambda S^t_W(M)+x$, $\forall\ \lambda \in C$,

Proof: (i) $$d(M+x, W+x, t) = \inf_{w\in W}\sup_{m\in M} N(x_1, x_2, ..., x_{n-1}, (m+x)-(w+x), t)$$
$$= \inf_{w\in W}\sup_{m\in M} N(x_1, x_2, ..., x_{n-1}, m-w, t) = d(M, W, t)$$

(ii) Clearly equality holds for $\lambda = 0$, so suppose that $\lambda \neq 0$. Then,

$$d(\lambda M, \lambda W, t) = \inf_{w\in W}\sup_{m\in M} N(x_1, x_2, ..., x_{n-1}, \lambda(m-w), t)$$
$$= \inf_{w\in W}\sup_{m\in M} N(x_1, x_2, ..., x_{n-1}, m-w, \frac{t}{|\lambda|}) = d(M, W, \frac{t}{|\lambda|})$$

(iii) $x+W \in S^t_{W+x}(M+x)$ if and only if,
$$\sup_{m+x\in M+x} N(x_1, x_2, ..., x_{n-1}, m+x-w-x, t) = d(M+x, W+x, t)$$

and by (i), the above equality holds if and only if,

$$\sup_{m\in M} N(x_1, x_2, ..., x_{n-1}, m-w, t) = d(M, W, t)$$

for all $w \in W$ and this shows that $w \in S^t_W(M)$. So $x+w \in S^t_W(M)+x$.

(iv) $y_0 \in S^{|\lambda|t}_{\lambda W}(\lambda M)$ if and only if $y_0 \in \lambda W$ and,
$$d(\lambda W, \lambda M, |\lambda|t) = \sup_{\lambda m\in \lambda M} N(x_1, x_2, ..., x_{n-1}, y_0 - \lambda m, |\lambda|t)$$
$$= \sup_{m\in M} N(x_1, x_2, ..., x_{n-1}, \frac{y_0}{\lambda} - m, t)$$

But by (ii), we have $d(\lambda M, \lambda W, |\lambda|t) = d(W, M, t)$. So we have $\frac{y_0}{\lambda} \in W$ and

$d(M, W, t) = \sup_{m\in M} N(x_1, x_2, ..., x_{n-1}, \frac{y_0}{\lambda} - m, t)$ or equivalently $\frac{y_0}{\lambda} \in S^t_W(M)$ and the proof is completed.

Corollary 3.8: Let A be a nonempty subset of a fuzzy anti-*n*-normed linear space (X, N) then the following statements are hold.

(i) *A* is simultaneous *t*-proximinal (respectively simultaneous *t*-Chebyshev) if and only if *A*+*y* is simultaneous *t*-proximinal (respectively simultaneous *t*-Chebyshev), for each $y \in X$,

(ii) *A* is simultaneous *t*-proximinal (respectively simultaneous *t*-Chebyshev) if and only if αA is simultaneous $|\alpha|t$ -proximinal (respectively simultaneous $|\alpha|t$ -Chebyshev), for each $\alpha \in C$.

Corollary 3.9: Let A be a nonempty subspace of a fuzzy anti-*n*-normed linear space *X* and *M* be a *F*-bounded subset of *X*. Then for $t > 0$,

(i) $d(A, M+y, t) = d(A, M, t)$, $\forall\ y \in A$,

(ii) $S^t_A(M+y) = S^t_A(M)+y$, $\forall\ y \in A$,

(iii) $d(A, \alpha M, |\alpha|t) = d(A, M, t)$, for $0 \neq \alpha \in C$,

(iv) $S^{|\alpha|t}_A(\alpha M) = \alpha S^t_A(M)$, for $0 \neq \alpha \in C$.

4. SIMULTANEOUS t-PROXIMINALITY AND SIMULTANEOUS t-CHEBYSHEVITY IN QUOTIENT SPACES:

In this section we give characterization of simultaneous *t*-proximinality and simultaneous *t*-Chebyshevity in quotient spaces.

Definition 4.1: Let (X,N) be a fuzzy anti-*n*-normed linear space, M be a linear manifold in *X* and let $Q: X \to X/M$ be the natural map $Qx = x+M$. We define

$$N(x_1, x_2, ..., x_{n-1}, x+M, t) = \inf\{N(x_1, x_2, ..., x_{n-1}, x+y, t) : y \in M\}, \quad t > 0$$

Theorem 4.2: If M is a closed subspace of a fuzzy anti-*n*-normed linear space (X,N) and $N(x_1, x_2, ..., x_{n-1}, x+M, t)$ is defined as above then

(a) N is a fuzzy anti-n-norm on X/M.
(b) $N(x_1, x_2, ..., x_{n-1}, Qx, t) \le N(x_1, x_2, ..., x_{n-1}, x, t)$.
(c) If (X,N) is a fuzzy anti-*n*-Banach space then so is $(X/M, N)$.

Proof: (a) It is clear that $N(x_1, x_2, ..., x_{n-1}, x+M, t) = 1$ for $t \le 0$.

Let $N(x_1, x_2, ..., x_{n-1}, x+M, t) = 0$ for $t > 0$. By definition there is a sequence $\{x_k\}$ in M such that $N(x_1, x_2, ..., x_{n-1}, x+x_k, t) \to 0$. So $x + x_k \to 0$ or equivalently $x_k \to (-x)$ and since *M* is closed so $x \in M$ and $x+M = M$, the zero element of X/M. On the other hand we have,

$$\begin{aligned} N(x_1, x_2, ..., x_{n-1}, (x+M)+(y+M), t) &= N(x_1, x_2, ..., x_{n-1}, (x+y)+M, t) \\ &\le N(x_1, x_2, ..., x_{n-1}, (x+m)+(y+n), t) \\ &\le \max\{N(x_1, x_2, ..., x_{n-1}, x+m, t_1), N(x_1, x_2, ..., x_{n-1}, y+n, t_2)\} \end{aligned}$$

for $m, n \in M$, $x_1, x_2, ..., x_{n-1}, x, y \in X$ and $t_1 + t_2 = t$. Now if we take infimum on both sides, we have,

$$N(x_1, x_2, ..., x_{n-1}, (x+M)+(y+M), t) \le \max\{N(x_1, x_2, ..., x_{n-1}, x+M, t_1), N(x_1, x_2, ..., x_{n-1}, y+M, t_2)\}.$$

Also we have,
$$\begin{aligned} N(x_1, x_2, ..., x_{n-1}, \alpha(x+M), t) &= N(x_1, x_2, ..., x_{n-1}, \alpha x+M, t) \\ &= \inf\{N(x_1, x_2, ..., x_{n-1}, \alpha x+\alpha y, t) : y \in M\} \\ &= \inf\{N(x_1, x_2, ..., x_{n-1}, x+y, \frac{t}{|\alpha|}) : y \in M\} \\ &= N(x_1, x_2, ..., x_{n-1}, x+M, \frac{t}{|\alpha|}) \end{aligned}$$

and the remaining properties are obviously true. Therefore N is a fuzzy anti-*n*-norm on X/M.

(b) We have,
$$\begin{aligned} N(x_1, x_2, ..., x_{n-1}, Qx, t) &= N(x_1, x_2, ..., x_{n-1}, x+M, t) \\ &= \inf\{N(x_1, x_2, ..., x_{n-1}, x+y, t) : y \in M\} \\ &\le N(x_1, x_2, ..., x_{n-1}, x, t) \end{aligned}$$

(c) Let $\{y_k + M\}$ be a Cauchy sequence in X/M. Then there exists $\varepsilon_k > 0$ such that $\varepsilon_k \to 0$ and $N(x_1, x_2, ..., x_{n-1}, (y_k+M)-(y_{k+1}+M), t) \le \varepsilon_k$. Let $z_1 = 0$. We choose $z_2 \in M$ such that,

$$N(x_1, x_2, ..., x_{n-1}, y_1-(y_2-z_2), t) \le \max\{N(x_1, x_2, ..., x_{n-1}, (y_1-y_2)+M, t), \varepsilon_1\}.$$

But $N(x_1, x_2,..., x_{n-1}, (y_1 - y_2) + M, t) \le \varepsilon_1$. Therefore,

$$N(x_1, x_2,..., x_{n-1}, y_1 - (y_2 - z_2), t) \le \max\{\varepsilon_1, \varepsilon_1\} = \varepsilon_1 .$$

Now suppose z_{k-1} has been chosen, $z_k \in M$ can be chosen such that

$N(x_1, x_2,..., x_{n-1}, (y_{k-1} + z_{k-1}) - (y_k + z_k), t) \le \max\{N(x_1, x_2,..., x_{n-1}, (y_{k-1} - y_k) + M, t), \varepsilon_{k-1}\}$ and therefore, $N(x_1, x_2,..., x_{n-1}, (y_{k-1} + z_{k-1}) - (y_k + z_k), t) \le \max\{\varepsilon_{k-1}, \varepsilon_{k-1}\} = \varepsilon_{k-1}$.

Thus, $\{y_k + z_k\}$ is Cauchy sequence in *X*. Since *X* is complete, there is an y_0 in *X* such that $y_k + z_k \to y_0$ in *X*. On the other hand $y_k + M = Q(y_k + z_k) \to Q(y_0) = y_0 + M$. Therefore every Cauchy sequence $\{y_k + M\}$ is convergent in X/M and so X/M is complete and $(X/M, N)$ is a fuzzy anti-*n*-Banach space.

Definition 4.3: Let *A* be a nonempty set in a fuzzy anti-*n*-normed linear space (X, N). For $x \in X$ and $t > 0$, we shall denote the set of all elements of *t*-best approximation to *x* from *A* by $P_A^t(x)$;

i.e., $\quad P_A^t(x) = \{y \in A : d(A, x, t) = N(x_1, x_2,..., x_{n-1}, y - x, t)\}$.

where, $\quad d(A, x, t) = \inf\{N(x_1, x_2,..., x_{n-1}, y - x, t) : y \in A\} = \inf_{y \in A} N(x_1, x_2,..., x_{n-1}, y - x, t)$.

If each $x \in X$ has at least (respectively exactly) one *t*-best approximation in *A* then *A* is called a *t*-proximinal (respectively *t*-chebyshev) set.

Lemma 4.4: Let (X, N) be a fuzzy anti-*n*-normed linear space and *M* be a *t*-proximinal subspace of *X*. For each nonempty *F*-bounded set *S* in X and $t > 0$,

$$d(S, M, t) = \sup_{s \in S} \inf_{m \in M} N(x_1, x_2,..., x_{n-1}, s - m, t)$$

Proof: Since *M* is *t*-proximinal it follows that for each $s \in S$ there exists $m_s \in P_M^t(S)$ such that for $t > 0$,

$$N(x_1, x_2,..., x_{n-1}, s - m_s, t) = \inf_{m \in M} N(x_1, x_2,..., x_{n-1}, s - m, t) .$$

So,
$$\begin{aligned} d(S, M, t) &= \inf_{m \in M} \sup_{s \in S} N(x_1, x_2,..., x_{n-1}, s - m, t) \\ &\le \sup_{s \in S} N(x_1, x_2,..., x_{n-1}, s - m_s, t) \\ &\le \sup_{s \in S} \inf_{m \in M} N(x_1, x_2,..., x_{n-1}, s - m, t) \\ &\le \inf_{m \in M} \sup_{s \in S} N(x_1, x_2,..., x_{n-1}, s - m, t) = d(S, M, t) \end{aligned}$$

This implies that, $\quad d(S, M, t) = \sup_{s \in S} \inf_{m \in M} N(x_1, x_2,..., x_{n-1}, s - m, t)$.

Example 4.5: Let $(X = R^2, \|\bullet, \bullet, ..., \bullet\|)$ be a anti-*n*-normed linear space and consider (X, N) as its standard induced fuzzy anti-*n*-normed linear space (Example 2.7). A nonempty subset *S* of *X* is *F*-bounded if and only if *S* is bounded in $(X, \|\bullet, \bullet, ..., \bullet\|)$. If we take $M = R$ we can easily prove that *M* is proximinal in $(X, \|\bullet, \bullet, ..., \bullet\|)$.

Lemma 4.6: Let (X, N) be a fuzzy anti-*n*-normed linear space, *M* be a *t*-proximinal subspace of X and S be an arbitrary subset of X then the following assertions are equivalent:

(i) *S* is a *F*-bounded subset of *X*.
(ii) S/M is a *F*-bounded subset of X/M .

Proof: Suppose that S be a F-bounded subset of X. Then there exist $t>0$, $0<r<1$ such that, $N(x_1,x_2,...,x_{n-1},x,t)<r$, for all $x\in S$. But,

$$N(x_1,x_2,...,x_{n-1},x+M,t)=\inf_{y\in M}N(x_1,x_2,...,x_{n-1},x+y,t)\le N(x_1,x_2,...,x_{n-1},x,t)\le r.$$

So, $(i)\Rightarrow(ii)$ is proved. Now to prove that $(ii)\Rightarrow(i)$. Let S/M be a F-bounded subset of X/M. Since M is t-proximinal, then for each $s\in S$ there exists $m_s\in M$ such that $m_s\in P_M^t(S)$. So for each $s\in S$,

$$N(x_1,x_2,...,x_{n-1},s-m_s,t)=\inf_{m\in M}N(x_1,x_2,...,x_{n-1},s-m,t) \tag{1}$$

Now from Lemma 4.4, we conclude that for $t>0$,

$$\begin{aligned}\sup_{s\in S}N(x_1,x_2,...,x_{n-1},s-m_s,t)&=\sup_{s\in S}\inf_{m\in M}N(x_1,x_2,...,x_{n-1},s-m,t)\\&=\inf_{m\in M}\sup_{s\in S}N(x_1,x_2,...,x_{n-1},s-m,t).\end{aligned}$$

Then for $0<r<1$ such that $\sup_{s\in S}N(x_1,x_2,...,x_{n-1},s-m_s,t)\le r$ and $t>0$ there exists $m_r\in M$ such that,

$$\sup_{s\in S}N(x_1,x_2,...,x_{n-1},s-m_r,t)\le\sup_{s\in S}N(x_1,x_2,...,x_{n-1},s-m_s,t)-r\le 0.$$

So by (1), for all $s\in S$ we have,

$$\begin{aligned}N(x_1,x_2,...,x_{n-1},s,t)&=N(x_1,x_2,...,x_{n-1},s-m_r+m_r,t)\\&\le\max\{N(x_1,x_2,...,x_{n-1},s-m_r,\tfrac{t}{2}),N(x_1,x_2,...,x_{n-1},m_r,\tfrac{t}{2})\}\\&\le\sup_{s\in S}\max\{N(x_1,x_2,...,x_{n-1},s-m_r,\tfrac{t}{2}),N(x_1,x_2,...,x_{n-1},m_r,\tfrac{t}{2})\}\\&\le\max\{(\sup_{s\in S}N(x_1,x_2,...,x_{n-1},s-m_s,\tfrac{t}{2})-r),N(x_1,x_2,...,x_{n-1},m_r,\tfrac{t}{2})\}\\&=\max\{(\sup_{s\in S}\inf_{m\in M}N(x_1,x_2,...,x_{n-1},s-m,\tfrac{t}{2})-r),N(x_1,x_2,...,x_{n-1},m_r,\tfrac{t}{2})\}\\&\le\max\{(\sup_{s\in S}N(x_1,x_2,...,x_{n-1},s+M,\tfrac{t}{2})-r),N(x_1,x_2,...,x_{n-1},m_r,\tfrac{t}{2})\}.\end{aligned}\tag{2}$$

Since S/M is F-bounded, by its definition we can find $0<r_0<1$ such that in the right hand side of (2) be less than or equal to r_0 and this completes the proof.

Lemma 4.7: Let M be a t-proximinal subspace of a fuzzy anti-n-normed linear space (X,N) and $W\supseteq M$ a subspace of X. Let K be F-bounded in X. If $w_0\in S_W^t(K)$, then

$$w_0+M\in S_{W/M}^t(K/M).$$

Proof: Since K is F-bounded by Lemma 4.6, K/M is F-bounded in X/M. Assume that $w_0\in S_W^t(K)$ and $w_0+M\notin S_{W/M}^t(K/M)$. Thus there exists $w'\in M$ such that for $t>0$,

$$\begin{aligned}\sup_{k\in K}N(x_1,x_2,...,x_{n-1},k-(w'+M),t)&<\sup_{k\in K}N(x_1,x_2,...,x_{n-1},k-(w_0+M),t)\\&\le\sup_{k\in K}N(x_1,x_2,...,x_{n-1},k-w_0,t)=d(K,W,t)\end{aligned}\tag{3}$$

On the other hand for each $k\in K$ and for $t>0$,

$$N(x_1,x_2,...,x_{n-1},k-(w'+M),t)=\inf_{m\in M} N(x_1,x_2,...,x_{n-1},k-(w'+m),t)$$

Then for each $0<\varepsilon<1$ and $k\in K$ there exists $m_k\in M$ such that for $t>0$,

$$N(x_1,x_2,...,x_{n-1},k-(w'+m_k),t)\le N(x_1,x_2,...,x_{n-1},k-(w'+M),t)+\varepsilon .$$

Since $w'+m_k\in M$ we conclude that

$$\begin{aligned} d(K,W,t) &\le \sup_{k\in K} N(x_1,x_2,...,x_{n-1},k-(w'+m_k),t) \\ &\le \sup_{k\in K} N(x_1,x_2,...,x_{n-1},k-(w'+M),t)+\varepsilon \end{aligned}$$

Thus, $$d(K,W,t)\le \sup_{k\in K} N(x_1,x_2,...,x_{n-1},k-(w'+M),t) \tag{4}$$

By (3) and (4) we get,

$$d(K,W,t)\le \sup_{k\in K} N(x_1,x_2,...,x_{n-1},k-(w'+M),t)<d(K,W,t),$$

and this is a contradiction. Therefore $w_0+M\in S^t_{W/M}(K/M)$ and the proof is completed.

Corollary 4.8: Let *M* be a *t*-proximinal subspace of a fuzzy anti-*n*-normed linear space (X,N) and $W\supseteq M$ a subspace *X*. If *W* is simultaneous *t*-proximinal then W/M is a simultaneous *t*-proximinal subspace of X/M .

Corollary 4.9: Let *M* be a *t*-proximinal subspace of a fuzzy anti-*n*-normed linear space (X,N) and $W\supseteq M$ a subspace *X*. If *W* is simultaneous *t*-proximinal then for each *F*-bounded set *K* in X,

$$Q(S^t_W(K))\subseteq S^t_{W/M}(K/M).$$

Theorem 4.10: Let *M* be a *t*-proximinal subspace of a fuzzy anti-*n*-normed linear space (X,N) and $W\supseteq M$ subspace of X. If *K* is *F*-bounded set in *X* such that $w_0+M\in S^t_{W/M}(K/M)$ and $m_0\in S^t_M(K-w_0)$, then $w_0+m_0\in S^t_W(K)$.

Proof: In view of Lemma 4.4, for $t>0$ we have,

$$\begin{aligned} \sup_{k\in K} N(x_1,x_2,...,x_{n-1},(k-w_0)-m_0,t) &= \inf_{m\in M}\sup_{k\in K} N(x_1,x_2,...,x_{n-1},(k-w_0)-m,t) \\ &= \sup_{k\in K}\inf_{m\in M} N(x_1,x_2,...,x_{n-1},k-(w_0+m),t) \\ &= \sup_{k\in K} N(x_1,x_2,...,x_{n-1},k-(w_0+M),t) \\ &\le \sup_{k\in K} N(x_1,x_2,...,x_{n-1},k-(w+M),t) \quad \forall\ w\in W \\ &\le \sup_{k\in K} N(x_1,x_2,...,x_{n-1},k-w,t) \quad \forall\ w\in W . \end{aligned}$$

Hence, $\sup_{k\in K} N(x_1,x_2,...,x_{n-1},k-(w_0+m_0),t)\le \sup_{k\in K} N(x_1,x_2,...,x_{n-1},k-w,t)\ \ \forall\ w\in W$.

But $w_0+m_0\in W$. Then $w_0+m_0\in S^t_W(K)$ and so the proof is completed.

Theorem 4.11: Let *M* be a *t*-proximinal subspace of a fuzzy anti-*n*-normed linear space (X,N) and $W\supseteq M$ a simultaneous *t*-proximinal subspace of *X*. Then for each *F*-bounded set *K* in *X*,

$$Q(S^t_W(K))=S^t_{W/M}(K/M)$$

Proof: By Corollary 4.9, we obtain

$$Q(S_W^t(K)) \subseteq S_{W/M}^t(K/M).$$

Also by Lemma 4.6, W/M is simultaneous *t*-proximinal in X/M. Now let, $w_0 + M \in S_{W/M}^t(K/M)$, where $w_0 \in W$. By simultaneous *t*-proximinality of *M* there exists $m_0 \in M$ such that $m_0 \in S_M^t(K - w_0)$. Then in view of Theorem 4.10, we conclude that $w_0 + m_0 \in S_W^t(K)$. Therefore $w_0 + M \in Q(S_W^t(K))$ and the proof is completed.

Corollary 4.12: Let *W* and *M* be subspaces of a fuzzy anti-*n*-normed linear space (X, N). If *M* is simultaneous *t*-proximinal then the following assertions are equivalent:

(i) W/M is simultaneous *t*-proximinal in X/M.
(ii) $W + M$ is simultaneous *t*-proximinal in *X*.

Proof: $(i) \Rightarrow (ii)$. Let *K* be an arbitrary *F*-bounded set in *X*. Then by Lemma 4.6, K/M is a *F*-bounded set in X/M. Since $(W+M)/M = W/M$ and *M* are simultaneous *t*-proximinal it follows that there exists $w_0 + M \in (W+M)/M$ and $m_0 \in M$ such that $w_0 + M \in S_{(W+M)/M}^t(K/M)$ and $m_0 \in S_M^t(K - w_0)$. By Theorem 4.10, we have $w_0 + m_0 \in S_{W+M}^t(K)$. This shows that $W + M$ is simultaneous *t*-proximinal in *X*.

$(ii) \Rightarrow (i)$. Since $W + M$ is simultaneous *t*-proximinal and $W + M \supseteq M$, by Corollary 4.8, we have $(W+M)/M = W/M$ is simultaneous *t*-proximinal.

Theorem 4.13: Let *W* and *M* be subspaces of a fuzzy anti-*n*-normed linear space (X, N). If *M* is simultaneous *t*-Chebyshev then the following assertions are equivalent:

(i) W/M is simultaneous *t*-Chebyshev in X/M.
(ii) $W + M$ is simultaneous *t*-Chebyshev in *X*.

Proof: $(i) \Rightarrow (ii)$, By hypothesis $(W+M)/M = W/M$ is simultaneous *t*-Chebyshev. Assume that (ii) is false. Then some *F*-bounded subset *K* of *X* has two distinct simultaneous *t*-best approximations such as l_0 and l_1 in $W + M$. Thus we have,

$$l_0,\ l_1 \in S_{W+M}^t(K). \tag{5}$$

Since $W + M \supseteq M$ by lemma 4.6, $l_0 + M$, $l_1 + M \in S_{(W+M)/M}^t(K/M) = S_{W/M}^t(K/M)$.

Since W/M is simultaneous *t*-Chebyshev, $l_0 + M = l_1 + M$. So there exists $0 \neq m_0 \in M$ such that $l_1 = l_0 + m_0$.

By (5) for all $t > 0$,

$$\begin{aligned}
\sup_{k\in K} N(x_1, x_2, ..., x_{n-1}, (k - l_0) - m_0, t) &= \sup_{k\in K} N(x_1, x_2, ..., x_{n-1}, k - l_1, t) \\
&= \sup_{k\in K} N(x_1, x_2, ..., x_{n-1}, k - l_0, t) \\
&= d(K, W + M, t) \\
&= d(K - l_0, W + M, t) \leq d(K - l_0, M, t)
\end{aligned}$$

This shows that both *m* and zero are simultaneous *t*-best approximations to $S - l_0$ from *M* and this is a contradiction.
$(ii) \Rightarrow (i)$. Assume that (i) does not hold. Then for some *F*-bounded subset *K* of *X*, K/M has two distinct simultaneous *t*-best approximations such as $w + M$ and

$w' + M$ in W/M. Thus $w - w' \notin M$. Since M is simultaneous t-proximinal there exists simultaneous t-best approximations m and m' to $K - w$ and $K - w'$ from M respectively. Therefore $m \in S_M^t(K - w)$ and $m' \in S_M^t(K - w')$. Since $W + M \supseteq M$, $w + M$ and $w' + M$ are in $S_{W/M}^t(K/M) = S_{(K+M)/M}^t(K/M)$, by Theorem 4.10, $w + m$ and $w' + m' \in S_{W+M}^t(K)$.

But $W + M$ is simultaneous t-Chebyshev. Thus $w + m = w' + m'$ and so $w - w' \in M$, which is a contradiction.

Corollary 4.14: Let M be simultaneous t-Chebyshev subspace of a fuzzy anti-n-normed linear space (X, N). If $W \supseteq M$ is a simultaneous t-Chebyshev subspace in X, then the following assertions are equivalent:

(i) W is simultaneous t-Chebyshev in X.
(ii) W/M is simultaneous t-Chebyshev in X/M.

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