MODULO ELEVEN: NUMBER THEORY (11A07; 11A41)

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ABSTRACT

The following paper provides a technique, with the help of which, one can easily and efficiently find remainders when a two or three-digit number is divided by 11. The paper provides logical and simple proofs to the formulae as well as verifies them with the help of examples.

Keywords: Number theory, modular arithmetic, congruence modulo, modulo 11, division algorithm.

INTRODUCTION

Modular Arithmetic

\[ a \mod n = r \]

When \( a = qn + r \), where \( q \) is the quotient and \( r \) is the remainder upon dividing \( a \) by \( n \), we write \( a \mod n = r \).

\begin{align*}
3 \mod 2 & = 1, \text{ since } 3 = 1 \times 2 + 1 \\
6 \mod 2 & = 0, \text{ since } 6 = 2 \times 3 + 0 \\
11 \mod 3 & = 2, \text{ since } 11 = 3 \times 3 + 2
\end{align*}

Definitions and Concepts Used

- \( \mod 11/\text{modulo 11} \) implies the remainder obtained when a number, here a two or three-digit number, is divided by 11.

- Let \( m \) be any multiple of 11, then \( m \mod 11 = 0 \).

- Division Algorithm for integers: Suppose \( b > 0 \) and \( a \) are integers. Then there exist unique integers \( q \) and \( r \) such that \( a = bq + r \), where \( 0 \leq r < b \). The number \( q \) is called the quotient and \( r \) is called the remainder. \([2]\)

- The set \( Z_n = \{0, 1, \ldots, n - 1\} \) for \( n \geq 1 \) is a group under addition modulo \( n \). For any \( j > 0 \) in \( Z_n \), the inverse of \( j \) is \( n - j \). \([3]\)

FORMULATION, PROOFS, and EXAMPLES

Let \( ab \) be a two-digit number, where \( a \) is the tens digit and \( b \) is the ones digit, then:

\[ ab \mod 11 = \begin{cases} 
 b - a & \text{if } a \leq b \\
11 - (a - b) & \text{if } a > b
\end{cases} \]

Proof:

i) Given \( a \leq b \),

\[ ab \text{ can be written as:} \]

\[ a \times 10 + b \]

Now, adding and subtracting \( a \) from (1), we get,

\[ a \times 11 + (b - a) \] \hspace{1cm} (2)

Equation (2) modulo eleven gives the remainder as \( b - a \).
ii) Given \( a > b \)

Equation (2) modulo eleven, in this case, leaves us with \((b-a) \mod 11\), where \(b-a < 0\). The remainder, however, cannot be negative and therefore here, \( b-a \) can be viewed as the additive inverse of \( a-b \), which will be \( 11-(a-b) \).

(The set of remainders when a number is divided by 11 is \( \{0,1,2,3,4,5,6,7,8,9,10\} \) which is the group \( \mathbb{Z}_{11} \) with the operation addition modulo 11. The additive inverse of an element \( j \) is given by \( 11 - j \)).

**Examples:**

1. Consider the number 24. Here, \( 2 < 4 \) and hence, the remainder is \( 4 - 2 = 2 \).

   It can be verified by dividing 24 by 11. \( 24 = 11 \times 2 + 2 \), by division algorithm, which clearly gives 2 as the remainder.

2. Consider the number 42. Here, \( 4 > 2 \) and hence, using the above formula, the remainder would be \( 11 - (4 - 2) = 9 \).

   It can be verified by dividing 42 by 11. \( 42 = 11 \times 3 + 9 \), by division algorithm, which clearly gives 9 as the remainder.

**Let \( abc \) be a three-digit number, where \( a \) is the hundreds digit, \( b \) is the tens digit and \( c \) is the ones digit, then:**

\[
abc \mod 11 = \begin{cases} 
    a + c - b & \text{if } a + c \geq b \text{ and } a + c - b \leq 11 \\
    (a + c) - b - 11 & \text{if } a + c \geq b \text{ and } a + c - b > 11 \\
    11 - (b - (a + c)) & \text{if } a + c < b 
\end{cases}
\]

**Proof:**

(i) \( a + c \geq b, (a + c) - b \leq 11 \)

\( abc \) can be written in the expanded form as:

\[
a \times 100 + b \times 10 + c
\]

Adding and subtracting \( 10a \) and \( b \) in (1), we get,

\[
110 \times a + 11 \times b + c - 10 \times a - b
\]

Writing \( -10 \times a \) as \( -11 \times a + a \) in (2), we get,

\[
110 \times a + 11 \times b + c - 11 \times a + a - b
\]

Equation (5) modulo 11, gives the remainder as \( a + c - b \).

(ii) \( a + c \geq b, (a + c) - b > 11 \)

Reducing equation (5) modulo 11, gives \( a + c - b \). Here, since, \( a + c - b > 11 \), it can be written as:

\[
11 + [(a + c - b) - 11], \text{ where } [(a + c - b) - 11] < 11
\]

(Since, the maximum value of \( a + c - b \) can be 18, when \( a = 9 \) and \( b = 0 \).)

Equation (6) modulo 11, gives the remainder as \( (a + c - b) - 11 \).

(iii) \( a + c < b \)

Again, reducing equation (5) modulo 11, we get, \( a + c - b \). Here, \( a + c - b < 0 \), therefore, it can be viewed as the additive inverse of \( b - (a + c) \), where \( b - (a + c) > 0 \), which is given by

\[
11 - [b - (a + c)].
\]

These three cases are exhaustive; any three digit number will fit into either of the three cases.

**Examples:**

1. Consider the number 148 Using the above formula, we get the remainder as \( 1 + 8 - 4 = 5 \).

   It can be verified using the division algorithm:

   \( 148 = 11 \times 13 + 5 \), which clearly gives the remainder as 5.
2. Consider the number 819. Using the above formula, since $8 + 9 - 1 = 16 > 11$, therefore the remainder is $16 - 11 = 5$.
Using division algorithm, we get, $819 = 11 \times 74 + 5$, which clearly gives the remainder as 5.

3. Consider the number 191. Using the above formula, the remainder will be $11 - \left(9 - (1+1)\right) = 4$.
Using division algorithm, we get, $191 = 11 \times 17 + 4$, which clearly gives the remainder as 4.

Let $a0b$ be a three-digit number, where $a$ is the hundreds digit, $0$ is the tens digit and $b$ is the ones digit, then:

$$a0b \mod 11 = \begin{cases} a + b & \text{if } a + b \leq 11 \\ a + b - 11 & \text{if } a + b > 11 \end{cases}$$

This is, in particular, for a three digit number whose tens digit is zero.

The formula for two-digits can also be obtained from the formula for three-digits by equating $a$ to 0.

REFERENCES