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APPROACH OF FRACTAL GEOMETRY IN FUNCTIONAL ANALYSIS

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ABSTRACT

In this paper we discuss about fractal geometry and give more precisely basic ideas and concepts on fractal geometry. Also, we discuss the approach of fractal geometry in functional analysis.

Keywords: *Fractal geometry, Fractal dimension, Space of Fractal.*

2000 Mathematical Subject Classification: *28A80, 11M41.*

1. INTRODUCTION

In this paper we will describe some of the wonderful new ideas in the area of mathematics known as fractal geometry. Today fractal geometry is completely new area of research in the field of science and technology. It has wide range of applications. Fractals in nature are so complicated and irregular that it is hopeless to model them by simply using classical geometrical objects.

This paper explores various concepts of fractal i.e. fractal dimension, their characteristics and their application in real life [4] and also, we define the space of fractal. As we can see fractals are incredibly complicated and often quite beautiful geometric shapes that can be generated by simple rules. We have tried to find out the mathematics behind these incredible geometric shapes called fractals. We have also discussed the space where fractals live.

1.1 Historical Part

Fractal geometry is a branch of mathematics concerned with irregular patterns made of parts that are in some way similar to the whole. In the 17th century, the introduction of calculus along with Euclidean geometry was widely used tool for analysis of smooth objects thereby ignoring complex structure. Therefore, most of the natural phenomena were overlooked to interpret, as they were difficult to express mathematically on account of complexity. It remained a challenge to explain nature's complexity, irregularity and self-similarity using the Euclidean geometry.

In 1975, Mandelbrot gave a term "Fractal" for irregular object which is derived from a Latin word "Fractus" (broken). He was the first ever scientist to study irregular object. Therefore, is famous as "Father of Fractals". Mandelbrot discovered fractals while he was working on an experiment to find the length of Britain's coastline (1967) [6].

Later in 1980's fractals have attracted interest of many scientists internationally. Many natural phenomena have been studied using fractal and fractal geometry as a major tool to describe complex structure [5]. Now a day's fractals are in vogue as they are used to make real life images using computer. Fractal geometry made it easy to describe model and analyze any irregular complex forms found in nature [2].

Main reason for using fractal geometry is that several natural objects can be approximated by fractals to a certain degree including clouds, mountain ranges, coastlines, vegetables etc. thus providing a baseline for simulating spatial patterns often found in nature. So, algorithms are developed using fractal geometry concepts to simulate and understand objects in the nature.

Fractals are not discovered until the invention of computers. It was virtually impossible to discover fractals before the advent of the computer because of their complexity and gargantuan output. Fractals existed only in theory before the computer and were published in a most basic from in 1918 by Gaston Julia. Later they published it in an advanced form in 1925. Due to this advanced formula Benoit Mandelbrot became famous.

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1.2. Definitions

There are some basic definitions of Fractal-

- A Fractal is a rough or fragmented geometric shape that can split into parts, each of which is (at least approximately) a reduced size copy of the whole, this property is called self-similarity.
- A Fractal is a geometry figure that repeats itself under several levels of magnification a shape, that appear irregular at all scales of length.
- A Fractal is a never-ending pattern that repeats itself at different scales, this property is called self-similarity.

1.3. Properties of Fractals

- **Self-similarity-** Self–similarity means we can magnify the Fractal object many times and after every step we can see the same shape. There are three types of self–similarity found in Fractals-
  1. Exact self-similarity.
  2. Quasi self-similarity.
  3. Statistical self-similarity.

- **Scale invariance-** A system, function, or statistic has a scale invariance if changing the scale by a certain amount does not change the system, function, or statistics shape or properties. For example, if you zoom in on a Koch Snowflake (see example 2.3), it looks the same. Mathematically the property of scale invariance is written as $f(AX) = A^K f(X)$ for fixed numbers A and K.

- **Non-integer dimension-** The non-integer dimension is a more difficult to explain. Classical geometry deals with objects of integer dimension; zero-dimensional point, one dimensional lines and curves, two-dimensional plan figures such as square and circles and three-dimensional solids such as cubes and spheres. However, many natural phenomena are better described using a dimension between two whole numbers. So, while a straight line has a dimension of one, a fractal has a dimension between one and two depending on how much space it takes up as it twists and curves.

1.4. Characteristics Features of a Fractals

- Classical methods of geometry are insufficient to describe Fractals because of its complexity.
- At all scales it has a fine structure.
- It possesses property of self-similarity.
- It possesses property of scale invariance.
- It is formed by simple iterative process.
- It has non integer dimension.
- It has a Hausdorff dimension which is greater than its Topological dimension.

1.5. Applications

1.5.1 **Fractal Medicine-** We can use Fractal math to quantify, describe, diagnose and perhaps soon to help cure diseases.

1.5.2 **Computer Science-** The most useful use of Fractals in computer science is the Fractal image compression. This kind of compression uses the fact that the real world is well described by Fractal geometry.

1.5.3 **Fluid Mechanics-** The study of turbulence in flows is very adapted to Fractals. Turbulent flows are chaotic and very difficult to model correctly.

1.5.4 **Telecommunications-** A new application is Fractal-shaped antennae that reduce greatly the size and the weight of the antennas.

1.5.5 **Surface Physics-** Fractals are used to describe the roughness of surfaces. A rough surface is characterized by a combination of two different Fractals.

1.5.6 **Landscapes-** Fractal landscapes are a very classic application of Fractals. If we look at a mountain, we will not find its shape being a cone, but instead we will find a more complicated shape with some smaller hills and valleys.

1.5.7 **Data Compression-** In December 1992, Microsoft released a compact disk entitled Encarta Encyclopedia. It contains thousands of articles, 7000 photographs, 100 animations, and 800 color maps. All of this is in less than 600 megabytes of data. It was possible only by the fractals and the answered lies in the mathematics of Fractal data compression.

1.5.8 **Market Analysis-** Benoit Mandelbrot introduced a new fractal theory which is helpful to analyze the market. After plotting price data of market for a month some rises and fall will be appeared in the graph. It this graph is plotted for weak or even for a day same rises and fall will be appeared. This is self- similar property of fractal. It is also called Brownian self-similarity.

1.5.9 **Fractal in Art-** The concept of Fractal can be used to create pictures which are more complicated in nature and they have the property of self-similarity. Mandelbrot set suggested by Benoit Mandelbrot is a good example of fractal science.

1.5.10 **Fractal in Bacteria Cultures-** The spreading of bacteria can be modeled by Fractals such as the diffusion fractals.

2. FRACTAL DIMENSION

Fractal dimension can be calculated by taking the limit of the quotient if the log change in object size and the log change in measurement scale, as the measurement scale approaches zero.

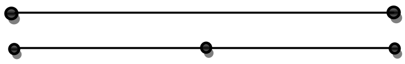
i.e.

$$\text{Fractal dimension} = \lim_{s \to 0} \frac{\log\ (change\ \ in\ object\ \ size)}{\log\ (change\ \ in\ measurment\ \ \ size)},$$

where $s =$ measurement scale.

Example 2.1 Consider a straight line. Now blow up the line by a factor of two. The line is now twice as long as before.

After blowing



$\therefore$ Fractal dimension $= \lim_{2 \to 0} \frac{\log 2}{\log 2} = 1.$

Corresponding to dimension 1.

Thus, we see that the fractal dimension of a rectifiable curve is 1.

Example 2.2: For a plane region of area A, a disk of radius ϵ covers an area of $\pi\epsilon^2$.

Hence $N(\epsilon) = \frac{A}{\pi\epsilon^2} = \frac{A}{\pi}\ (\epsilon)^{-2}$ disks are needed. The fractal dimension of a plane region is

$$\begin{aligned}
\delta \ &= -\lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log \epsilon} \\
&= -\lim_{\epsilon \to 0} \frac{\log \dfrac{A}{\pi} (\epsilon)^{-2}}{\log \epsilon} \\
&= -\lim_{\epsilon \to 0} \left[\frac{\log \frac{A}{\pi} + (-2) \log \epsilon}{\log \epsilon} \right] \\
&= 0 + 2 = 2.
\end{aligned}$$

Hence the fractal dimension is 2.

Example 2.3: Koch Snowflake - The Koch curve is simply constructed using an iterated procedure beginning with the initiator of the set as the unit line segment (step $k = 0$ in the figure 2.1). The unit line segment is divided into third and middle third removed. The middle third is then replaced with two equal segments, both one third in length, which form an equilateral triangle (step $k = 1$); this step is the generator of the curve. At the next step ($k = 2$), the middle third is removed from each of the four segments and each is replaced with two equal segments as before. This process is repeated an infinite number of times to produce the Koch curve.

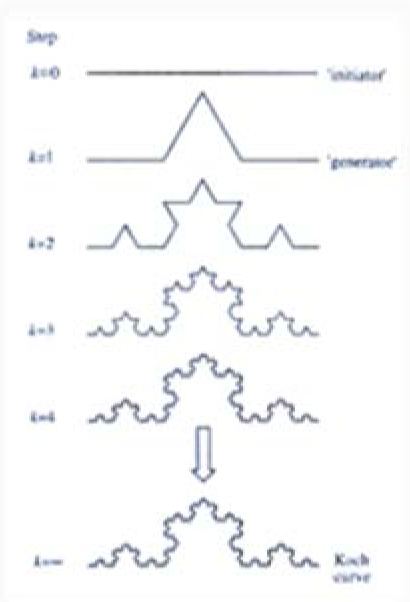


Figure-2.1: Construction of Koch curve

The Koch curve is a fractal dimension object possessing a fractal dimension. Each smaller segment of the Koch curve is an exact replica of the whole curve. At each scale there are four sub-segments making up the curve, each one a one third reduction of the original curve. Thus $N(\epsilon) = 4,\ \epsilon = \frac{1}{3}$ and the fractal dimension based on expression.

$$\delta = -\lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log \epsilon} \quad \Rightarrow \quad \delta = -\lim_{\epsilon \to 0} \frac{\log 4}{\log \frac{1}{3}}$$

$$\delta = \lim_{\epsilon \to 0} \frac{\log 4}{\log 3} \quad \Rightarrow \quad \delta = 1.2618 \ldots.$$

That is, the Koch curve has a dimension greater than that of the unit line and less than that of the unit area.

3. APPROACH IN FUNCTIONAL ANALYSIS

First, we define the concept of space in Mathematics. A space is an algebraic structure like (X, d) where X is any non-empty set and d is function. If this function satisfied some axioms, then algebraic structure (X, d) is called a space.

Example 3.1: Let X be a non-empty set and d be a metric function, if this metric function 'd' satisfied the following axioms-

1. $d(x, y) = d(y, x)\ \forall\, x, y \in X,$
2. $0 < d(x, y) < \infty\ \forall\, x, y \in X,\ x \neq y,$
3. $d(x, x) = 0 \quad \forall\, x \in X,$
4. $d(x, y) \leq d(x, z) + d(z, y)\ \forall\, x, y, z \in X.$

Then (X, d) is called metric space.

Fractal geometry is concerned with the description, classification, analysis and observation of subsets of metric spaces (X, d). The metric spaces are usually, but not always, of an inherently simple geometrical character; the subsets are typically geometrically "complicated". There are a number of general properties for the subsets of metric spaces, which occur over and over again, which are very basic, and which form part of the vocabulary for describing fractal sets and other subsets of metric spaces. The properties which are invariant under metric space equivalence.

Now we introduce the space of fractals.

Let (X, d) be a complete metric space and $\mathcal{H}(X)$ denotes the space where elements are the compact subsets of X, other then empty. Then, we define following definitions:

Definition 3.1: Let (X, d) be a complete metric space, $x \in X$, and $B \in \mathcal{H}(X)$. Define

$$d(x, B) = Min\{d(x, y) : y \in B\}.$$

Then $d(x, B)$ is the distance from the point x to the set B.

Definition 3.2: Let (X, d) be a complete metric space. Let $A, B \in \mathcal{H}(X)$. Define

$$d(A, B) = Max\{d(x, B) : x \in A\}.$$

$d(A, B)$ is the distance from the set $A \in \mathcal{H}(X)$ to the set $B \in \mathcal{H}(X)$.

Definition 3.3: Let (X, d) be a complete metric space. Then the Hausdroff distance between points A and B in $\mathcal{H}(X)$ is defined by

$$h(A, B) = d(A, B) \vee d(B, A).$$

The completeness of the space of Fractals-

We refer to $(\mathcal{H}(X), h)$ as "the space of fractals". Fractals are not defined by a short legalistic statement, but by the many pictures and contexts which refer to them. **Any subset of $(\mathcal{H}(X), h)$ is a fractal**. However, as with the concept of "a space", more meaning is suggested than is formalized that **the space of fractals $(\mathcal{H}(X), h)$ is a complete metric space.**

4. CONCLUSION

This paper has described various fundamental concepts, characteristics and properties of fractal geometry theory. The calculation of fractal dimension has also been discussed. This paper provides discussion for the study of fractal and fractal geometry and also space of fractal as a result of continuous interest of the researchers can use this review in various fields of science and technology.

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